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Spreading dynamics of a discrete Nicholson's blowflies equation with distributed delay

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This paper is focused on spreading dynamics for a discrete Nicholson's blowflies model with time convolution kernel. This problem arises in the invasive activity of blowflies scattered in discrete spatial environment and has distributed maturated age. We found that for a *general* convolution kernel, the model can exhibit travelling wave phenomena in a discrete spatial habitat. In particular, we determine the minimal wave speed of travelling waves by deriving the non-existence of travelling waves, and we demonstrate that the minimal wave speed can determine the long time behaviour of solutions with compact initial function. Moreover, we prove that *all* travelling waves are strictly increasing, which implies that the waveforms remain monotone in the propagation process. Some numerical simulations are also presented to confirm the analytical results.

Keywords: minimal wave speed; spreading speed; travelling waves; monotonicity; Nicholson's blowflies model

2020 *Mathematics Subject Classification* 34A33; 35R10; 34R10; 92D25

1. Introduction

Since Nicholson [16] declared his significant work regarding competition for food in laboratory population of Australian sheep-blowfly (*Lucia cuprina*) in 1954, numerous models have been developed to simulate Nicholson's data of blowflies (see e.g. [6, 8, 11, 17]). By taking the spatial continuous diffusion and distributed maturation periods into account, Gourley and Ruan [6] proposed and analysed a diffusive Nicholson's blowflies equation with distributed delay

$$u_t'(t,x) = D\Delta u - \tau u(t,x) + \beta\tau\left(\int_{-\infty}^{t} k(t-s)u(s,x)\,\mathrm{d}s\right)\exp\left[-\int_{-\infty}^{t} k(t-s)u(s,x)\,\mathrm{d}s\right], \quad (1.1)$$

where $\beta, \tau > 0$, $(t,x) \in [0,\infty) \times \Omega$, Ω is either some finite domain or the whole $\mathbb{R}^n$, and f is non-negative and integrable with $\int_0^\infty f(t)\,\mathrm{d}t = 1$. They showed that the zero state $u = 0$ is globally stable if $\beta < 1$ and the unique non-zero state $u^* = \ln\beta$ is globally stable when $1 < \beta \leqslant e$. For the spatial domain $\Omega = \mathbb{R}$, Gourley [5] further showed the existence of travelling wave solution for (1.1) with the following special kernels

$$k(t) = \frac{1}{\tau}\,\mathrm{e}^{-t/\tau} \quad \text{and} \quad k(t) = \frac{t}{\tau^2}\,\mathrm{e}^{-t/\tau}.$$

By recasting the wave equations into a four-dimensional system of non-delay ODEs and employing the geometric singular perturbation method, Gourley [5] showed that (1.1) has a travelling wave solution for $\tau > 0$ sufficiently small. The existence of travelling wave solutions for some other special kernels was also studied in [11, 18], in which the theory of second-order functional differential equations was used to establish the existence of monotone travelling wave solutions when $1 < \beta \leqslant e$. However, the problems on the existence of the minimal wave speed and monotonicity of travelling waves were not addressed in [5, 11, 18].

Note that the original Nicholson's blowflies equation modelling the population of Australian sheep-blowfly (*Lucia cuprina*) is the ordinary differential equation (see [8, 16])

$$u_t'(t) = -\tau u(t) + \beta\tau u(t-1)\exp[-u(t-1)] \tag{1.2}$$

after rescaling, which does not contain the spatial structure. The generalized Nicholson's blowflies equation (1.1) is based on the assumption that the blowflies scattered in a continuous spatial environment and subjected to continuous random diffusion. In practice, however, blowflies do not necessarily spread continuously since they can fly from one place to another. It has been observed that for the aggregated dispersion, discrete models are more suitable than continuous models to describe the invasion phenomenon [1]. Models involving discrete equations are referred to as 'patch models', which can be found in many scientific disciplines, such as materials science, pattern formation, neural networks and population biology (see the survey by Chow and Mallet-Paret [2]). Besides, they are also natural outcomes of discretizing the corresponding spatial continuous model. In the literature, some discretizations of classical continuous models have been proposed, such as the discrete Fisher's equation [28], discrete Nagumo equation [27], discrete Allen–Cahn equation [3] and discrete Lotka–Volterra equation [7], etc., and there have been many studies focussed on the study of propagation dynamics of various types of discrete equations, see [4, 9, 10, 14, 19–26] and the references therein. However, the approximation of continuous models by spatial discretization is a delicate issue, and it is well known that there could exist essential differences between a discrete model and its associated continuous version.

To explore the spreading dynamics of the blowflies living in discrete environment with distributed maturated age, we consider the following formulation of the original Nicholson' blowflies equation in (1.2).

$$\begin{aligned} u_j'(t) &= D[u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)] - \tau u_j(t) \\ &\quad + \beta\tau\left(\int_{-\infty}^{t} k(t-s)u_j(s)\,\mathrm{d}s\right)\exp\left[-\int_{-\infty}^{t} k(t-s)u_j(s)\,\mathrm{d}s\right], \quad j \in \mathbb{Z}. \end{aligned} \tag{1.3}$$

It can also be regarded as the space discrete approximation of continuous model (1.1). The main purpose of this study is to explore whether the discrete model

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(1.3) can exhibit propagation phenomenon like the continuous model (1.1), and what the minimal wave speed is if it exists, and whether it can determine the long time asymptotic behaviour of (1.3).

Compared with (1.1), the wave equation of (1.3) is a mixed-type which contains both back- and forward-shifts. This mixed structure makes the study of existence of travelling waves of discrete model (1.3) more difficult. Meanwhile, it is hard to transform it into ODEs without delay such that, as the spatial continuous case studied in [**5**], the geometric singular perturbation theorem works. Moreover, the theorem of second-order functional differential equations used in [**11**, **18**] cannot be applied directly to (1.3) because of the effect of the difference operator. In this paper, we shall present a different method to investigate the existence of travelling wave solutions for the discrete model (1.3). Specifically, to prove the existence of travelling wave solutions, we first apply the monotone dynamical system approach to show the existence of travelling waves for an auxiliary system with finite distributed delay. Then, by establishing some priori estimates, we extend this existence result to (1.3) by employing the delay approximation technique and some comparison arguments. In particular, by analysing the spreading properties of the solution, we can show the non-existence of travelling wave solutions. This method enables us to obtain the minimal wave speed of travelling wave solutions and show that it can determine the long time behaviour of the solution with compact initial function. After the existence of travelling waves is discussed, we further consider the monotonicity problem of the travelling waves. By adapting the sliding method, we theoretically prove that *all* travelling waves of (1.3) are strictly increasing, which implies that the waveforms of the travelling waves remain monotone increasing during the propagation process. As far as we know, this is the first time that the monotonicity of travelling wave solutions is theoretically proved for Nicholson's blowflies model with distributed (infinite) delay.

The paper is organized as follows. In §2, we establish some preliminary results on the well posedness and comparison principle of solutions. In §3, we prove the existence of travelling waves and spreading speed, and also show the consistency of the minimal wave speed and the spreading speed. In §4, we prove that the travelling wave solutions are strictly monotone. Some numerical simulation results are presented in the last section to illustrate the analytical results established.

2. Preliminaries

In this section, we first establish some preliminary results on the well posedness and comparison principle of solutions for model (1.3).

Let $\mathbb{X} := \{a = \{a_i\}_{i\in\mathbb{Z}};\ \|a\|_{\mathbb{X}} = \sup_{i\in\mathbb{Z}} |a_i| < \infty\}$. Denote by $\mathbb{C} := C((-\infty, 0]; \mathbb{X})$. For $\psi = \{\psi_i\}_{i\in\mathbb{Z}}$ and $\phi = \{\phi_i\}_{i\in\mathbb{Z}}$ in $\mathbb{C}$, we write $\psi \geqslant \phi$ ($\psi \gg \phi$) if $\psi_i(s) \geqslant \phi_i(s)$ ($\psi_i(s) > \phi_i(s)$) for any $i \in \mathbb{Z}$, $s \in (-\infty, 0]$, and $\psi > \phi$ if $\psi \geqslant \phi$ but $\psi \neq \phi$. In a similar way, we write $a \geqslant (\gg, >) b$ for a, b in $\mathbb{X}$. For given $\omega \in \mathbb{X}$ with $\omega \geqslant 0$, denote $\mathbb{C}_\omega = \{\psi \in \mathbb{C} : 0 \leqslant \psi(\theta) \leqslant \omega,\ \forall\theta \in (-\infty, 0]\}$ and $\mathbb{X}_\omega = \{a \in \mathbb{X} : 0 \leqslant a \leqslant \omega\}$.

LEMMA 2.1. *Assume $1 < \beta \leqslant e$. For any initial function $\phi = \{\phi_j\}_{j\in\mathbb{Z}}$ in $\mathbb{C}_{u^*}$, (1.3) has a unique solution $u = \{u_j\}_{j\in\mathbb{Z}}$ with $u_j \in C([0, +\infty), [0, u^*])$.*

Proof. Note that (1.3) can be rewritten as

$$u_j'(t) = -(\tau + 2D)u_j(t) + M(u_j)(t), \quad j \in \mathbb{Z},\, t \geqslant 0. \tag{2.1}$$

Then for the initial function $\phi = \{\phi_j\}_{j\in\mathbb{Z}}$ with $\phi_j \in C((-\infty, 0], [0, u^*])$, the initial value problem of (1.3) is equivalent to

$$\begin{cases} u_j(t) = \mathrm{e}^{-(\tau+2D)t}\phi_j(0) + \int_0^t \mathrm{e}^{-(\tau+2D)(t-z)} M(u_j)(z)\,\mathrm{d}z, & j \in \mathbb{Z},\, t \geqslant 0, \\ u_j(t) = \phi_j(t), \quad j \in \mathbb{Z},\, t \in (-\infty, 0], \end{cases} \tag{2.2}$$

where

$$\begin{aligned} M(u_j)(t) = D[u_{j+1}(t) + u_{j-1}(t)] + \beta\tau \left(\int_0^\infty k(s) u_j(t-s)\,\mathrm{d}s \right) \\ \times \exp\left[- \int_0^\infty k(s) u_j(t-s)\,\mathrm{d}s \right]. \end{aligned}$$

Define an operator $\mathcal{T} = \{\mathcal{T}_j\}_{j\in\mathbb{Z}}$ on Ω by

$$\mathcal{T}_j(u)(t) = \mathrm{e}^{-(\tau+2D)t}\phi_j(0) + \int_0^t \mathrm{e}^{-(\tau+2D)(t-z)} M(u_j)(z)\,\mathrm{d}z, \quad j \in \mathbb{Z},\ t \geqslant 0,$$

where $\Omega := \{u = (u_j)_{j\in\mathbb{Z}} | u_j \in C([0, +\infty), [0, u^*]),\ u_j(t) = \phi_j(t), t \in (-\infty, 0]\}$. Note that $0 \leqslant M(u_j)(t) \leqslant 2Du^* + \beta\tau u^* \mathrm{e}^{-u^*} = (\tau + 2D)u^*$, and

$$0 \leqslant \mathcal{T}_j(u)(t) \leqslant \mathrm{e}^{-(\tau+2D)t} u^* + (\tau + 2D)u^* \int_0^t \mathrm{e}^{-(\tau+2D)(t-z)}\,\mathrm{d}z = u^*.$$

