

RIGIDITY OF HILBERT METRICS

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We study the groups of isometries for Hilbert metrics on bounded open convex domains in \mathbb{R}^n and show that if \mathcal{C} is such a set with a *strictly* convex boundary, the Hilbert geometry is asymptotically Riemannian at infinity. As a consequence of this result, we prove there are no Hausdorff quotients of \mathcal{C} by isometry subgroups with finite *volume* except when $\partial\mathcal{C}$ is an ellipsoid.

INTRODUCTION

Let \mathcal{C} be a bounded open convex domain in \mathbb{R}^n with boundary $\partial\mathcal{C}$ and $\|\cdot\|$ be the canonical Euclidean norm in \mathbb{R}^n . It is then possible to define a distance $d_{\mathcal{C}}$ on \mathcal{C} , the so-called Hilbert metric (discovered by D. Hilbert in 1894), as follows. Given two distinct points p and q in \mathcal{C} , let a and b be the intersection points of the straight line defined by p and q with $\partial\mathcal{C}$ so that $p = ta + (1-t)b$ and $q = sa + (1-s)b$ with $0 < s < t < 1$. Then $d_{\mathcal{C}}(p, p) = 0$ and $d_{\mathcal{C}}(p, q) = \ln[a, p, q, b]$, where $[a, p, q, b] = (1-t)s/((1-s)/t) > 1$ is the cross ratio of the ordered collinear points $\{a, p, q, b\}$. The fact $d_{\mathcal{C}}$ is a distance comes from basic properties of the cross ratio and the metric space $(\mathcal{C}, d_{\mathcal{C}})$ thus obtained is a complete non-compact geodesic metric space whose topology is the one induced by the canonical topology of \mathbb{R}^n and in which the affine open segments joining two points of the boundary are geodesics isometric to $(\mathbb{R}, |\cdot|)$. On the other hand, the distance $d_{\mathcal{C}}$ is associated to the Finsler metric $F_{\mathcal{C}}$ on \mathcal{C} given for $p \in \mathcal{C}$ and $v \in T_p\mathcal{C} = \mathbb{R}^n$ by $F_{\mathcal{C}}(p, v) = \|v\| \left((1/\|p - p^-\|) + (1/\|p - p^+\|) \right)$, where p^- (respectively p^+) is the intersection point of the half line $p + \mathbb{R}^-v$ (respectively $p + \mathbb{R}^+v$) with $\partial\mathcal{C}$. For further information, we refer to [3, 4, 6, 9] for an introduction to the subject. In the present paper, we study the subgroups Γ of the group $\text{Isom}(\mathcal{C}, d_{\mathcal{C}})$ of isometries of $(\mathcal{C}, d_{\mathcal{C}})$ with *proper* actions on \mathcal{C} (that is, such that the quotient topological space \mathcal{C}/Γ is Hausdorff) and prove the following *rigidity* result:

THEOREM. *Let \mathcal{C} be a bounded open convex domain in \mathbb{R}^n whose boundary $\partial\mathcal{C}$ is a hypersurface of class C^3 which is strictly convex (in the sense the Hessian is positive*

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definite). Then if $\partial\mathcal{C}$ is not an ellipsoid, any subgroup of the isometry group $\text{Isom}(\mathcal{C}, d_{\mathcal{C}})$ whose action on \mathcal{C} is proper is finite.

We then deduce an important consequence involving the volume measure associated to the Finsler metric $F_{\mathcal{C}}$ on \mathcal{C} which is defined by $\mu(f) = \int_{\mathcal{C}} f(p)\sigma(p)dp$ for every continuous function $f : \mathcal{C} \rightarrow \mathbb{R}$ with compact support, where $\sigma(p)$ is the Finsler density for the metric $F_{\mathcal{C}}$. Recall here that for any Finsler manifold (F, M) the density function σ over M is defined as follows. Given an arbitrary Riemannian metric g on M , for all $p \in M$ the value $\sigma(p)$ is equal to the square root of the ratio of the *Euclidean* volume of the ball $\{v \in T_pM : g(p) \cdot (v, v) \leq 1\}$ by the *Euclidean* volume of the ball $\{v \in T_pM : F(p, v) \leq 1\}$ in the Euclidean vector space $(T_pM, g(p))$. This definition does not depend on the choice of g and it generalises the well known Riemannian density (indeed, if F is Riemannian, then σ is nothing else than the famous $\sqrt{\det(g_{ij})}$). For more information about the Finsler density and some problems related to this notion, one can have a look at [14].

COROLLARY. *Let \mathcal{C} be as above. Then if $\partial\mathcal{C}$ is not an ellipsoid, $(\mathcal{C}, d_{\mathcal{C}})$ does not allow quotients of finite volume (and thus compact quotients) by subgroups of $\text{Isom}(\mathcal{C}, d_{\mathcal{C}})$ whose actions on \mathcal{C} are proper.*

Hilbert metrics are important objects that people have been interested in for many reasons. First of all, they give a range of basic and rich enough examples of geodesic metric spaces (see the investigations of Busemann in [3, 4]). They next generalise in Finsler geometry the Riemannian hyperbolic spaces which are obtained in the case $\partial\mathcal{C}$ is an ellipsoid and correspond to Klein's model (see [3, 4]). On the other hand, Hilbert geometries are considered in affine geometry from the 'projective transformation groups' point of view (see [9] and the recent preprint of Benoist [1]). The question we are dealing with in the present paper has already been tackled in the literature for the *compact* case. Namely, the non-existence of compact quotients has been proved in the framework of affine geometry by Benzécri in [2] (see also the illuminating Lecture Notes of Goldman [9]). Using a dynamical systems approach, a proof of this result has also been given by Egloff ([5, Theorem 3.59]) in dimension two and by Foulon in higher dimensions as a consequence of a more general rigidity theorem (see [7, 8]). The situation when the boundary of \mathcal{C} is no longer *strictly* convex has been investigated in [5, Proposition 3.2], [9] and [10], and it is known that if \mathcal{C} is for example a triangle, there *exist* co-compact subgroups of $\text{Isom}(\mathcal{C}, d_{\mathcal{C}})$.