It follows that $\mathcal{T}(\Omega) \subseteq \Omega$. Choose $\mu \geqslant L_f$, where L_f is the Lipschitz constant of $f(u) = \beta\tau u\,\mathrm{e}^{-u}$ on $[0, u^*]$. Define a complete metric space (Ω, d_μ) with

$$d_\mu(u, v) := \|u - v\|_\mu, \quad \forall u, v \in \Omega, \quad \|u\|_\mu := \sup_{j\in\mathbb{Z}, t\geqslant 0} |u_j(t)| \mathrm{e}^{-\mu t}.$$

For any $u, v \in \Omega$, we have

$$\begin{aligned} \|M(u_j) - M(v_j)\|_\mu &= \sup_{j\in\mathbb{Z}, t\geqslant 0} |M(u_j)(t) - M(v_j)(t)| \mathrm{e}^{-\mu t} \\ &\leqslant \sup_{j\in\mathbb{Z}, t\geqslant 0} \{ D(|u_{j+1}(t) - v_{j+1}(t)| + |u_{j-1}(t) - v_{j-1}(t)|) \\ &\quad + \beta\tau \Big| \left(\int_0^\infty k(s) u_j(t-s)\,\mathrm{d}s \right) \exp\left[- \int_0^\infty k(s) u_j(t-s)\,\mathrm{d}s \right] \\ &\quad - \left(\int_0^\infty k(s) v_j(t-s)\,\mathrm{d}s \right) \exp\left[- \int_0^\infty k(s) v_j(t-s)\,\mathrm{d}s \right] \Big| \Big\} \mathrm{e}^{-\mu t} \\ &\leqslant 2D\|u - v\|_\mu + L_f \int_0^\infty k(s)\,\mathrm{e}^{-\mu s}\,\mathrm{d}s \|u - v\|_\mu \\ &\leqslant (2D + L_f)\|u - v\|_\mu. \end{aligned}$$

Thus,

$$\begin{aligned}\|\mathcal{T}_j(u)-\mathcal{T}_j(v)\|_\mu &= \sup_{j\in\mathbb{Z},t\geqslant 0}|\mathcal{T}_j(u)(t)-\mathcal{T}_j(v)(t)|\,\mathrm{e}^{-\mu t}\\ &\leqslant \sup_{j\in\mathbb{Z},t\geqslant 0}\mathrm{e}^{-\mu t}\int_0^t \mathrm{e}^{-(\tau+2D)(t-z)}|M(u_j)(z)-M(v_j)(z)|\,\mathrm{d}z\\ &\leqslant \sup_{j\in\mathbb{Z},t\geqslant 0}\mathrm{e}^{-(\tau+2D+\mu)t}\int_0^t \mathrm{e}^{(\tau+2D+\mu)z}(2D+L_f)\|u-v\|_\mu\,\mathrm{d}z\\ &\leqslant \kappa\|u-v\|_\mu,\end{aligned}$$

where $\kappa := \frac{2D+L_f}{\tau+2D+\mu} < 1$. By the contraction map theorem, we obtain that $\mathcal{T}$ has a unique fixed point u in Ω, which is a solution of (1.3). □

Definition 2.2. *A function $u(t)=\{u_j(t)\}_{j\in\mathbb{Z}}$ with $u_j(t)\in C(\mathbb{R},[0,u^*])$ is called a supersolution of (1.3) if there holds*

$$\begin{aligned}u_j'(t) &\geqslant D[u_{j+1}(t)+u_{j-1}(t)-2u_j(t)]-\tau u_j(t)\\ &\quad+\beta\tau\left(\int_{-\infty}^t k(t-s)u_j(s)\,\mathrm{d}s\right)\exp\left[-\int_{-\infty}^t k(t-s)u_j(s)\,\mathrm{d}s\right],\ j\in\mathbb{Z},\end{aligned}\tag{2.3}$$

for any $j\in\mathbb{Z}$, $t>0$; and a function $u(t)=\{u_j(t)\}_{j\in\mathbb{Z}}$ with $u_j(t)\in C(\mathbb{R},[0,u^*])$ is called a subsolution of (1.3) if there holds

$$\begin{aligned}u_j'(t) &\leqslant D[u_{j+1}(t)+u_{j-1}(t)-2u_j(t)]-\tau u_j(t)\\ &\quad+\beta\tau\left(\int_{-\infty}^t k(t-s)u_j(s)\,\mathrm{d}s\right)\exp\left[-\int_{-\infty}^t k(t-s)u_j(s)\,\mathrm{d}s\right],\ j\in\mathbb{Z},\end{aligned}\tag{2.4}$$

for any $j\in\mathbb{Z}$, $t>0$.

Lemma 2.3. *Let $u^1,u^2\in C(\mathbb{R},\mathbb{X}_{u^*})$ be, respectively, the subsolution and supersolution of (1.3) with $u^1(\theta)\leqslant u^2(\theta)$ for $\theta\in(-\infty,0]$. Then, $u^1(t)\leqslant u^2(t)$ for any $t\geqslant 0$.*

Proof. Let $u_j(t):=u_j^1(t)-u_j^2(t)$ for $j\in\mathbb{Z},t\in[0,+\infty)$. Define $y(t):=\sup_{j\in\mathbb{Z}}u_j(t)$, $t\in[0,+\infty)$. Next, we show that $y(t)\leqslant 0$ for all $t\geqslant 0$ to finish the proof. Suppose that there exists $t_0>0$ such that $y(t_0)>0$ and

$$y(t_0)\,\mathrm{e}^{-M_0t_0}=\sup_{t\geqslant 0}y(t)\,\mathrm{e}^{-M_0t}>y(s)\,\mathrm{e}^{-M_0s}\quad\text{for}\quad s\in[0,t_0),$$

where M_0 is chosen such that $M_0>L_f\int_0^\infty k(s)\,\mathrm{e}^{-M_0s}\,\mathrm{d}s$, and L_f is the Lipschitz constant of $f(u)=\beta\tau u\,\mathrm{e}^{-u}$ on $[0,u^*]$. Then, there exists a sequence $\{j_k\}_{k=1}^\infty$ such that $u_{j_k}(t_0)>0$ and $\lim_{k\to+\infty}u_{j_k}(t_0)=y(t_0)$. Let $\{t_k\}_{k=1}^\infty\subseteq[0,t_0]$ be a sequence

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such that $u_{j_k}(t_k)\,\mathrm{e}^{-M_0 t_k} = \max_{t\in[0,t_0]} u_{j_k}(t)\,\mathrm{e}^{-M_0 t}$. Note that

$$u_{j_k}(t_0)\,\mathrm{e}^{-M_0 t_0} \leqslant u_{j_k}(t_k)\,\mathrm{e}^{-M_0 t_k} \leqslant y(t_k)\,\mathrm{e}^{-M_0 t_k} \leqslant y(t_0)\,\mathrm{e}^{-M_0 t_0}. \tag{2.5}$$

Then, we have $\lim_{k\to+\infty} y(t_k)\,\mathrm{e}^{-M_0 t_k} = y(t_0)\,\mathrm{e}^{-M_0 t_0}$, and then $\lim_{k\to+\infty} t_k = t_0$. By (2.5), we also have

$$u_{j_k}(t_0)e^{M_0(t_k-t_0)} \leqslant u_{j_k}(t_k) \leqslant y(t_0)\,\mathrm{e}^{M_0(t_k-t_0)}.$$

Thus, $\lim_{k\to+\infty} u_{j_k}(t_k) = y(t_0)$. Note that for every $k \geqslant 1$,

$$0 \leqslant \left(u_{j_k}(t)\,\mathrm{e}^{-M_0 t}\right)'|_{t=t_k^-} = [u'_{j_k}(t_k) - M_0 u_{j_k}(t_k)]\,\mathrm{e}^{-M_0 t_k},$$

which implies that $M_0 u_{j_k}(t_k) \leqslant u'_{j_k}(t_k)$. Since $u'_{j_k}(t_k) = (u^1_{j_k}(t) - u^2_{j_k}(t))'|_{t=t_k}$, by the definition of supersolution and subsolution, we have

$$\begin{aligned}
u'_{j_k}(t_k) &\leqslant D[u_{j_k+1}(t_k) + u_{j_k-1}(t_k) - 2u_{j_k}(t_k)] - \tau u_{j_k}(t_k) \\
&\quad + L_f \int_0^\infty k(s)|u^1_{j_k}(t_k - s) - u^2_{j_k}(t_k - s)|\,\mathrm{d}s \\
&\leqslant D[u_{j_k+1}(t_k) + u_{j_k-1}(t_k) - 2u_{j_k}(t_k)] - \tau u_{j_k}(t_k) \\
&\quad + L_f \int_0^{t_k} k(s)[u^1_{j_k}(t_k - s) - u^2_{j_k}(t_k - s)]\,\mathrm{d}s \\
&\leqslant D[y(t_k) + y(t_k) - 2u_{j_k}(t_k)] - \tau u_{j_k}(t_k) + L_f \int_0^{t_k} k(s) y(t_k - s)\,\mathrm{d}s.
\end{aligned} \tag{2.6}$$

Since $y(t_0)\,\mathrm{e}^{-M_0(t_0-s)} > y(s)$ for $s \in [0, t_0)$, it follows that

$$\begin{aligned}
M_0 u_{j_k}(t_k) &\leqslant D[y(t_k) + y(t_k) - 2u_{j_k}(t_k)] - \tau u_{j_k}(t_k) \\
&\quad + L_f y(t_0) \int_0^{t_k} g(s)\,\mathrm{e}^{-M_0(s+t_0-t_k)}\,\mathrm{d}s.
\end{aligned} \tag{2.7}$$

Letting $k \to \infty$ in (2.7), we obtain that

$$M_0 y(t_0) \leqslant -\tau y(t_0) + L_f y(t_0) \int_0^{t_0} k(s)\,\mathrm{e}^{-M_0 s}\,\mathrm{d}s \leqslant L_f y(t_0) \int_0^\infty k(s)\,\mathrm{e}^{-M_0 s}\,\mathrm{d}s,$$

which is a contradiction since $M_0 > L_f \int_0^\infty k(s)\,\mathrm{e}^{-M_0 s}\,\mathrm{d}s$. Thus, we have $y(t) \leqslant 0$ for $t \in [0, +\infty)$. This completes the proof. □

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3. Travelling waves and spreading speed

In this section, we study the existence of travelling wave solutions and spreading speed for (1.3). For this, we first consider the delay truncated system of (1.3)

$$u_j'(t) = D[u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)] - \tau u_j(t) + \beta\tau \left(\int_0^r k(s)u_j(t-s)\,\mathrm{d}s\right) \exp\left[-\int_0^r k(s)u_j(t-s)\,\mathrm{d}s\right],\ j \in \mathbb{Z}, \tag{3.1}$$

where $r > 0$ is a given finite number. Since $\beta > 1$ and $\int_0^\infty k(s)\,\mathrm{d}s = 1$, we choose r sufficiently large so that $\hat{k}\beta > 1$, where $\hat{k} = \int_0^r k(s)\,\mathrm{d}s$. Then, (3.1) has two constant steady states 0 and $u_r^* = \frac{1}{\hat{k}} \ln \beta\hat{k}$. Clearly, $u_r^* \to u^*$ as $r \to \infty$. For such a truncated system, we shall show that

THEOREM 3.1. *Assume $1 < \beta \leqslant e$. Let $u_j(t)$ be the solution of (3.1). Then, we have*

(i) *For any $c > c_r^*$, if $u^0 \in \mathcal{C}_{u_r^*}$ with $u^0(s, j) = 0$ for $s \in [-r, 0]$ and j outside a bounded interval, then $\lim\limits_{t\to\infty, |j|\geqslant ct} u_j(t) = 0$.*

(ii) *For any $c < c_r^*$, if $u^0 \in \mathcal{C}_{u_r^*}\backslash\{0\}$, then $\lim\limits_{t\to\infty, |j|\leqslant ct} u_j(t) = u_r^*$.*

(iii) *For any $c \geqslant c_r^*$, (3.1) admits a travelling wave $u_j(t) = \phi(j + ct)$ such that $\phi(\xi)$ is monotone on $\xi \in \mathbb{R}$, $\phi(-\infty) = 0$ and $\phi(+\infty) = u_r^*$, while for any $c < c_r^*$, there exists no such travelling wave.*

Here $c_r^* = \inf\limits_{\lambda\in(0,\infty)} \frac{\mu(\lambda)}{\lambda}$, where $\mu = \mu(\lambda)$ is determined by

$$\mu - D\left[\mathrm{e}^{-\lambda} + \mathrm{e}^{\lambda} - 2\right] + \tau - \beta\tau \int_0^r k(s)\,\mathrm{e}^{-\mu s}\,\mathrm{d}s = 0.$$

To prove theorem 3.1, we apply the results on monotone dynamical systems [**13**], in which the spreading speed and travelling waves are established for abstract semiflows $\{Q_t\}_{t\geqslant 0}$ satisfying some prescribed assumptions (cf. (A1)–(A5) below).