The starting point of the present paper was to know whether there are quotients of $\text{Isom}(\mathcal{C}, d_{\mathcal{C}})$ with finite *volume* and the strategy we use here to prove that in general such quotients do *not* exist involves very basic tools in metric and Finsler geometry. Indeed, the key idea of the proof consists in showing that the closer to $\partial\mathcal{C}$ we are, the less the metric $d_{\mathcal{C}}$ on \mathcal{C} is different from a Riemannian one (this is given by Proposition 1.3 below). Therefore, if an infinite group of isometries for $d_{\mathcal{C}}$ had a proper action on \mathcal{C} , then every point would be sent to the boundary $\partial\mathcal{C}$ and the Hilbert metric would be as close as

desired to a Riemannian metric and thus Riemannian itself. Then we conclude $\partial\mathcal{C}$ is an ellipsoid by combining results due to different authors ([3, 11]) in the following general statement:

THEOREM. *Let \mathcal{C} be any bounded open convex domain in \mathbb{R}^n . If the metric $d_{\mathcal{C}}$ is Riemannian, then $\partial\mathcal{C}$ is an ellipsoid.*

To be complete let us point out that in December 1999 we discussed a first version of this work (see [15]) with Édith Socié-Méthou who proposed to use an approach for this problem that is different from ours and that she later developed in her Ph.D. thesis ([12]) and in [13].

1. GEOMETRY AT INFINITY

In this part, we give two independent technical lemmas before proving that for a bounded open convex domain $\mathcal{C} \subset \mathbb{R}^n$ with a strictly convex boundary of class C^3 the Hilbert metric $d_{\mathcal{C}}$ is asymptotically Riemannian at infinity (that is, when moving towards the boundary $\partial\mathcal{C}$). Our approach for doing this consists in a local approximation of $\partial\mathcal{C}$ by a parabola (Lemma 1.1) in order to show the geometry of $(\mathcal{C}, d_{\mathcal{C}})$ behaves near $\partial\mathcal{C}$ as if it were Klein's geometry in the Euclidean unit ball near its boundary (Lemma 1.2). We then give Proposition 1.3.

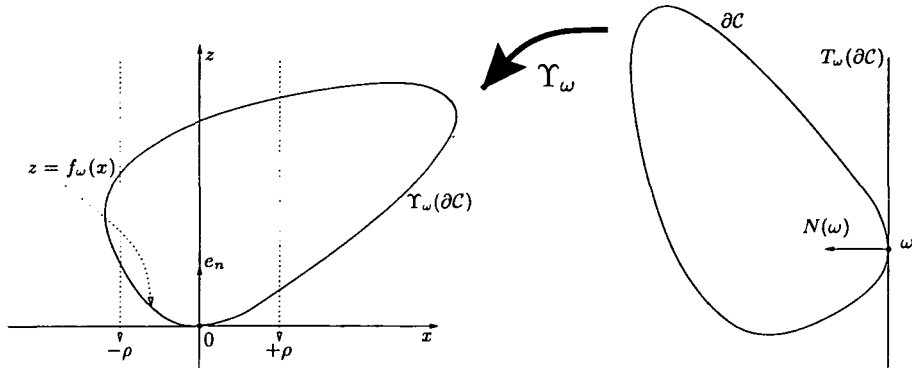
Throughout the section, the canonical Euclidean norms in \mathbb{R}^{n-1} and \mathbb{R}^n will be both denoted by $\|\cdot\|$ and the open ball in $(\mathbb{R}^{n-1}, \|\cdot\|)$ centred at 0 with radius $r > 0$ by $B(0, r)$. On the other hand, (e_1, \dots, e_n) will be the canonical basis in \mathbb{R}^n and $\langle \cdot | \cdot \rangle$ the canonical scalar product.

The first lemma shows that for each point $\omega \in \partial\mathcal{C}$ there is a ball in $(\mathbb{R}^n, \|\cdot\|)$ with centre ω and radius *independent* of ω in which the boundary of \mathcal{C} can be written as the graph of a function defined in \mathbb{R}^{n-1} (after an appropriate coordinate change depending on ω) having the property that its value at x lies between $(1 - M\|x\|)\|x\|^2$ and $(1 + M\|x\|)\|x\|^2$, where $M > 0$ is a number which does *not* depend on ω but *only* on \mathcal{C} .

LEMMA 1.1. *Let \mathcal{C} be a bounded open convex domain in \mathbb{R}^n whose boundary $\partial\mathcal{C}$ is a hypersurface of class C^3 that is strictly convex (in the sense the Hessian is positive definite) and denote $N : \mathcal{C} \rightarrow \mathbb{R}^n$ the normal vector field over $\partial\mathcal{C}$ pointing inwards.*

Then there are positive constants $0 < M$ and $0 < \rho \leq 1/M$ with a family $(\Upsilon_{\omega})_{\omega \in \partial\mathcal{C}}$ of affine isometries in $(\mathbb{R}^n, \|\cdot\|)$ together with a family $(f_{\omega})_{\omega \in \partial\mathcal{C}}$ of functions defined in $B(0, \rho) \subset \mathbb{R}^{n-1}$ such that for each $\omega \in \partial\mathcal{C}$ we have the following:

- (i) $\Upsilon_{\omega}(\mathcal{C}) \subset \{(x, z) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid z > 0\}$, $\Upsilon_{\omega}(\omega) = 0$ and $\vec{\Upsilon}_{\omega} \cdot N(\omega) = e_n$, where $\vec{\Upsilon}_{\omega}$ is the linear part of Υ_{ω} .
- (ii) For all $(x, z) \in B(0, \rho) \times \mathbb{R}$, $z = f_{\omega}(x) \implies (x, z) \in \Upsilon_{\omega}(\partial\mathcal{C})$.
- (iii) For all $x \in B(0, \rho)$, $(1 - M\|x\|)\|x\|^2 \leq f_{\omega}(x) \leq (1 + M\|x\|)\|x\|^2$.