Let $\mathcal{C}$ be the set of all bounded and continuous functions from $[-r, 0] \times \mathbb{Z}$ to $\mathbb{R}$. Clearly, any element in the space $\bar{\mathcal{C}} := C([-\tau, 0], \mathbb{R})$ can be regarded as a function in $\mathcal{C}$. For any $\xi > 0$, we set $[0, \xi] := \{u \in \mathbb{R} : 0 \leqslant u \leqslant \xi\}$ and $\mathcal{C}_\xi = \{u \in \mathcal{C} : 0 \leqslant u \leqslant \xi\}$. Similarly, define $\bar{\mathcal{C}}_\xi$. For any $u(\theta) = \{u(\theta, j)\}_{j\in\mathbb{Z}} =: \{u_j(\theta)\}_{j\in\mathbb{Z}}$, $v(\theta) = \{v(\theta, j)\}_{j\in\mathbb{Z}} := \{v_j(\theta)\}_{j\in\mathbb{Z}}$, we write $u(\theta) \geqslant v(\theta)\,(u(\theta) \gg v(\theta))$ provided $u_j(\theta) \geqslant v_j(\theta)\,(u_j(\theta) \gg v_j(\theta))$, $\forall\, j \in \mathbb{Z}$, $\theta \in [-r, 0]$, and $u > v$ provided $u \geqslant v$ but $u \neq v$. We equip $\mathcal{C}$ with the compact open topology, that is, $u^m \to u$ in $\mathcal{C}$ means that the sequence of u_j^m converges to u_j, as $m \to +\infty$, uniformly for j in any compact set of $\mathbb{Z}$. Define

$$\|u\| = \sum_{k=1}^{\infty} \frac{\max_{|j|\leqslant k,\, \theta\in[-r,0]} |u_j(\theta)|}{2^k},\ \forall u\ \in \mathcal{C}.$$

Then $(\mathcal{C}, \|\cdot\|)$ is a normed space. It follows that the topology in the metric space $(\mathcal{C}_\xi, \|\cdot\|)$ is the same as the compact open topology in $\mathcal{C}_\xi$. Moreover, $\mathcal{C}_\xi$ is a complete metric space.

Define a reflection operator R by $R(\phi)(x) = \phi(-x)$, and for each $y \in \mathbb{Z}$, a translation operator T_y by $T_y(\phi)(x) = \phi(x-y)$ for all $x \in \mathbb{R}$, respectively. For a given map $Q : \mathcal{C}_{u_r^*} \to \mathcal{C}_{u_r^*}$, we impose the following hypotheses on it:

(A1) $Q[R[u]] = R[Q[u]]$, $T_y[Q[u]] = Q[T_y[u]]$.

(A2) $Q : \mathcal{C}_{u_r^*} \to \mathcal{C}_{u_r^*}$ is monotone, that is, if $u \geqslant w$, then $Q[u] \geqslant Q[w]$.

(A3) The set $Q[\mathcal{C}_{u_r^*}](0,\cdot)$ is precompact in the space $C(\mathbb{Z},\mathbb{R})$ equipped with compact open topology, and there is an equivalent norm $\|\cdot\|^*$ in $\bar{\mathcal{C}}$ such that for any number $l \geqslant 0$, there exists $n = n(l) \in [0,1)$ such that for any $I = [a,b]_{\mathbb{Z}}$ of the length l and any $\mathcal{U} \subset \mathcal{C}_{u_r^*}$ with $\mathcal{U}(0,\cdot)$ precompact in $C(\mathbb{Z},\mathbb{R})$, we have $\alpha((Q[\mathcal{U}])_I) \leqslant n\alpha(\mathcal{U}_I)$, where α is the Kuratowski measure of non-compactness on $\mathcal{C}_I$.

(A4) $Q : \bar{\mathcal{C}}_{u_r^*} \to \bar{\mathcal{C}}_{u_r^*}$ where $\bar{\mathcal{C}}_{u_r^*} = \{u \in C([-r,0],\mathbb{R}) : 0 \leqslant u \leqslant u_r^*\}$, has exactly two fixed points 0 and u_r^*, and $\lim_{t\to\infty} Q[z] = u_r^*$, for $z \in [0, u_r^*] \setminus \{0\}$.

(A5) $Q : \mathcal{C}_{u_r^*} \to \mathcal{C}_{u_r^*}$ is continuous with respect to the compact open topology.

For any $u^0 = \{u_j^0\}_{j\in\mathbb{Z}} \in \mathcal{C}_{u_r^*}$, (3.1) has a unique global solution $u(t,u^0) = \{u_j(t,u^0)\}_{j\in\mathbb{Z}}$ with $u_j(\theta,u^0) = u_j^0(\theta)\ \forall j \in \mathbb{Z}, \theta \in [-r,0]$ and $0 \leqslant u(t,u^0) \leqslant u_r^*, \forall t \geqslant 0$. Let Q_t be the associated solution map of (3.1) at $t \geqslant 0$. Then,

$$Q_t(u^0)(\theta) = u(t+\theta, u^0),\ \forall\, \theta \in [-r,0],\ u^0 = \{u_j^0\}_{j\in\mathbb{Z}} \in \mathcal{C}_{u_r^*}.$$

LEMMA 3.2. *Assume* $1 < \beta \leqslant e$. *Then,* $\{Q_t\}_{t\geqslant 0}$ *is a monotone semiflow in* $\mathcal{C}_{u_r^*}$ *such that for each* $t > 0$, Q_t *satisfies all the hypotheses (A1)–(A5) for each* $t > 0$.

Proof. We easily observe that each time-t map is monotone and satisfies (A1) and (A2). Now we introduce

$$L_t[u^0](\theta) = \begin{cases} u^0(t+\theta) - u^0(0), & t+\theta < 0, \\ 0, & t+\theta \geqslant 0, -r \leqslant \theta \leqslant 0. \end{cases}$$

Let $S_t := Q_t - L_t,\ t \geqslant 0$. It is not difficult to show that for any given $\gamma > 0$, there is an equivalent norm $\|\cdot\|^*$ of $\bar{\mathcal{C}}$ such that for any $\psi \in \bar{\mathcal{C}}$ there holds $\|L_t(\psi)\|^* \leqslant \mathrm{e}^{-\gamma t}\|\psi\|^*, \forall t \geqslant 0$ (cf. [**13**]). Let $t > 0$ be given. By direct calculation, we can show that $Q_t[\mathcal{C}_{u_r^*}](0,\cdot) = u(t,\cdot,\mathcal{C}_{u_r^*})$ is precompact in $C(\mathbb{Z},\mathbb{R})$, and that $S(t)[\mathcal{U}]$ is precompact in $\mathcal{C}_{u_r^*}$ for any $\mathcal{U} \subset \mathcal{C}_{u_r^*}$ with $\mathcal{U}(0,\cdot)$ precompact in $\mathcal{C}_{u_r^*}$. Therefore, for any interval $I = [a,b]_{\mathbb{Z}}$ of the length l, we have

$$\alpha((Q_t[\mathcal{U}])_I) \leqslant \alpha((L_t[\mathcal{U}])_I) + \alpha((S_t[\mathcal{U}])_I) \leqslant \mathrm{e}^{-\gamma t}\alpha(\mathcal{U}_I),$$

where α is the Kuratowski measure of non-compactness on the space $\bar{\mathcal{C}}$. This implies that for each $t > 0$, Q_t satisfies (A3) with $n = e^{-\gamma t}$. To verify monostable structure,

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we consider a spatially homogeneous model associated with (3.1)

$$u'(t) = -\tau u(t) + \beta\tau\left(\int_0^r k(s)u(t-s)\,\mathrm{d}s\right)\exp\left(-\int_0^r k(s)u(t-s)\,\mathrm{d}s\right). \tag{3.2}$$

Clearly (3.2) has exact two fixed points: 0 and u_r^*. We linearize (3.2) at $u = 0$, and obtain

$$u'(t) = -\tau u(t) + \beta\tau\int_0^r k(s)u(t-s)\,\mathrm{d}s =: Pu_t.$$

We notice that $s(P) > 0$, where $s(P) = \max\{\mathrm{Re}\,\mu : \beta\tau\int_0^r k(s)\,\mathrm{e}^{-\mu s}\,\mathrm{d}s - \mu - \tau = 0\}$. Then, the equilibrium $u = 0$ is unstable. Together with the monotonicity of Q_t, item (A4) holds. In the following, we will show that $Q_t(u^0)$ is continuous in $(t, u^0) \in \mathbb{R}_+ \times \mathcal{C}_{u_r^*}$. We divide the proof into two steps.

Step 1. For any initial data $u^0 = \{u_j^0\}_{j\in\mathbb{Z}} \in \mathcal{C}_{u_r^*}$, we have $u_j(t, u^0) \in \mathcal{C}_{u_r^*}$. Moreover, there exists a C, which is independent of (j, u^0), such that $|\frac{\mathrm{d}u_j(t,u^0)}{\mathrm{d}t}| \leqslant C$, $\forall (t, j) \in [0, \infty) \times \mathbb{Z}$. Therefore, for any $j \in \mathbb{Z}$, $u_j(t, u^0)$ is uniformly continuous for $t \in [0, t_0 + 1]$ with $t_0 \geqslant 0$. As $u^0 \in \mathcal{C}_{u_r^*}$, then we know for each $j \in \mathbb{Z}$, the uniform continuity result holds for $u_j(t, u^0)$ on $t \in [-r, t_0 + 1]$. For any $\varepsilon > 0$, we can select $N = N(\varepsilon) > 0$ such that $\sum_{n=N+1}^{\infty} \frac{u_r^*}{2^{n-1}} < \frac{\varepsilon}{2}$. For the above $\varepsilon > 0$ and $t_0 \geqslant 0$, there exists a $\eta = \eta(\varepsilon, t_0) \in (0, \min\{1, t_0 + r\})$ such that for any $|t - t_0| < \eta$ and $j \in [-N, N]$, we have $|u_j(t + \theta) - u_j(t_0 + \theta)| < \frac{\varepsilon}{2}$. Hence,

$$\begin{aligned}
\|Q_t(u^0) - Q_{t_0}(u^0)\| &= \sum_{n=1}^{\infty} \frac{\max_{|j|\leqslant n,\,\theta\in[-r,0]} |u_j(t+\theta, u^0) - u_j(t_0+\theta, u^0)|}{2^n} \\
&= \left(\sum_{n=1}^{N} + \sum_{n=N+1}^{\infty}\right) \frac{\max_{|j|\leqslant n,\theta\in[-r,0]} |u_j(t+\theta, u^0) - u_j(t_0+\theta, u^0)|}{2^n} \\
&\leqslant \sum_{n=1}^{N} \frac{\varepsilon}{2^{n+1}} + \sum_{n=N+1}^{\infty} \frac{2u_r^*}{2^n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

We finish the proof of step 1, which implies that for given $u^0 \in \mathcal{C}_{r^*}$, $Q_t(u^0)$ is continuous in $t \geqslant 0$.

Step 2. In this step, for any given $[0, t_0]$ with $t_0 > 0$, we will show that $Q_t(u^0)$ is uniformly continuous in $u^0 \in \mathcal{C}_{r^*}$. Let $u^0(s) = \phi(s) = \{\phi_j(s)\}_{j\in\mathbb{Z}}$ and $s \in [-r, 0]$. It suffices to show if $\phi^{(h)} \to \phi$ $(h \to \infty)$ in $\mathcal{C}_{u_r^*}$, then $Q_t(\phi^{(h)}) \to Q_t(\phi)$ in $\mathcal{C}_{u_r^*}$ uniformly on $[0, t_0]$.

In order to obtain the above result, we first prove there exists a subsequence $\{Q_t(\phi^{(h_n^{(n)})})\}_{n\in\mathbb{N}} \subset \mathcal{C}_{u_r^*}$ which converges uniformly on $[0, t_0]$. Let $u^{(h)}(t) = Q_t(\phi^{(h)})$, where $u^{(h)}(t) = \{u_j(t, \phi^{(h)})\}_{j\in\mathbb{Z}}$ is a solution of (3.1) with $\phi^{(h)}(s) = \{\phi_j^{(h)}(s)\}_{j\in\mathbb{Z}}$, $\forall s \in [-r, 0]$, $\phi^{(h)} \to \phi$ in $\mathcal{C}_{u_r^*}$ as $h \to \infty$. For each j, then $0 \leqslant u_j(t, \phi^{(h)}) \leqslant u_r^*$ and $|u_j(t_1, \phi^{(h)}) - u_j(t_2, \phi^{(h)})| \leqslant C|t_1 - t_2|$, $\forall\ t, t_1, t_2 \in [0, t_0]$. Note that $\{u_j(t, \phi^{(h)})\}_{h=1}^{\infty} \subset C([0, t_0], \mathbb{R})$ is bounded and equicontinuous. Hence, there exists a convergent subsequence $\{u_{-1}(t, \phi^{(h_{-1,n})})\}_{n=1}^{\infty} \subset \{u_{-1}(t, \phi^{(h)})\}_{h=1}^{\infty}$

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which converges uniformly on $[0, t_0]$. Furthermore, there also exists a convergent subsequence $\{u_0(t, \phi^{(h_{-1,0,n})})\}_{n=1}^{\infty} \subset \{u_0(t, \phi^{(h_{-1,n})})\}_{n=1}^{\infty}$ with $\{h_{-1,0,n}\}_{n=1}^{\infty} \subset \{h_{-1,n}\}_{n=1}^{\infty}$. Analogously, we see there admits a convergent subsequence $\{u_1(t, \phi^{(h_{-1,0,1,n})})\}_{n=1}^{\infty} \subset \{u_1(t, \phi^{(h_{-1,0,n})})\}_{n=1}^{\infty}$ with $\{h_{-1,0,1,n}\}_{n=1}^{\infty} \subset \{h_{-1,0,n}\}_{n=1}^{\infty}$. For simplicity, write $h_{-1,0,1,n} = h_n^{(1)}$. Thus, $\{u_{-1}(t, \phi^{h_n^{(1)}})\}_{n=1}^{\infty}$, $\{u_1(t, \phi^{h_n^{(1)}})\}_{n=1}^{\infty}$ and $\{u_0(t, \phi^{h_n^{(1)}})\}_{n=1}^{\infty}$ are uniformly convergent on $[0, t_0]$. Hence, the sequence $\{u_j(t, \psi^{h_n^{(2)}})\}_{n=1}^{\infty}$ is uniformly convergent at $j = 0, \pm 1$ on $[0, t_0]$.

From the above process, we know that at each j, there exists $\{u_j(t, \phi^{h_n^{(n)}})\}_{n=1}^{\infty} \subset \{u_j(t, \phi^{(h)})\}_{h=1}^{\infty}$ which converges uniformly on $[0, t_0]$. Note that for each j, $\phi_j^{(h)} \to \phi_j$ in $C([-r, 0], \mathbb{R})$ as $\phi^{(h)} \to \phi$ in $\mathcal{C}_{u_r^*}$ and $h \to \infty$. It follows for each j, there exists $\{\phi_j^{h_n^{(n)}}\}_{n=1}^{\infty} \subset \{\phi_j^{(h)}\}_{h=1}^{\infty}$ which is convergent in $C([-r, 0], \mathbb{R})$. We immediately have for each j, $\{u_j(t, \phi^{(h_n^{(n)})})\}_{n=1}^{\infty}$ is uniformly convergent on $[-r, t_0]$. Therefore, for each j, $\{u_j(t+\theta, \phi^{(h_n^{(n)})})\}_{n=1}^{\infty}$ is uniformly convergent for $\theta \in [-r, 0]$ on $[0, t_0]$. It further indicates that for each j, $u_j^{(h_n^{(n)})}(t + \cdot) = Q_t(\phi^{(h_n^{(n)})})(j, \cdot)$ is uniformly convergent in $C([-r, 0], \mathbb{R})$ for $t \in [0, t_0]$. Let $u_j(t)$ be a limit point of $u_j^{(h_n^{(n)})}(t)$ in $C([-r, 0], \mathbb{R})$. Then $u_j(t) \in \mathcal{C}_{u_r^*}$. For any $\varepsilon > 0$, let $N_1 = N_1(\varepsilon) > 0$ such that $\sum_{i=N_1+1}^{\infty} \frac{u_r^*}{2^{i-1}} < \frac{\varepsilon}{4}$. For the above ε, we choose sufficiently large $N_2 = N_2(\varepsilon) > 0$ such that $|u_j^{(h_n^{(n)})}(t+\theta) - u_j(t+\theta)| < \frac{\varepsilon}{2}$ whenever $n \geqslant N_2$, $t \in [0, t_0]$ and $j \in [-N_1, N_1]$. Then for $n \geqslant N_2$,

$$
\begin{aligned}
\|u_j^{(h_n^{(n)})}(t) - u_j(t)\| &= \sum_{i=1}^{\infty} \frac{\max_{|j|\leqslant i,\, \theta\in[-r,0]} |u_j(t+\theta, \phi^{(h_n^{(n)})}) - u_j(t+\theta)|}{2^i} \\
&= \left(\sum_{i=1}^{N_1} + \sum_{i=N_1+1}^{\infty}\right) \frac{\max_{|j|\leqslant i, \theta\in[-r,0]} |u_j(t+\theta, \phi^{(h_n^{(n)})}) - u_j(t+\theta)|}{2^i} \\
&\leqslant \sum_{i=1}^{N_1} \frac{\varepsilon}{2^i} + \sum_{i=N_1+1}^{\infty} \frac{2u_r^*}{2^i} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

Secondly, we show that if $Q_t(\phi^{(h_m)})$ converges uniformly for $t \in [0, t_0]$, then $\lim_{n\to\infty} Q_t(\phi^{(h_n)}) = Q_t(\phi)$ uniformly for $t \in [0, t_0]$. In fact, if $u(t) = \{u_j(t)\}_{j\in\mathbb{Z}}$ is a limit point of $Q_t(\phi^{(h_n)})$ as $n \to \infty$, then $u(t)$ is a solution of (3.1) for $\forall\, t \in [0, t_0]$, that is, $u(t) = Q_t(\phi)$, $\forall\, t \in [0, t_0]$. Thus, if $\lim_{h\to\infty} \phi^{(h)} = \phi$, then $\lim_{h\to\infty} Q_t(\phi^{(h)}) = Q_t(\phi)$ uniformly for $t \in [0, t_0]$. As a result, there holds $\lim_{\phi\to\phi_0} Q_t(\phi) = Q_t(\phi_0)$ uniformly on $[0, t_0]$. This is the end of step 2. It follows from the above two steps that, together with $\|Q_t(\phi) - Q_{t_0}(\phi_0)\| \leqslant \|Q_t(\phi) - Q_t(\phi_0)\| + \|Q_t(\phi_0) - Q_{t_0}(\phi_0)\|$, $Q_t(u^0)$ is continuous in $(t, u^0) \in \mathbb{R}^+ \times \mathcal{C}_{u_r^*}$. Clearly, $Q_0 = I$, and $Q_{t'+t''} = Q_{t'} \circ Q_{t''}$. Hence, Q_t is a semiflow of (3.1) in $\mathcal{C}_{u_r^*}$. We finish the proof of this lemma. □

***Proof of theorem* 3.1.** By lemma 3.2, it follows that the semiflow Q_t generated by (3.1) satisfies all the hypotheses (A1)–(A5) for each $t > 0$. Then, we can apply the spreading theorems established in [**13**] to obtain the existence of a minimal wave speed c_r^* and its coincidence with spreading speed, that is, the conclusions (i)–(iii) are valid. We next focus on the computation of c_r^*. Consider a linearized system of (3.1) at $u = 0$

$$u_j'(t) = D\left[u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)\right] - \tau u_j(t) + \beta\tau \int_0^r k(s) u_j(t-s)\,\mathrm{d}s. \quad (3.3)$$

Obviously, if u is a solution of (3.3), then u is a supersolution of (3.1). Let M_t be the solution map associated with (3.3). Then $Q_t(u^0) \leqslant M_t(u^0)$, $\forall u^0 \in \mathcal{C}_{u_r^*}$. Secondly, we introduce the following linear system with parameter $\varepsilon \in (0, 1)$

$$u_j'(t) = D\left[u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)\right] - \tau u_j(t) + (1-\varepsilon)\beta\tau \int_0^r k(s) u_j(t-s)\,\mathrm{d}s. \quad (3.4)$$

Let M_t^ε be the solution map associated with (3.4). Then $Q_t(u^0) \geqslant M_t^\epsilon(u^0)$, $\forall u^0 \in \mathcal{C}_{u_r^*}, \forall\, t \in [0, 1]$. Letting $\varepsilon \to 0$, the desired result follows. Thus, it suffices to estimate the spreading speed for M_t. For each $\varphi \in C([-r, 0], \mathbb{R})$, set $\eta(t, \varphi)$ to be the solution of

$$\eta'(t) = D\left[\mathrm{e}^{-\lambda} + \mathrm{e}^{\lambda} - 2\right]\eta(t) - \tau\eta(t) + \beta\tau \int_0^r k(s)\eta(t-s)\,\mathrm{d}s, \quad (3.5)$$

with $\eta(\theta, \varphi) = \varphi(\theta)$ and $\theta \in [-r, 0]$. Then $u(t) = \{u_j(t)\}_{j\in\mathbb{Z}}$ where $u_j = \mathrm{e}^{-\lambda j}\eta(t, \varphi)$ is a solution of (3.3). It follows that $B_\lambda^t(\varphi)(\theta) := M_t[\varphi\,\mathrm{e}^{-\lambda j}](\theta, 0) = \eta(t + \theta, \varphi)$, $\forall \theta \in [-r, 0]$. Then B_λ^t is the solution map of (3.5). We can further verify that $B_\lambda^{m_0}$ is compact, strongly positive, for any $m_0 > r$. Since (3.5) is cooperative and irreducible, there exists a real root $\mu = \mu(\lambda)$ of $\mu - D[\mathrm{e}^{-\lambda} + e^{\lambda} - 2] + \tau - \beta\tau \int_0^r k(s)\,\mathrm{e}^{-\mu s}\,\mathrm{d}s = 0$, which also has the greatest real part among all roots. Observe that $\mu(\lambda) \geqslant D[\mathrm{e}^{-\lambda} + \mathrm{e}^{\lambda} - 2] - \tau$. It yields $\mu(\lambda) = \infty$ as $\lambda \to \infty$.