PROOF: For each $\omega \in \partial\mathcal{C}$ let us define the ‘shape’ $\mathcal{S}_\omega(\mathcal{C})$ of \mathcal{C} with respect to ω as the orthogonal projection of \mathcal{C} onto the affine tangent space $T_\omega(\partial\mathcal{C})$ and denote $e(\omega) \geq 0$ the Euclidean distance from ω to $\partial(\mathcal{S}_\omega(\mathcal{C}))$. If there were a sequence of points $(\omega_k)_{k \geq 0}$ in $\partial\mathcal{C}$ such that $\lim_{k \rightarrow +\infty} e(\omega_k) = 0$, by compactness of $\partial\mathcal{C}$ we could find ω in this boundary with $e(\omega) = 0$. But this is not possible because $\partial\mathcal{C}$ is differentiable at ω . So there is a constant $r > 0$ such that $e(\omega) \geq 2r$ for all $\omega \in \partial\mathcal{C}$.

Moreover, if we introduce the ball $\mathcal{B}_\omega = \{m \in T_\omega(\partial\mathcal{C}) : \|m - \omega\| < 2r\}$ in $T_\omega(\partial\mathcal{C})$, the intersection of the full open cylinder $\{m + sN(\omega) : m \in \mathcal{B}_\omega, s \in \mathbb{R}\}$ with $\partial\mathcal{C}$ has exactly two connected components one of which contains ω and that we will denote \mathcal{U}_ω . We then immediately get from the convexity of \mathcal{C} that for each $m \in \mathcal{B}_\omega$ there is a unique $s = \varphi_\omega(m) \in \mathbb{R}$ such that $m + sN(\omega) \in \mathcal{U}_\omega$ and so we get a map $\varphi_\omega : \mathcal{B}_\omega \rightarrow \mathbb{R}$ of class \mathcal{C}^3 .

Let us now fix an open set \mathcal{W} in \mathbb{R}^n such that $\Gamma = \mathcal{W} \cap \partial\mathcal{C}$ is non-empty and parallelisable.

We can then find a family $(\Upsilon_\omega)_{\omega \in \Gamma}$ of affine isometries in $(\mathbb{R}^n, \|\cdot\|)$ depending smoothly on $\omega \in \Gamma$ such that $\Upsilon_\omega(\omega) = 0$ and $\vec{\Upsilon}_\omega \cdot N(\omega) = e_n$.

Since \mathcal{C} is a star-shaped set with respect to one (arbitrary) of its point and has a boundary which is a hypersurface of class \mathcal{C}^3 , there exists a function $F : \mathbb{R}^n \rightarrow \mathbb{R}$ with the same smoothness such that $\partial\mathcal{C} = F^{-1}(0)$ and the family $(F_\omega)_{\omega \in \Gamma}$ defined by $F_\omega = F \circ \Upsilon_\omega^{-1}$ thus satisfies $\Upsilon_\omega(\partial\mathcal{C}) = F_\omega^{-1}(0)$.

Next define for each $\omega \in \Gamma$ the function $g_\omega : B(0, r) \subset \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ by $g_\omega(u) = \varphi_\omega(\Upsilon_\omega^{-1}(u, 0))$; as we have $F_\omega(u, g_\omega(u)) = 0$ for all $u = (u_1, \dots, u_{n-1}) \in B(0, 2r)$, we get by differentiation that all the third order partial derivatives of g_ω with respect to u_1, \dots, u_{n-1} are rational expressions of the partial derivatives of the function $F_\omega : (u, z) \in \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R} \mapsto F_\omega(u, z)$ with respect to u_1, \dots, u_{n-1}, z computed at the point $(u, z) = (u, g_\omega(u))$.

Therefore, by continuity of these partial derivatives with respect to both u and ω , if \mathcal{O} is an open set in \mathbb{R}^n whose closure is in \mathcal{W} , there is a constant $A > 0$ such that for all

$\omega \in \mathcal{O} \cap \partial\mathcal{C}$ and all $u \in \overline{B}(0, r)$ (closed ball) we have from the Taylor expansion

$$\sum_{i=1}^{n-1} \lambda_i(\omega)u_i^2 - A\|u\|^3 \leq g_\omega(u) \leq \sum_{i=1}^{n-1} \lambda_i(\omega)u_i^2 + A\|u\|^3,$$

where $\lambda_1(\omega), \dots, \lambda_{n-1}(\omega)$ are the principal curvatures of $\partial\mathcal{C}$ at the point $\omega \in \partial\mathcal{C}$.

The strict convexity and the compactness of $\partial\mathcal{C}$ together with the continuity of $\lambda_1, \dots, \lambda_{n-1}$ over $\partial\mathcal{C}$ imply there exists $\alpha > 0$ such that $0 < \lambda_i(\omega) \leq 1/\alpha^2$ for all $\omega \in \partial\mathcal{C}$ and all $i \in \{1, \dots, n-1\}$; so we can define $f_\omega(x) = g_\omega(\sqrt{\lambda_1(\omega)}x_1, \dots, \sqrt{\lambda_{n-1}(\omega)}x_{n-1})$ for each $\omega \in \mathcal{O} \cap \partial\mathcal{C}$ and all $x = (x_1, \dots, x_{n-1})$ in $B(0, \rho) \subset \mathbb{R}^{n-1}$, where $\rho = \min\{r\alpha, \alpha^3/A\} > 0$. We then get a family of functions $(f_\omega)_{\omega \in \mathcal{O} \cap \partial\mathcal{C}}$ that satisfies (ii) and (iii) of Lemma 1.1 with $M = A/\alpha^3 > 0$.

Finally, as $\partial\mathcal{C}$ is compact, it can be recovered by a finite number of open sets like \mathcal{O} above and this proves Lemma 1.1. □

The aim of the second lemma is to give us an estimate of the value $F_{\mathcal{C}}(m_0, v)$ of the Finsler metric $F_{\mathcal{C}}$ at a point $m_0 \in \mathcal{C}$ for a vector $v \in \mathbb{R}^n$ in terms of the Euclidean distance from m_0 to $\partial\mathcal{C}$ and the direction of v . More precisely, let us consider a bounded open convex domain K in \mathbb{R}^2 such that there exist positive numbers $M > 0, r \geq M + 2$ and a function $\xi : (-r, r) \rightarrow \mathbb{R}$ which satisfies $\text{Graph}(\xi) \subset \partial K$ together with

$$(1) \quad \frac{1}{4}a(r)t^2 \leq \xi(t) \leq \frac{1}{4}b(r)t^2 \text{ for all } |t| < r,$$

where $a(r) = 1 - M/r > 0$ and $b(r) = 1 + M/r > 0$. For each $c \in \mathbb{R}$ denote by \mathcal{P}_c the parabola in \mathbb{R}^2 whose equation is $y = ct^2/4$. Then define $H(r) \in \mathbb{R}$ so that the point $(r, H(r) + 1)$ is in $\mathcal{P}_{a(r)}$ and let $h(r) \in \mathbb{R}$ so that $h(r) + 1$ is the second component of the intersection point between $\mathcal{P}_{b(r)}$ and the straight line passing through $m_0 = (0, 1)$ and $(r, H(r) + 1)$. Hence,

$$H(r) = \frac{1}{4}a(r)r^2 - 1 \text{ and } h(r) = \frac{2H(r)}{b(r)r^2} \left(H(r) + \sqrt{H(r)^2 + b(r)r^2} \right).$$

As $r \geq M + 2$, we have $H(r) > 0$ and $h(r) > 0$, which implies the point m_0 is in K by (1) and the convexity of K .

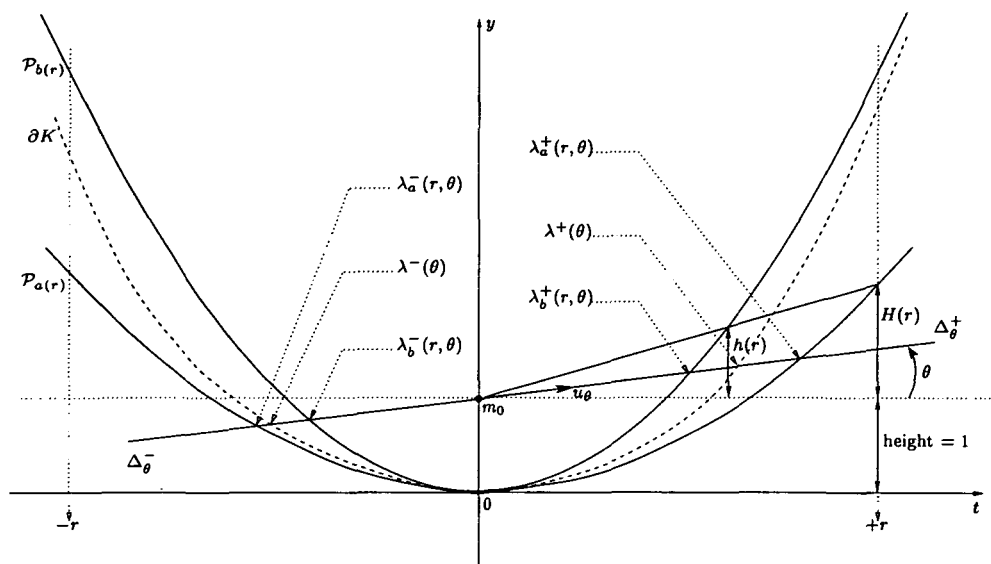
For $\theta \in (-\pi/2, \pi/2)$ we can therefore consider $F_K(m_0, u_\theta)$, where $u_\theta = (\cos \theta, \sin \theta)$, and control this quantity in terms of $r \in [M + 2, +\infty)$ uniformly in θ :

LEMMA 1.2. *For all $\theta \in (-\pi/2, \pi/2)$ we have*

$$\varphi(r) \leq F_K(m_0, u_\theta) \leq \psi(r),$$

where φ and ψ are functions with the property that $\lim_{r \rightarrow +\infty} \varphi(r) = \lim_{r \rightarrow +\infty} \psi(r) = 1$.

REMARK. This result asserts that the bigger r is (and then the closer is ∂K to the parabola with equation $y = t^2/4$ within $(-r, r) \times \mathbb{R}$), the closer to the unit canonical Euclidean sphere is the Finsler sphere $\{v \in \mathbb{R}^2 : F_K(m_0, v) = 1\}$ at the point m_0 .