Consider $\psi \in C([-r, 0], \mathbb{R})$ by $\psi(\theta) = \mathrm{e}^{\mu(\lambda)\theta}$ and $\theta \in [-r, 0]$. Clearly $\eta(t, \psi) = \mathrm{e}^{\mu(\lambda)t}$, $\forall\, t \geqslant 0$. Thus, $B_\lambda^t(\psi) = \eta(t + \cdot, \psi) = \mathrm{e}^{\mu(\lambda)t}\psi$, $\forall\, t \geqslant 0$. Then B_λ^t admits a principal eigenvalue $\mathrm{e}^{\mu(\lambda)t}$ associated with a positive eigenfunction ψ. Set $t = 1$. Then $\mathrm{e}^{\mu(\lambda)}$ is the principal eigenvalue of B_λ^1. By [**12**, lemma 3.7], $\mu(\lambda)$ is convex on $\mathbb{R}$. Moreover, we claim that $\mathrm{e}^{\mu(0)} > 1$, where $\mathrm{e}^{\mu(0)}$ is the principal eigenvalue of B_λ^1 at $\lambda = 0$. Indeed, let $F(\mu) = \beta\tau \int_0^r k(s)\,\mathrm{e}^{-\mu s}\,\mathrm{d}s - \mu - \tau = 0$. It follows from $F(0) = \beta\tau - \tau > 0$, $F(\infty) = -\infty$ and $\frac{\partial F(\mu)}{\partial \mu} < 0$ that there exists a unique positive root of $F(\mu) = 0$. Since $\mu(0)$ is a root of $F(\mu) = 0$, then $\mu(0) > 0$ and $\mathrm{e}^{\mu(0)} > 1$. Define $\Phi(\lambda) := \frac{\mu(\lambda)}{\lambda}$. From the above discussions, we know that $\Phi(\lambda)$ reaches the minimum value at some finite $\lambda_r^* < \infty$. By [**12**, theorems 3.5 and 3.10], the spreading speed for $\{Q_t\}_{t\geqslant 0}$ is $c_r^* = \inf\limits_{\lambda\in(0,+\infty)} \Phi(\lambda) = \Phi(\lambda_r^*)$. Moreover, (c_r^*, λ_r^*) is uniquely determined by $\mathcal{R}_1(\lambda, c) = 0$ and $\frac{\partial \mathcal{R}_1}{\partial \lambda}(\lambda, c) = 0$, $\lambda > 0$, where

$$\mathcal{R}_1(\lambda, c) = c\lambda - D\left[\mathrm{e}^{-\lambda} + \mathrm{e}^{\lambda} - 2\right] + \tau - \beta\tau \int_0^r k(s)\,\mathrm{e}^{-c\lambda s}\,\mathrm{d}s.$$

This completes the proof. □

Let

$$\mathcal{R}(\lambda, c) := D[\mathrm{e}^{\lambda} + \mathrm{e}^{-\lambda} - 2] - c\lambda - \tau + \beta\tau \int_0^\infty k(s)\,\mathrm{e}^{-\lambda cs}\,\mathrm{d}s = 0.$$

The following properties can be easily verified by direct calculation.

LEMMA 3.3. *There exists unique $c^* > 0$ such that*

(i) $\mathcal{R}(\lambda^*, c^*) = 0$, $\frac{\partial}{\partial\lambda}\mathcal{R}(\lambda^*, c^*) = 0$ *for some* $\lambda^* > 0$, *and* $\lim_{r\to\infty} c_r^* = c^*$.

(ii) *for every* $c > c^*$, $\mathcal{R}(\lambda, c) = 0$ *has the smallest positive root* $\lambda_1(c)$ *and* $\mathcal{R}(\lambda, c) < 0$ *for some* $\lambda > 0$.

THEOREM 3.4. *Assume* $1 < \beta \leqslant e$. *For* $\phi \in C((-\infty, 0], [0, u^*]_{\mathbb{X}})$. *Let* $u(t) = u(t; \phi)$ *be the unique solution of* (1.3) *with* $u(t) = \phi(t)$ *for* $t \in (-\infty, 0]$. *Then the following conclusions are valid:*

(i) *If* $u_j(t) = 0$ *for* (j, t) *outside a bounded set of* $\mathbb{Z} \times (-\infty, 0]$, *then*

$$\lim_{t\to\infty,\ |j|\geqslant ct} u_j(t) = 0 \quad \textit{for any} \quad c > c^*.$$

(ii) *If* $u(t) \not\equiv 0$ *for* $t \in (-\infty, 0]$, *then*

$$\lim_{t\to\infty, |j|\leqslant ct} u_j(t) = u^* \quad \textit{for any} \quad 0 < c < c^*.$$

Proof. By lemma 3.3, for any $c > c^*$, there is some $\lambda > 0$ such that $\mathcal{R}(\lambda, c) < 0$. For given $\alpha > 0$. Define $\overline{u}(t) = \{\overline{u}_j(t)\}_{j\in\mathbb{Z}}$ with $\overline{u}_j(t) := \alpha\,\mathrm{e}^{\lambda(ct-\eta j)}$, where $\eta = 1$, or $\eta = -1$. Then, we have

$$\begin{aligned}
&D[u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)] - \tau u_j(t) + \beta\tau \int_0^\infty k(s) u_j(t-s)\,\mathrm{d}s - u_j'(t)\\
&= D[\alpha\,\mathrm{e}^{\lambda(ct-\eta(j+1))} + \alpha\,\mathrm{e}^{\lambda(ct-\eta(j-1))} - 2\alpha\,\mathrm{e}^{\lambda(ct-\eta j)}] - \tau\alpha\,\mathrm{e}^{\lambda(ct-\eta j)}\\
&\quad + \beta\tau \int_0^\infty k(s)\alpha\,\mathrm{e}^{\lambda(c(t-s)-\eta j)}\,\mathrm{d}s - \alpha c\lambda\,\mathrm{e}^{\lambda(ct-\eta j)}\\
&= \alpha\,\mathrm{e}^{\lambda(ct-\eta j)}\mathcal{R}(\lambda, c) < 0.
\end{aligned} \tag{3.6}$$

For any given $c > c^*$, we choose $c^* < \hat{c} < c$ and $\hat{\lambda} > 0$ such that $\mathcal{R}(\hat{\lambda}, \hat{c}) < 0$. By the assumption, we can choose suitable $\alpha > 0$ such that

$$\phi_j(t) \leqslant \alpha\,\mathrm{e}^{\hat{\lambda}(\hat{c}t-\eta j)} \quad \text{for} \quad j \in \mathbb{Z}, t \in (-\infty, 0].$$

Note that $u(t; \phi) = \{u_j(t; \phi_j)\}_{j\in\mathbb{Z}}$ is a subsolution of

$$u_j'(t) = D[u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)] - \tau u_j(t) + \beta\tau \int_0^\infty k(s) u_j(t-s)\,\mathrm{d}s. \tag{3.7}$$

By the comparison principle, we have $u(t;\phi) \leqslant \overline{u}(t)$ for $t \geqslant 0$, that is, $u_n(t;\phi_j) \leqslant \alpha\,\mathrm{e}^{\hat{\lambda}(\hat{c}t-\eta j)}$ for $j \in \mathbb{Z}, t \in [0,\infty)$. Denote $\eta = \mathrm{sgn}\{j\}, j \neq 0$. It then follows that

$$u_j(t;\phi_j) \leqslant \alpha\,\mathrm{e}^{\hat{\lambda}(\hat{c}t-|j|)} \quad \text{for} \quad j \in \mathbb{Z}, t \in [0,\infty).$$

Since $c > \hat{c}$, we have

$$\lim_{t\to\infty, |j|\geqslant ct} u_j(t;\phi_j) = 0.$$

We next prove (ii). For any given $c \in (0, c^*)$. Because of $\lim_{r\to\infty} c_r^* = c^*$, we may choose r sufficiently large such that $c_r^* > c$. Let $\hat{\phi}(\theta) = \{\hat{\phi}_j(\theta)\}_{j\in\mathbb{Z}}$ with

$$\hat{\phi}_j(\theta) = \min\{\phi_j(\theta), u_r^*\} \quad \text{for} \quad j \in \mathbb{Z},\ \theta \in [-r, 0].$$

Observe that $u(t;\phi) = \{u_j;\phi_j\}_{j\in\mathbb{Z}}$ is a supersolution of (3.1). The comparison principle indicates that $u(t;\hat{\phi}) \leqslant u(t;\phi)$ for $t \in [0,\infty)$, where $u(t;\hat{\phi}) = \{u_j;\hat{\phi}_j\}_{j\in\mathbb{Z}}$ is a solution of (3.1) with $u(\theta) = \hat{\phi}(\theta)$ for $\theta \in (-\infty, 0]$. Thus, by (ii) of theorem 3.1, we obtain

$$u_r^* \leqslant \liminf_{t\to\infty, |j|\leqslant ct} u_j(t;\phi_j) \leqslant \limsup_{t\to\infty, |j|\leqslant ct} u_j(t;\phi_j) \leqslant u^*. \tag{3.8}$$

Letting $r \to \infty$ in (3.8), we have $\lim_{t\to\infty, |j|\leqslant ct} u_j(t;\phi_j) = u^*$ since $u_r^* \to u^*$ as $r \to \infty$. This completes the proof. □

Next, we study the existence of travelling wave solutions for system (1.3). Consider the wave equation

$$\begin{aligned} c\phi'(\xi) &= D[\phi(\xi+1) + \phi(\xi-1) - 2\phi(\xi)] - \tau\phi(\xi) \\ &\quad + \beta\tau\left(\int_0^\infty k(s)\phi(\xi - cs)\,\mathrm{d}s\right)\exp\left[-\int_0^\infty k(s)\phi(\xi-cs)\,\mathrm{d}s\right], \quad j \in \mathbb{Z}, \end{aligned} \tag{3.9}$$

with

$$\phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = u^*. \tag{3.10}$$

THEOREM 3.5. *Assume* $1 < \beta \leqslant e$. *For any* $c \geqslant c^*$, (1.3) *has a monotone travelling wave solution* $u_j(t) = \phi(j + ct)$ *satisfying* $\phi(-\infty) = 0$ *and* $\phi(+\infty) = u^*$, *while for any* $0 < c < c^*$, (1.3) *has no travelling wave solution* $\phi(j+ct)$ *such that* $\phi(-\infty) = 0$ *and* $\phi(+\infty) = u^*$.