PROOF: For $\theta \in (-\pi/2, \pi/2)$ introduce the half lines $\Delta_\theta^+ = m_0 + \mathbb{R}^+ u_\theta$ and $\Delta_\theta^- = m_0 + \mathbb{R}^- u_\theta$ and let $\lambda_a^+(r, \theta)$ (respectively $\lambda_a^-(r, \theta)$) be the Euclidean distance between m_0 and the intersection point $\Delta_\theta^+ \cap P_{a(r)}$ (respectively $\Delta_\theta^- \cap P_{a(r)}$). Define $\lambda_b^+(r, \theta)$ (respectively $\lambda_b^-(r, \theta)$) in a similar way for $P_{b(r)}$ and denote by $\lambda^+(\theta)$ (respectively $\lambda^-(\theta)$) the Euclidean distance between m_0 and the intersection point $\Delta_\theta^+ \cap \partial K$ (respectively $\Delta_\theta^- \cap \partial K$).

Using these notations, we then have $F_K(m_0, u_\theta) = (1/\lambda^-(\theta)) + (1/\lambda^+(\theta))$ from the definition of F_K (see Introduction) and a straightforward computation gives

$$(2) \quad \lambda_a^+(r, \theta) = \frac{2}{\sqrt{\sin^2 \theta + a(r) \cos^2 \theta - \sin \theta}} \quad \text{and} \quad \lambda_a^-(r, \theta) = \frac{2}{\sqrt{\sin^2 \theta + a(r) \cos^2 \theta + \sin \theta}}$$

with the analogous formulas for $\lambda_b^+(r, \theta)$ and $\lambda_b^-(r, \theta)$ by changing $a(r)$ into $b(r)$.

There are now three cases to be considered.

FIRST CASE. $|\tan \theta| \leq H(r)/r$.

In this situation the half line Δ_θ^+ (respectively Δ_θ^-) cuts ∂K on the curve $\{(t, \xi(t)) : |t| < r\}$ and we have by (1)

$$\lambda_a^-(r, \theta) \leq \lambda^-(\theta) \leq \lambda_b^-(r, \theta) \quad \text{and} \quad \lambda_b^+(r, \theta) \leq \lambda^+(\theta) \leq \lambda_a^+(r, \theta)$$

which implies

$$(3) \quad \frac{1}{\lambda_a^+(r, \theta)} + \frac{1}{\lambda_b^-(r, \theta)} \leq \frac{1}{\lambda^-(\theta)} + \frac{1}{\lambda^+(\theta)} \leq \frac{1}{\lambda_a^-(r, \theta)} + \frac{1}{\lambda_b^+(r, \theta)}$$

Since $0 < a(r) \leq b(r)$, we then have by (2) the inequality

$$\begin{aligned} \frac{1}{\lambda_a^+(r, \theta)} + \frac{1}{\lambda_b^-(r, \theta)} &\geq \sqrt{\sin^2 \theta + a(r) \cos^2 \theta} \\ &= \sqrt{1 + (a(r) - 1) \cos^2 \theta} = \sqrt{1 - (M \cos^2 \theta)/r} \end{aligned}$$

and thus

$$(4) \quad \frac{1}{\lambda_a^+(r, \theta)} + \frac{1}{\lambda_b^-(r, \theta)} \geq \sqrt{a(r)}.$$

On the other hand,

$$\sqrt{\sin^2 \theta + b(r) \cos^2 \theta} = \sqrt{b(r) + (1 - b(r)) \sin^2 \theta} = \sqrt{b(r) - (M \cos^2 \theta)/r} \leq \sqrt{b(r)},$$

which leads to

$$(5) \quad \frac{1}{\lambda_a^-(r, \theta)} + \frac{1}{\lambda_b^+(r, \theta)} \leq \sqrt{b(r)}.$$

Finally, from (3), (4) and (5) we get

$$\sqrt{a(r)} \leq \frac{1}{\lambda^-(\theta)} + \frac{1}{\lambda^+(\theta)} \leq \sqrt{b(r)}.$$

SECOND CASE. $\tan \theta > H(r)/r$.

The convexity of K and (1) imply the intersection point $\Delta_\theta^+ \cap \partial K$ lies in $[0, +\infty) \times [h(r) + 1, +\infty)$ and therefore $\lambda^+(\theta) \geq h(r)$. As the point $\Delta_\theta^- \cap \partial K$ is still on the curve $\{(t, \xi(t)) : |t| < r\}$, we can write by condition (1) that $\lambda_a^-(r, \theta) \leq \lambda^-(\theta) \leq \lambda_b^-(r, \theta)$ in order to get

$$(6) \quad \frac{1}{\lambda_b^-(r, \theta)} \leq \frac{1}{\lambda^-(\theta)} \leq \frac{1}{\lambda^-(\theta)} + \frac{1}{\lambda^+(\theta)} \leq \frac{1}{\lambda^-(\theta)} + \frac{1}{h(r)} \leq \frac{1}{\lambda_a^-(r, \theta)} + \frac{1}{h(r)}.$$

From $\tan \theta > H(r)/r$ (together with $\theta \in (-\pi/2, \pi/2)$), we have $\sin \theta > (1/\sqrt{1 + H(r)^2/r^2})$ and then

$$(7) \quad \frac{1}{\lambda_b^-(r, \theta)} = \frac{1}{2} \left(\sin \theta + \sqrt{\sin^2 \theta + b(r) \cos^2 \theta} \right) \geq \sin \theta \geq \frac{1}{\sqrt{1 + H(r)^2/r^2}}.$$

On the other hand,

$$(8) \quad \begin{aligned} \frac{1}{\lambda_a^-(r, \theta)} &= \frac{1}{2} \left(\sin \theta + \sqrt{\sin^2 \theta + a(r) \cos^2 \theta} \right) \\ &\leq \frac{1}{2} \left(1 + \sqrt{1 + (a(r) - 1) \cos^2 \theta} \right) = \frac{1}{2} \left(1 + \sqrt{1 - (M \cos^2 \theta)/r} \right) \leq 1. \end{aligned}$$

Hence, (6), (7) and (8) imply

$$\frac{1}{\sqrt{1 + H(r)^2/r^2}} \leq \frac{1}{\lambda^-(\theta)} + \frac{1}{\lambda^+(\theta)} \leq 1 + \frac{1}{h(r)}.$$

THIRD CASE. $\tan \theta < -H(r)/r$.