Proof. For any $c > c^*$, since $\lim_{r\to\infty} c_r^* = c^*$ and $\lim_{r\to\infty} u_r^* = u^*$, there exists $\hat{r} \geqslant r_0$ such that $u_r^* > \frac{1}{2}u^*$ and $c > c_r^*$ for all $r \geqslant \hat{r}$. By (iii) of theorem 3.1, for each $r > \hat{r}$, equation (1.3) has a monotone travelling wave $\phi_r(j + ct)$ connecting 0 and

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u_r^*. Namely, $\phi_r(\xi)$ satisfies the following wave equation

$$
\begin{aligned}
c\phi_r^{'}(\xi) &= D[\phi_r(\xi+1)+\phi_r(\xi-1)-2\phi_r(\xi)]-\tau\phi_r(\xi) \\
&\quad + \beta\tau\left(\int_0^r k(s)\phi_r(\xi-cs)\,\mathrm{d}s\right)\exp\left[-\int_0^r k(s)\phi_r(\xi-cs)\,\mathrm{d}s\right], \quad j\in\mathbb{Z},
\end{aligned}
\tag{3.11}
$$

with

$$
\phi_r(-\infty)=0 \quad \text{and} \quad \phi_r(+\infty)=u_r^*.
$$

Since $\phi_r(\xi)$ is bounded on $\mathbb{R}$, it can be seen from (3.11) that $|\phi_r^{'}(\xi)| \leqslant B$ for some $B>0$. We next show that $\phi_r^{'}(\xi)$ is equicontinuous. For any $\xi_1,\xi_2\in\mathbb{R}$, and $r>\hat{r}$, we have

$$
\begin{aligned}
&|\phi_r^{'}(\xi_1)-\phi'_r(\xi_2)| \\
&= c^{-1}\Big|D[\phi_r(\xi_1+1)-\phi_r(\xi_2+1)]+[\phi_r(\xi_1-1)-\phi_r(\xi_2-1)]-2[\phi_r(\xi_1)-\phi_r(\xi_2)] \\
&\quad -\tau[\phi_r(\xi_1)-\phi_r(\xi_2)]+\beta\tau\left(\int_0^r k(s)\phi_r(\xi_1-cs)ds\right)\exp\left[-\int_0^r k(s)\phi_r(\xi_1-cs)\,\mathrm{d}s\right] \\
&\quad -\beta\tau\left(\int_0^r k(s)\phi_r(\xi_2-cs)\,\mathrm{d}s\right)\exp\left[-\int_0^r k(s)\phi_r(\xi_2-cs)\,\mathrm{d}s\right]\Big| \\
&\leqslant c^{-1}\Big\{D[|\phi_r(\xi_1+1)-\phi_r(\xi_2+1)|+|\phi_r(\xi_1-1)-\phi_r(\xi_2-1)|+2|\phi_r(\xi_1)-\phi_r(\xi_2)|] \\
&\quad +\tau|\phi_r(\xi_1)-\phi_r(\xi_2)|+\beta\tau\int_0^r k(s)|\phi_r(\xi_1-cs)-\phi_r(\xi_2-cs)|\,\mathrm{d}s\Big\}.
\end{aligned}
$$

Because $\phi_r(\xi)$ is equicontinuous, then $\phi_r^{'}(\xi)$ is equicontinuous. Using Arzéla–Ascoli theorem, we obtain that there exists a subsequence $\{\phi_{r_n}(\xi)\}$ of $\phi_r(\xi)$ that converges uniformly with $\phi(\xi)$ on compact set as $n\to\infty$. By Lebesgue dominated convergence theorem, we obtain

$$
\begin{aligned}
c\phi^{'}(\xi) &= D[\phi(\xi+1)+\phi(\xi-1)-2\phi(\xi)]-\tau\phi(\xi) \\
&\quad + \beta\tau\left(\int_0^\infty k(s)\phi(\xi-cs)\,\mathrm{d}s\right)\exp\left[-\int_0^\infty k(s)\phi(\xi-cs)\,\mathrm{d}s\right],
\end{aligned}
\tag{3.12}
$$

by letting $r=r_n\to\infty$ in (3.11). This implies that $u_j(t):=\phi(j+ct)$ is a solution of (1.3). Clearly, $\phi(\xi)$ is non-decreasing on $\mathbb{R}$, and satisfies $\phi(-\infty)=0$ and $\phi(+\infty)=u^*$. For $c=c^*$, let $\{c_i\}_{i=1}^\infty$ be a sequence with $c_i>c^*$ and $\lim\limits_{n\to\infty} c_i=c^*$. Then, the above results show that for every c_i, (3.9) has a monotone solution $\phi_i(\xi)$ such that $\lim\limits_{\xi\to-\infty}\phi_i(\xi)=0$ and $\lim\limits_{\xi\to\infty}\phi_i(\xi)=u^*$. Using the above arguments again, we can obtain that there is a monotone function $\phi_*(\xi)$ such that $\lim\limits_{i\to\infty}\phi_i(\xi)=\phi_*(\xi)$ pointwise, and $\phi_*(\xi)$ satisfies (3.9) with $c=c_*$. Therefore, (1.3) has a monotone travelling wave solution $\phi_*(j+c^*t)$ connecting 0 and u^*.

For $\hat{c} \in (0, c^*)$. Suppose by contradiction that (1.3) has a travelling wave solution $\phi(j + \hat{c}t)$ such that $\phi(-\infty) = 0$ and $\phi(+\infty) = u^*$. Let

$$\varphi_j(\theta) := \phi(j + \hat{c}\theta), \quad j \in \mathbb{Z}, \quad \theta \in (-\infty, 0].$$

Define $\varphi(\theta) = \{\varphi_j(\theta)\}_{j\in\mathbb{Z}}$, $\theta \in (-\infty, 0]$. Obviously, $\varphi \in C((-\infty, 0], [0, u^*]_{\mathbb{X}})$ and $\varphi(\theta) \not\equiv 0$ for $\theta \in (-\infty, 0]$. Let $u(t) = \{u_j(t, \varphi)\}_{j\in\mathbb{Z}}$ be the solution of (1.3) with the initial data φ. The uniqueness of solution indicates that $u_j(t) := \phi(j + \hat{c}t)$ for $t \geqslant 0$. By (ii) of theorem 3.4, it follows that

$$\lim_{t\to\infty, |j|\leqslant ct} u_j(t) = \lim_{t\to\infty, |j|\leqslant ct} \phi(j + \hat{c}t) = u^* \quad \text{for any} \quad 0 < c < c^*. \tag{3.13}$$

Let $c_0 > 0$ be such that $\hat{c} < c_0 < c^*$. Note that $c_0 t - 1 < [c_0 t] \leqslant c_0 t$, $\forall t \geqslant 0$. Then $-[c_0 t] + \hat{c}t \to -\infty$ as $t \to \infty$. Letting $j = -[c_0 t]$ in equality (3.13), we have $u^* = \lim_{t\to\infty} \phi(-[c_0 t] + \hat{c}t) = 0$, which is a contradiction. This completes the proof. $\square$

4. Monotonicity of travelling waves

In this section, we further study the monotonicity problem of travelling wave solutions. We shall prove that any travelling wave solution is strictly monotone increasing.

LEMMA 4.1. *Assume $1 < \beta \leqslant e$. Then, any travelling wave of (3.9) and (3.10) satisfies $\phi(\xi) \leqslant \ln\beta$ on $\mathbb{R}$.*

Proof. If $\phi(\xi) > \ln\beta$ at some point, then it must attain a global maximum at some point $\xi_0 \in \mathbb{R}$, i.e. $\phi(\xi_0) > \ln\beta$ and $\phi'(\xi_0) = 0$. We claim that $\phi(\xi_0) \leqslant 1$. In fact, by (3.9) and the fact $ue^{-u} \leqslant \frac{1}{e}$ for $u \geqslant 0$, we obtain

$$\begin{aligned}\phi(\xi_0) &\leqslant \beta\left(\int_0^\infty k(s)\phi(\xi_0 - cs)\,\mathrm{d}s\right)\exp\left[-\int_0^\infty k(s)\phi(\xi_0 - cs)\,\mathrm{d}s\right]\\ &\leqslant \frac{\beta}{e} \leqslant 1.\end{aligned}$$

Then, we have

$$\begin{aligned}0 = c\phi'(\xi_0) &= D[\phi(\xi_0 + 1) + \phi(\xi_0 - 1) - 2\phi(\xi_0)] - \tau\phi(\xi_0)\\ &\quad + \beta\tau\left(\int_0^\infty k(s)\phi(\xi_0 - cs)\,\mathrm{d}s\right)\exp\left[-\int_0^\infty k(s)\phi(\xi_0 - cs)\,\mathrm{d}s\right]\\ &\leqslant -\tau\phi(\xi_0) + \beta\tau\,\mathrm{e}^{-\phi(\xi_0)}.\end{aligned} \tag{4.1}$$

This implies that $\phi(\xi_0) \leqslant \ln\beta$, which is a contradiction. $\square$

LEMMA 4.2. *Let ϕ be any travelling wave of (3.9) and (3.10). Then, $\phi(\xi)\,\mathrm{e}^{-\rho\xi} \leqslant \Lambda$ for some $\rho, \Lambda > 0$, $\xi \in \mathbb{R}$.*

Proof. Choose sufficiently large $r > 0$ and small $\varepsilon > 0$ such that $\gamma := (\beta\tau - \varepsilon)\int_0^r k(s)\,\mathrm{d}s > \tau$. Note that there exists small $\theta > 0$ such that $\beta\tau u\,\mathrm{e}^{-u} \geqslant (\beta\tau - \varepsilon)u$

for $u \leqslant \theta$. Choose $M \gg 1$ such that $\phi(\xi) \leqslant \theta$ for $\xi \leqslant -M$. Let $\xi^0 = -M + cr$. Then, for $\xi \leqslant \xi^0$, we have

$$\begin{aligned}
&\beta\tau \left(\int_0^\infty k(s)\phi(\xi - cs)\,\mathrm{d}s\right) \exp\left[-\int_0^\infty k(s)\phi(\xi - cs)\,\mathrm{d}s\right] \\
&\geqslant \beta\tau \left(\int_0^r k(s)\phi(\xi - cs)\,\mathrm{d}s\right) \exp\left[-\int_0^r k(s)\phi(\xi - cs)\,\mathrm{d}s\right] \\
&\geqslant (\beta\tau - \varepsilon)\int_0^r k(s)\phi(\xi - cs)\,\mathrm{d}s.
\end{aligned}$$

Thus, we obtain that for $\xi \leqslant \xi^0$,

$$\begin{aligned}
c\phi^{'}(\xi) &= D[\phi(\xi+1) + \phi(\xi-1) - 2\phi(\xi)] - \tau\phi(\xi) \\
&\quad + \beta\tau \left(\int_0^\infty k(s)\phi(\xi - cs)\,\mathrm{d}s\right) \exp\left[-\int_0^\infty k(s)\phi(\xi - cs)\,\mathrm{d}s\right] \\
&\geqslant D[\phi(\xi+1) + \phi(\xi-1) - 2\phi(\xi)] - \tau\phi(\xi) + (\beta\tau - \varepsilon)\int_0^r k(s)\phi(\xi - cs)\,\mathrm{d}s \\
&= D[\phi(\xi+1) + \phi(\xi-1) - 2\phi(\xi)] + (\gamma - \tau)\phi(\xi) \\
&\quad + (\beta\tau - \varepsilon)\int_0^r k(s)[\phi(\xi - cs) - \phi(\xi)]\,\mathrm{d}s \qquad (4.2)
\end{aligned}$$

By $\int_{-\infty}^{\xi}[\phi(z-y) - \phi(z)]\,\mathrm{d}z = -y\int_0^1 \phi(\xi - y\nu)\,\mathrm{d}\nu$ for any $y \in \mathbb{R}$, it follows that

$$\begin{aligned}
c\phi(\xi) \geqslant D[\int_0^1 \phi(\xi+\nu)\,\mathrm{d}\nu - \int_0^1 \phi(\xi-\nu)\,\mathrm{d}\nu] + (\gamma - \tau)\int_{-\infty}^{\xi}\phi(z)\,\mathrm{d}z \\
- (\beta\tau - \varepsilon)\int_0^r csk(s)\int_0^1 \phi(\xi - \nu cs)\,\mathrm{d}\nu\,\mathrm{d}s.
\end{aligned}$$