Applying the same arguments as previously (where the roles of Δ_θ^- and Δ_θ^+ have been exchanged), we also get here

$$\frac{1}{\sqrt{1 + H(r)^2/r^2}} \leq \frac{1}{\lambda^-(\theta)} + \frac{1}{\lambda^+(\theta)} \leq 1 + \frac{1}{h(r)}.$$

CONCLUSION. If we define $\varphi(r) = \min\left\{1/\sqrt{1 + H(r)^2/r^2}, \sqrt{a(r)}\right\}$ and $\psi(r) = \max\left\{1 + (1/h(r)), \sqrt{b(r)}\right\}$, the three cases above say that for all $\theta \in (-\pi/2, \pi/2)$ we have

$$\varphi(r) \leq \frac{1}{\lambda^-(\theta)} + \frac{1}{\lambda^+(\theta)} = F_K(m_0, u_\theta) \leq \psi(r),$$

which proves Lemma 1.2 since $\lim_{r \rightarrow +\infty} \varphi(r) = \lim_{r \rightarrow +\infty} \psi(r) = 1$. □

Considering a convex set \mathcal{C} as described at the beginning of the present section, we are now able to prove the key idea of this paper which states that the closer to $\partial\mathcal{C}$ we are, the less the metric $d_{\mathcal{C}}$ on \mathcal{C} is different from a Riemannian one. In other words, we show the closer to $\partial\mathcal{C}$ a point $p \in \mathcal{C}$ is, the closer to an ellipsoid centred at p is the unit sphere $\{v \in \mathbb{R}^n : F_{\mathcal{C}}(p, v) = 1\}$ of the norm $F_{\mathcal{C}}(p, \cdot)$ in $T_p(\mathcal{C}) \simeq \mathbb{R}^n$.

PROPOSITION 1.3. *Let \mathcal{C} be a bounded open convex domain in \mathbb{R}^n whose boundary $\partial\mathcal{C}$ is a hypersurface of class C^3 that is strictly convex. For any $p \in \mathcal{C}$ let $\delta(p) > 0$ be the Euclidean distance from p to $\partial\mathcal{C}$. Then there exists a family $(\vec{\ell}_p)_{p \in \mathcal{C}}$ of linear transformations in \mathbb{R}^n such that*

$$\lim_{\delta(p) \rightarrow 0} \frac{F_{\mathcal{C}}(p, v)}{\|\vec{\ell}_p(v)\|} = 1 \text{ uniformly in } v \in \mathbb{R}^n \setminus \{0\}.$$

REMARK. The proposition means that the unit sphere of the norm $F_{\mathcal{C}}(p, \cdot)$ approaches the ellipsoid defined by the unit sphere of the Euclidean norm $\|\cdot\| \circ \vec{\ell}_p$ in \mathbb{R}^n as $\delta(p)$ goes to zero.

PROOF: Let $p \in \mathcal{C}$ sufficiently close to $\partial\mathcal{C}$ such that $\delta = \delta(p) < \rho$ (see Lemma 1.1) and that there is a unique $\omega \in \partial\mathcal{C}$ satisfying $\|p - \omega\| = \delta$.

Define $\Phi_p \in \text{GL}(\mathbb{R}^n)$ by $\Phi_p(x, z) = (X, Z) = (2x/\sqrt{\delta}, z/\delta)$ from $\mathbb{R}^{n-1} \times \mathbb{R}$ to $\mathbb{R}^{n-1} \times \mathbb{R}$; it sends $\Upsilon_\omega(p) = \delta e_n$ to $m_0 = (0, \dots, 0, 1)$ and changes inequality (iii) in Lemma 1.1 applied to ω into

$$\begin{aligned} \frac{1}{4}(1 - M\sqrt{\delta}\|X\|/2)\|X\|^2 &\leq f_\omega(\sqrt{\delta}X)/\delta \\ &\leq \frac{1}{4}(1 + M\sqrt{\delta}\|X\|/2)\|X\|^2 \text{ for all } X \in B(0, 1/\delta^{1/4}). \end{aligned}$$

Hence we deduce

$$(9) \quad \frac{1}{4}(1 - M\delta^{1/4})\|X\|^2 \leq f_\omega(\sqrt{\delta}X)/\delta \leq \frac{1}{4}(1 + M\delta^{1/4})\|X\|^2 \text{ for all } X \in B(0, 1/\delta^{1/4}).$$

Now introduce $r = 1/\delta^{1/4}$, fix $k \in \{1, \dots, n - 1\}$ and consider the function $\xi(t) = f_\omega(\sqrt{\delta}te_k)/\delta$ for $t \in (-r, r)$; in restriction to the 2-plane $\Pi_k = \{te_k + Ze_n : (t, Z) \in \mathbb{R}^2\}$, condition (9) means

$$(10) \quad \frac{1}{4}a(r)t^2 \leq \xi(t) \leq \frac{1}{4}b(r)t^2 \text{ for all } |t| < r,$$

where $a(r) = 1 - M/r$ and $b(r) = 1 + M/r$.

If we then define $\ell_p \in \text{Aff}(\mathbb{R}^n)$ by $\ell_p = \Phi_p \circ \Upsilon_\omega$ and focus on the convex set $K = \partial(\ell_p(\mathcal{C})) \cap \Pi_k$, we are exactly in the situation of Lemma 1.2 (where condition (1) is given by (10) above) from which we get

$$\varphi(r) \leq \mathcal{F}_p(m_0, \cos \theta e_k + \sin \theta e_n) \leq \psi(r) \text{ for all } \theta \in (-\pi/2, \pi/2),$$

where \mathcal{F}_p is the Finsler metric associated to $\ell_p(\mathcal{C})$ and φ, ψ are functions such that $\lim_{r \rightarrow +\infty} \varphi(r) = \lim_{r \rightarrow +\infty} \psi(r) = 1$.

As $\mathcal{F}_p(m_0, \cdot)$ is a symmetric, continuous and positive homogeneous function over \mathbb{R}^n , the last inequalities mean

$$\varphi(r)\|v\| \leq \mathcal{F}_p(m_0, v) \leq \psi(r)\|v\| \text{ for all } v \in \Pi_k.$$

But $k \in \{1, \dots, n - 1\}$ has been chosen arbitrarily and so we actually have

$$(11) \quad \varphi(r)\|v\| \leq \mathcal{F}_p(m_0, v) \leq \psi(r)\|v\| \text{ for all } v \in \mathbb{R}^n$$

because $\varphi(r)$ and $\psi(r)$ do not depend on k .

Since ℓ_p is an affine transformation, it is an isometry from $(\mathcal{C}, d_{\mathcal{C}})$ to $(\ell_p(\mathcal{C}), d_{\ell_p(\mathcal{C})})$ or equivalently an isometry from $(\mathcal{C}, F_{\mathcal{C}})$ to $(\ell_p(\mathcal{C}), \mathcal{F}_p)$, that is $\mathcal{F}_p(\ell_p(m), \vec{\ell}_p(v)) = F_{\mathcal{C}}(m, v)$ for all $m \in \mathcal{C}$ and all $v \in \mathbb{R}^n$, where $\vec{\ell}_p$ is the linear part of ℓ_p .

Therefore, as $\ell_p(p) = m_0$, we finally obtain

$$\varphi(r)\|\vec{\ell}_p(v)\| \leq F_{\mathcal{C}}(p, v) \leq \psi(r)\|\vec{\ell}_p(v)\| \text{ for all } v \in \mathbb{R}^n$$

and Proposition 1.3 follows since $r = 1/\delta^{1/4} \rightarrow +\infty$ as $\delta \rightarrow 0$. □

2. PROOF OF THE THEOREM

We give in this section the proof of the central *rigidity* result announced in the Introduction as a consequence of Proposition 1.3 for a bounded open convex domain \mathcal{C} in \mathbb{R}^n with a strictly convex boundary of class C^3 .

THEOREM 2.1. *If $\partial\mathcal{C}$ is not an ellipsoid, then any subgroup Γ of the isometry group $\text{Isom}(\mathcal{C}, d_{\mathcal{C}})$ with a proper action on \mathcal{C} is finite.*

Before showing this result, we establish the following lemma:

LEMMA 2.2. *Let \mathcal{C} be any bounded open strictly convex domain in \mathbb{R}^n and Γ any subgroup of $\text{Isom}(\mathcal{C}, d_{\mathcal{C}})$ with a proper action on \mathcal{C} . Then if there is a point in \mathcal{C} with a finite orbit, Γ has to be finite too.*

PROOF OF LEMMA 2.2: We shall consider \mathcal{C} in $P(\mathbb{R}^{n+1})$ via the classical imbedding $\mathbb{R}^n \hookrightarrow P(\mathbb{R}^{n+1})$ (see for example [9, Section 2]). Then the strict convexity of \mathcal{C} implies that $\text{Isom}(\mathcal{C}, d_{\mathcal{C}}) < \text{PGL}(\mathbb{R}^{n+1})$ according to [10, Proposition 3]. Let $p_0 \in \mathcal{C}$ with a finite orbit $\Gamma \cdot p_0$ and pick $n+1$ points p_1, \dots, p_{n+1} in \mathcal{C} such that $(p_0, p_1, \dots, p_{n+1})$ is a projective frame. Next define the sequence $(\Gamma_j)_{0 \leq j \leq n+1}$ of subgroups in Γ by

$$\Gamma_0 = \text{Stab}_{\Gamma}(p_0) \text{ and } \Gamma_j = \text{Stab}_{\Gamma_{j-1}}(p_j) \text{ for all } j \in \{1, \dots, n+1\},$$

where $\text{Stab}_G(x)$ denotes the stabiliser of the point x under the action of the group G . We therefore have $\Gamma_{n+1} < \dots < \Gamma_1 < \Gamma_0 < \Gamma$ with $\Gamma_{n+1} = \{Id_{\mathcal{C}}\}$ because $\Gamma_{n+1} < \text{PGL}(\mathbb{R}^{n+1})$ fixes the projective frame $(p_0, p_1, \dots, p_{n+1})$.

As for each $j \in \{0, \dots, n\}$ we have $d_{\mathcal{C}}(p_j, \gamma \cdot p_{j+1}) = d_{\mathcal{C}}(p_j, p_{j+1})$ for all $\gamma \in \Gamma_j$, the orbit $\Gamma_j \cdot p_{j+1}$ lies in a compact set and thus if it were infinite, it would have an accumulation point; but this is not possible because the action is proper. Hence, $\Gamma_j \cdot p_{j+1}$ is finite for all $j \in \{0, \dots, n\}$.