Thus,

$$\begin{aligned}
(\gamma - \tau)\int_{-\infty}^{\xi}\phi(z)\,\mathrm{d}z \leqslant D[\int_0^1 \phi(\xi-\nu)\,\mathrm{d}\nu - \int_0^1 \phi(\xi+\nu)\,\mathrm{d}\nu] + c\phi(\xi) \\
+ (\beta\tau - \varepsilon)\int_0^r csk(s)\int_0^1 \phi(\xi - \nu cs)\,\mathrm{d}\nu\,\mathrm{d}s. \qquad (4.3)
\end{aligned}$$

Because ϕ is bounded on $\mathbb{R}$, there is $B > 0$ such that $(\gamma - \tau)\int_{-\infty}^{\xi}\phi(z)\,\mathrm{d}z \leqslant B$. Then, we can define $\widetilde{\phi}(\xi) := \int_{-\infty}^{\xi}\phi(z)\,\mathrm{d}z$, $\xi \in \mathbb{R}$. By (4.3), we obtain that

$$\begin{aligned}
(\gamma - \tau)\int_{-\infty}^{\xi}\widetilde{\phi}(z)\,\mathrm{d}z \leqslant D[\int_0^1 \widetilde{\phi}(\xi-\nu)\,\mathrm{d}\nu - \int_0^1 \widetilde{\phi}(\xi+\nu)\,\mathrm{d}\nu] + c\widetilde{\phi}(\xi) \\
+ (\beta\tau - \varepsilon)\int_0^r csk(s)\int_0^1 \widetilde{\phi}(\xi - \nu cs)\,\mathrm{d}\nu\,\mathrm{d}s. \qquad (4.4)
\end{aligned}$$

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Note that $\int_0^1 \widetilde{\phi}(\xi-\nu)\,\mathrm{d}\nu - \int_0^1 \widetilde{\phi}(\xi+\nu)\,\mathrm{d}\nu \leqslant 0$. We have

$$(\gamma-\tau)\int_{-\infty}^{\xi} \widetilde{\phi}(z)\,\mathrm{d}z \leqslant c\widetilde{\phi}(\xi) + (\beta\tau-\varepsilon)\int_0^r csk(s)\int_0^1 \widetilde{\phi}(\xi-\nu cs)\,\mathrm{d}\nu\,\mathrm{d}s. \tag{4.5}$$

Thus, for some $\delta > 0$ there holds

$$(\gamma-\tau)\int_{-\infty}^{\xi} \widetilde{\phi}(z)\,\mathrm{d}z \leqslant \delta\widetilde{\phi}(\xi).$$

Choose $q_1 > 0$ such that $q_2 = \frac{\delta}{q_1(r-\tau)} < 1$. Then, we have

$$\widetilde{\phi}(\xi-q_1) \leqslant \frac{1}{q_1}\int_{\xi-q_1}^{\xi} \widetilde{\phi}(z)\,\mathrm{d}z \leqslant \frac{1}{q_1}\int_{-\infty}^{\xi} \widetilde{\phi}(z)\,\mathrm{d}z \leqslant q_2\widetilde{\phi}(\xi). \tag{4.6}$$

Define $\gamma = \frac{1}{q_1}\ln\frac{1}{q_2}$, and $\omega(\xi) = \widetilde{\phi}(\xi)\,\mathrm{e}^{-\gamma\xi}$. Then,

$$\omega(\xi-q_1) = \widetilde{\phi}(\xi-q_1)\,\mathrm{e}^{-\gamma(\xi-q_1)} \leqslant q_2\widetilde{\phi}(\xi)\,\mathrm{e}^{-\gamma(\xi-q_1)} = \omega(\xi)$$

which implies that $\omega(\xi)$ is bounded for $\xi \leqslant 0$. Thus, there exists some $\Lambda_1 > 0$ such that $\widetilde{\phi}(\xi)\,\mathrm{e}^{-\rho\xi} \leqslant \Lambda_1$ for $\xi \in \mathbb{R}$. We further obtain that there exists $\Lambda > 0$ such that

$$\phi(\xi)\,\mathrm{e}^{-\rho\xi} \leqslant \Lambda \quad \text{for some} \quad \Lambda > 0, \quad \xi \in \mathbb{R}.$$

□

LEMMA 4.3. *Let ϕ be any travelling wave of (3.9) and (3.10). If ϕ is monotone increasing on $(-\infty, -\ell]$ for some $\ell \geqslant 0$, then there exists $\eta_0 \geqslant 0$ such that*

$$\phi(\xi+\eta) \geqslant \phi(\xi) \quad \textit{for} \quad \eta \geqslant \eta_0,\ \xi \in \mathbb{R}. \tag{4.7}$$

Proof. Let $f(u) := ue^{-u}$. Since $f'(u^*) = e^{-u^*}(1-u^*) < \frac{1}{\beta}$, there exists $\delta > 0$ such that

$$f'(u) < \frac{1}{\beta} \quad \text{for} \quad u \in [u^*-\delta, u^*] \quad \text{and} \quad a := \max_{u\in[u^*-\delta,u^*]} f'(u) < \frac{1}{\beta}. \tag{4.8}$$

Choose $\chi > \ell$ large enough so that

$$\phi(\xi) \leqslant \delta \quad \text{for} \quad \xi \leqslant -\chi, \quad u^* - \phi(\xi) \leqslant \delta \quad \text{for} \quad \xi \geqslant \chi. \tag{4.9}$$

Let $\gamma := \inf_{\xi\in[-\chi,+\infty)} \phi(\xi) > 0$. Note that there exists $\chi_0 > 0$ such that $\phi(\xi) < \gamma$ for $\xi \leqslant -\chi_0$. Then, for every $\eta \geqslant 0$, we have $\phi(\xi+\eta) > \phi(\xi)$ for $\xi \in [-\chi-\eta, -\chi_0]$. Note by $\phi(+\infty) = u^*$ and $\sup_{\xi\in[-\chi_0,\chi+\mu]} \phi(\xi) < u^*$ for any $\mu > 0$, there exists $\eta_0 \geqslant$

0 such that for every $\eta \geqslant \eta_0$, $\phi(\xi+\eta) > \phi(\xi)$ for $\xi \in [-\chi_0, \chi+\mu]$. Thus,

$$\phi(\xi+\eta) > \phi(\xi) \quad \text{for} \quad \xi \in [-\chi-\eta, \chi+\mu].$$

Note also that for every $\eta \geqslant 0$, $\phi(\xi+\eta) \geqslant \phi(\xi)$ for $\xi \in (-\infty, -\chi-\eta]$. Thus, for any $\eta \geqslant \eta_0$, we have

$$\phi(\xi+\eta) \geqslant \phi(\xi) \quad \text{for} \quad \xi \in (-\infty, \chi+\mu]. \tag{4.10}$$

Fix $\mu \geqslant cr$ in (4.10), where $r > 0$ is a fixed number such that $\max_{u\in[0,u^*]} f^{'}(u) \int_r^\infty k(s)ds < \frac{1}{\beta} - a$. Define

$$\hat{\theta} = \inf\{\theta \geqslant 0 | \phi(\xi+\eta) + \theta \geqslant \phi(\xi) \quad \text{for} \quad \xi \in \mathbb{R}\}.$$

Obviously, $\hat{\theta}$ is well-defined, and $\phi(\xi+\eta) + \hat{\theta} \geqslant \phi(\xi)$ for $\xi \in \mathbb{R}$. We next claim that $\hat{\theta} = 0$ to finish the proof. Suppose that $\hat{\theta} > 0$. Define $\omega(\xi) := \phi(\xi+\eta) + \hat{\theta} - \phi(\xi)$. Clearly, $\omega(\pm\infty) = \hat{\theta} > 0$. There exists ξ_* such that $0 = \omega(\xi_*) = \min\limits_{\xi\in\mathbb{R}} \omega(\xi)$. Then, we have

$$\begin{aligned}
0 &= c[\phi^{'}(\xi_*+\eta) - \phi^{'}(\xi_*)] \\
&= D\{[\phi(\xi_*+\eta+1) - \phi(\xi_*+1)] + [\phi(\xi_*+\eta-1) - \phi(\xi_*-1)] \\
&\quad - 2[\phi(\xi_*+\eta) - \phi(\xi_*)]\} \\
&\quad - \tau[\phi(\xi_*+\eta) - \phi(\xi_*)] + \beta\tau \left(\int_0^\infty k(s)\phi(\xi_*+\eta-cs)\,\mathrm{d}s\right) \\
&\quad \times \exp\left[-\int_0^\infty k(s)\phi(\xi_*+\eta-cs)\,\mathrm{d}s\right] \\
&\quad - \beta\tau \left(\int_0^\infty k(s)\phi(\xi_*-cs)\,\mathrm{d}s\right) \exp\left[-\int_0^\infty k(s)\phi(\xi_*-cs)\,\mathrm{d}s\right] \\
&\geqslant \tau\hat{\theta} + \beta\tau \left(\int_0^\infty k(s)\phi(\xi_*+\eta-cs)\,\mathrm{d}s\right) \exp\left[-\int_0^\infty k(s)\phi(\xi_*+\eta-cs)\,\mathrm{d}s\right] \\
&\quad - \beta\tau \left(\int_0^\infty k(s)\phi(\xi_*-cs)\,\mathrm{d}s\right) \exp\left[-\int_0^\infty k(s)\phi(\xi_*-cs)\,\mathrm{d}s\right].
\end{aligned} \tag{4.11}$$

By (4.10), we know that $\xi_* > \chi+\mu \geqslant \chi+cr$. In view of (4.9), we obtain $\phi(\xi_*+\eta-cs), \phi(\xi_*-cs) \in [u^*-\delta, u^*]$ for any $s \leqslant r$. Since $\max_{u\in[0,u^*]} f^{'}(u) \int_r^\infty k(s)ds <$

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$\frac{1}{\beta} - a$, it follows that

$$
\begin{aligned}
&\beta\tau\left(\int_0^\infty k(s)\phi(\xi_* + \eta - cs)\,\mathrm{d}s\right)\exp\left[-\int_0^\infty k(s)\phi(\xi_* + \eta - cs)\,\mathrm{d}s\right] \\
&\quad - \beta\tau\left(\int_0^\infty k(s)\phi(\xi_* - cs)\,\mathrm{d}s\right)\exp\left[-\int_0^\infty k(s)\phi(\xi_* - cs)\,\mathrm{d}s\right] \\
&= \beta\tau f^{'}(\zeta_1)\int_0^r k(s)[\phi(\xi_* + \eta - cs) - \phi(\xi_* - cs)]\,\mathrm{d}s \\
&\quad + \beta\tau f^{'}(\zeta_2)\int_r^\infty k(s)[\phi(\xi_* + \eta - cs) - \phi(\xi_* - cs)]\,\mathrm{d}s \\
&> -\beta\tau a\hat{\theta} - \beta\tau\left(\frac{1}{\beta} - a\right)\hat{\theta} = -\tau\hat{\theta},
\end{aligned}
$$

where $\zeta_1 \in [u^* - \delta, u^*]$, $\zeta_2 \in [0, u^*]$. Thus, we have

$$
\begin{aligned}
&\tau\hat{\theta} + \beta\tau\left(\int_0^\infty k(s)\phi(\xi_* + \eta - cs)\,\mathrm{d}s\right)\exp\left[-\int_0^\infty k(s)\phi(\xi_* + \eta - cs)\,\mathrm{d}s\right] \\
&\quad - \beta\tau\left(\int_0^\infty k(s)\phi(\xi_* - cs)\,\mathrm{d}s\right)\exp\left[-\int_0^\infty k(s)\phi(\xi_* - cs)\,\mathrm{d}s\right] \\
&> \tau\hat{\theta} - \tau\hat{\theta} = 0,
\end{aligned}
\tag{4.12}
$$

which is a contradiction according to (??). Therefore, $\hat{\theta} = 0$. This completes the proof. □

THEOREM 4.4. *Assume $1 < \beta \leqslant e$. Any travelling wave ϕ of (3.9) and (3.10) is strictly monotone increasing. Moreover, each travelling wave satisfies*

$$\phi(\xi) = p(\xi)\,\mathrm{e}^{\lambda_1(c)\xi} + O(\mathrm{e}^{(\lambda_1(c)+\sigma)\xi}), \quad \xi \to -\infty, \tag{4.13}$$

where σ is some positive number, $p(\xi)$ is a polynomial of order k, and $k = 0$ if $c > c^*$ and $k = 1$ if $c = c^*$.

Proof. Let ϕ be any solution of (3.9) and (3.10) with $c \geqslant c^*$. Thanks to lemma 4.2, we can define the bilateral Laplace transform $\mathcal{L}(\lambda) = \int_{-\infty}^\infty \mathrm{e}^{-\lambda\xi}\phi(\xi)\,\mathrm{d}\xi$, $0 < \lambda < \rho$. Applying the Laplace transform to (3.9), we obtain

$$\mathcal{R}(\lambda, c)\mathcal{L}(\lambda) = \int_{-\infty}^\infty \mathrm{e}^{-\lambda\xi}h(\xi)\,\mathrm{d}\xi := P(\lambda), \tag{4.14}$$

where $h(\xi) := \beta\tau\int_0^\infty k(s)\phi(\xi - cs)\,\mathrm{d}s(1 - \exp^{-\int_0^\infty k(s)\phi(\xi-cs)\,\mathrm{d}s})$. It is easy to see that the analytic strip of $P(\lambda)$ is broader than that of $\mathcal{L}(\lambda)$. Then, it can be deduced from $\mathcal{L}(\lambda) = P(\lambda)/\mathcal{R}(\lambda, c)$ that analyticity of $\mathcal{L}(\lambda)$ can be extended to the strip $0 < \lambda < \chi$ until χ is the root of $\mathcal{R}(\lambda, c) = 0$. Then, $\mathcal{L}(\lambda)$ is analytic in the strip $0 < \lambda < \lambda_1(c)$ and $P(\lambda)$ is analytic in the strip $0 < \lambda < \lambda_1(c) + \varrho$ for some $\varrho > 0$.

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Note that (3.9) can be written as

$$c\phi'(\xi) = L[\phi](\xi) - h(\xi), \tag{4.15}$$

where $L[\phi](\xi) := D[\phi(\xi+1) + \phi(\xi-1) - 2\phi(\xi)] - \tau\phi(\xi) + \beta\tau \int_0^\infty k(s)\phi(\xi - cs)\,\mathrm{d}s$. Choose $a < \lambda_1(c) < b < \lambda_1(c) + \varrho$, we have

$$\phi(\xi) = O(\mathrm{e}^{a\xi}), \quad h(\xi) = O(\mathrm{e}^{b\xi}), \quad \text{as} \quad \xi \to -\infty.$$

Then, applying [**15**, proposition 6.1], we have

$$\phi(\xi) = p(\xi)\,\mathrm{e}^{\lambda_1(c)\xi} + O(\mathrm{e}^{(b-\varepsilon)\xi}), \quad \xi \to -\infty. \tag{4.16}$$

where p is a polynomial of order k and $k+1$ is the multiplicity of $\lambda_1(c)$ as the pole of $P(\lambda)/\mathcal{R}(\lambda, c)$. The asymptotic representation (4.13) has been proved. Next, we prove that any wave profile is strictly monotone increasing. Suppose ϕ is any travelling wave of (3.9) and (3.10) with $c \geqslant c^*$. In view of the asymptotic representation (4.13) and (3.9), we know that $\phi'(\xi) > 0$ at minus infinity which implies that $\phi(\xi)$ is monotone increasing on $\xi \in (-\infty, -\ell]$ for some $\ell > 0$. By lemma 4.3, there exists $\eta_0 \geqslant 0$ such that

$$\phi(\xi + \eta) \geqslant \phi(\xi) \quad \text{for} \quad \eta \geqslant \eta_0, \ \xi \in \mathbb{R}.$$

Define

$$\tilde{\eta} = \inf\{\eta_0 \geqslant 0 | \phi(\xi + \eta) \geqslant \phi(\xi) \quad \text{for} \quad \eta \geqslant \eta_0, \ \xi \in \mathbb{R}\}.$$

We claim that $\tilde{\eta} = 0$. Suppose by contradiction that $\tilde{\eta} > 0$. Then, we must have $\phi(\xi + \tilde{\eta}) > \phi(\xi)$ for $\xi \in \mathbb{R}$. Because ϕ is uniformly continuous on compact set and is monotone at minus infinite, then for any $\vartheta > 0$, there is small $\varepsilon > 0$ such that

$$\phi(\xi + (\tilde{\eta} - \varepsilon)) > \phi(\xi) \quad \text{for} \quad \xi \in (-\infty, \vartheta].$$

Proceeding the arguments below (4.10) in the proof of lemma 4.3, we would obtain $\phi(\xi + (\tilde{\eta} - \varepsilon)) \geqslant \phi(\xi)$ for $\xi \in \mathbb{R}$, which is a contradiction by the definition of $\tilde{\eta}$. It then follows that $\phi(\xi + \eta) \geqslant \phi(\xi)$ for $\xi \in \mathbb{R}$ and $\eta \geqslant 0$. Thus, $\phi(\xi)$ is non-decreasing. Note that $\phi(\xi) = \frac{1}{c}\,\mathrm{e}^{-\frac{\tau+2D}{c}\xi} \int\limits_{-\infty}^{\xi} \mathrm{e}^{\frac{\tau+2D}{c}z} H[\phi](z)\,\mathrm{d}z$, where $H[\phi](\xi) = D[\phi(\xi+1) + \phi(\xi-1)] + \beta\tau(\int_0^\infty k(s)\phi(\xi - cs)\,\mathrm{d}s)\exp[-\int_0^\infty k(s)\phi(\xi - cs)\,\mathrm{d}s]$. Differentiating $\phi(\xi) = \frac{1}{c}\,\mathrm{e}^{-\frac{\tau+2D}{c}\xi} \int\limits_{-\infty}^{\xi} \mathrm{e}^{\frac{\tau+2D}{c}z} H[\phi](z)\,\mathrm{d}z$ yields $\phi'(\xi) > 0$. □

5. Discussion and simulation

In this paper, we study a discrete Nicholson's blowflies equation with distributed delay which arises naturally from the consideration that the blowflies are distributed in discrete environment and have distributed maturated age. We prove the existence of travelling wave solutions for this patch model. Our method is based on the monotone dynamical system approach, together with the delay approximation technique. In particular, we obtain the minimal wave speed of travelling wave solutions, and

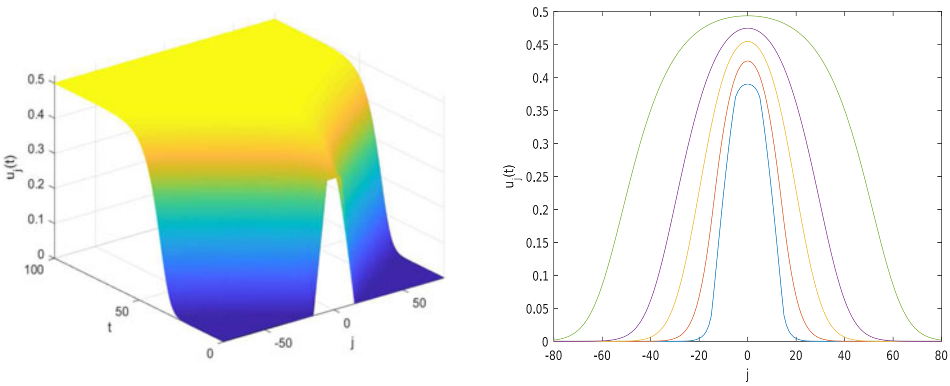

Figure 1. Spread of $u_j(t)$ in different views.

show that it coincides with the spreading speed. Furthermore, we prove theoretically that all the travelling wave solutions are strictly monotone increasing for general kernels. This implies that the shape of the travelling wave solutions remains monotone increasing during the propagation process for the case $1 < \beta \leqslant e$. However, the propagation dynamics is unclear in the case where $\beta > e$ (i.e. the ratio of the per capita daily adult death and the maximum per capita daily egg production is less than $1/e$). We conjecture that the travelling wave solution still exists, but not necessarily monotone in this case. The theoretical study in this case will be more challenging, and is currently under investigation. In this section, we shall carry out some numerical simulations, which will confirm the theoretical results established in our current work.

Let us consider (1.3) by taking $D = 0.5$, $\tau = 1$, $\beta = \sqrt{e}$ and $k(s) = \frac{1}{2}\,\mathrm{e}^{-\frac{1}{2}s}$. Then $u^* = 0.5$. We choose the initial data $u_j(\theta)$ as

$$u_j(\theta) = \begin{cases} 0.375, & |j| \leqslant 5, \theta \in [-50, 0], \\ \dfrac{3}{80}(15 - |j|), & 5 \leqslant |j| \leqslant 15, \theta \in [-50, 0], \\ 0, & \text{else}. \end{cases}$$

Figure 1 shows the spatial spread of the solution through the selected initial data $u_j(\theta)$. To demonstrate the travelling wave phenomenon, we choose the initial function as

$$u_j(\theta) = \begin{cases} 0.5, & j \geqslant 35,\ \theta \in [-50, 0], \\ \dfrac{1}{40}(j - 15), & 15 \leqslant j \leqslant 35,\ \theta \in [-50, 0], \\ 0, & \text{else}. \end{cases}$$

Our numerical experiment shows that the solution evolves promptly to a travelling wave (see figure 2). We see that the shape of the travelling wave remains strictly monotone in the propagation process.

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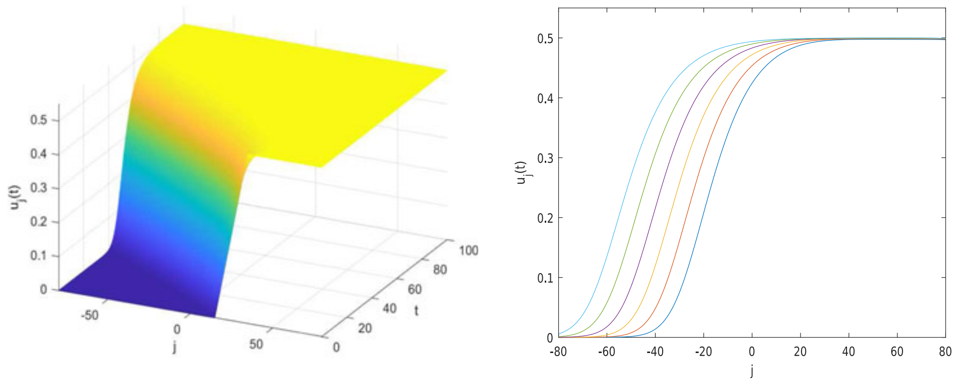

Figure 2. Travelling wave phenomenon observed for $u_j(t)$ in different views.

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