Since $\Gamma/\Gamma_0 \cong \Gamma \cdot p_0$ and $\Gamma_j/\Gamma_{j+1} \cong \Gamma_j \cdot p_{j+1}$ (as sets) for all $j \in \{0, \dots, n\}$, we finally get that all the indexes $[\Gamma : \Gamma_0], [\Gamma_0 : \Gamma_1], \dots, [\Gamma_n : \Gamma_{n+1}]$ are finite. So, Γ is finite. \square

PROOF OF THEOREM 2.1: Consider an infinite subgroup Γ of the isometry group $\text{Isom}(\mathcal{C}, d_{\mathcal{C}})$ whose action on \mathcal{C} is proper and choose a point p in \mathcal{C} . As Γ is not finite, the orbit $\Gamma \cdot p$ is infinite by Lemma 2.2 and thus if it were contained in a compact set, it would have an accumulation point which is not possible since the action is proper. So the Euclidean distance between $\Gamma \cdot p$ and $\partial\mathcal{C}$ is zero, which means there exists a sequence $(\gamma_k)_{k \geq 0}$ in Γ such that the limit of the Euclidean distance between $p_k = \gamma_k \cdot p$ and $\partial\mathcal{C}$ is zero as k goes to infinity. Then, from Proposition 1.3, there is a sequence $(\vec{\ell}_k)_{k \geq 0}$ in $\text{GL}(\mathbb{R}^n)$ which satisfies

$$\lim_{k \rightarrow +\infty} \frac{F_{\mathcal{C}}(p_k, T_{p_k} \gamma_k(v))}{\|\vec{\ell}_k(T_{p_k} \gamma_k(v))\|} = 1 \text{ uniformly in } v \in \mathbb{R}^n \setminus \{0\},$$

where $\|\cdot\|$ still denotes the canonical Euclidean norm in \mathbb{R}^n .

But since $\gamma_k \in \text{Isom}(\mathcal{C}, d_{\mathcal{C}})$, we have $F_{\mathcal{C}}(p_k, T_{p_k} \gamma_k(v)) = F_{\mathcal{C}}(p, v)$ and the equality above writes

$$\lim_{k \rightarrow +\infty} \|\mathcal{L}_k(v)\| = F_{\mathcal{C}}(p, v) \text{ uniformly in } v \in \mathbb{R}^n \setminus \{0\}$$

with $\mathcal{L}_k = \vec{\ell}_k \circ T_{p_k} \gamma_k$.

As this limit is uniform in v , the sequence $(\mathcal{L}_k)_{k \geq 0}$ is bounded in $L(\mathbb{R}^n)$ (the space of linear endomorphisms of \mathbb{R}^n endowed with the operator norm associated to $\|\cdot\|$) and we can therefore assume it converges to a limit $\mathcal{L} \in L(\mathbb{R}^n)$. We then obtain $F_{\mathcal{C}}(p, v) = \|\mathcal{L}(v)\|$

for all $v \in \mathbb{R}^n$ (hence $\mathcal{L} \in \text{GL}(\mathbb{R}^n)$), which means the Finsler metric $F_{\mathcal{C}}$ is Riemannian at the point $p \in \mathcal{C}$ (the norm $\|\cdot\| \circ \mathcal{L}$ is indeed Euclidean in \mathbb{R}^n) and since this is true for every choice of p , we get that $F_{\mathcal{C}}$ (or equivalently the corresponding Hilbert metric $d_{\mathcal{C}}$) is a Riemannian metric on \mathcal{C} . \square

At this stage, recall the following fact (see for example [3, page 85]):

THEOREM 2.3 (E. Beltrami, 1866). *Let a connected open set X of the projective space $P(\mathbb{R}^{n+1})$ be metrised so that the metric is Riemannian and the geodesics lie on projective lines. Then the sectional curvature of this Riemannian metric is constant.*

Using this, we deduce the very useful result:

THEOREM 2.4. *Let \mathcal{C} be any bounded open convex domain in \mathbb{R}^n . If the metric $d_{\mathcal{C}}$ is Riemannian, then $\partial\mathcal{C}$ is an ellipsoid.*

PROOF OF THEOREM 2.4: From Theorem 2.3 with $X = \mathcal{C}$, the sectional curvature of the Riemannian metric $d_{\mathcal{C}}$ is constant and thus non-positive since the space $(\mathcal{C}, d_{\mathcal{C}})$ is not compact. Then, by a theorem of Busemann ([3, page 269, Theorem 41.6]), the metric space $(\mathcal{C}, d_{\mathcal{C}})$ has non-positive curvature in the sense of Busemann (see [3, page 237 for the definition]) and this finally implies $\partial\mathcal{C}$ is an ellipsoid from a result due to Kelly and Straus ([11]). \square

This ends the proof of Theorem 2.1. \square

REMARK. Although Theorem 2.4 seems generally to have been accepted, we are unaware of any proof in the literature.

We now determine whether there are quotients of $\text{Isom}(\mathcal{C}, d_{\mathcal{C}})$ with finite volume:

COROLLARY 2.5. *Let \mathcal{C} be as in Theorem 2.1. Then if $\partial\mathcal{C}$ is not an ellipsoid, $(\mathcal{C}, d_{\mathcal{C}})$ does not allow quotients of finite volume by subgroups of $\text{Isom}(\mathcal{C}, d_{\mathcal{C}})$ whose actions on \mathcal{C} are proper.*

PROOF: The volume here is the one μ associated to the Finsler metric $F_{\mathcal{C}}$ on \mathcal{C} and described in the Introduction. Now if a subgroup Γ of $\text{Isom}(\mathcal{C}, d_{\mathcal{C}})$ has a proper action on \mathcal{C} , it is finite since $\partial\mathcal{C}$ is not an ellipsoid according to Theorem 2.1. So, if there were a fundamental domain $D \subset \mathcal{C}$ with $\mu(D)$ finite, we would get that $\mu(\mathcal{C}) = \sum_{\gamma \in \Gamma} \mu(\gamma \cdot D)$ is finite too; but this is not true. (If we indeed use Lemma 1.1 and Lemma 1.2 over the intersection U of \mathcal{C} with a small enough ball in $(\mathbb{R}^n, \|\cdot\|)$ centred at any given point in $\partial\mathcal{C}$, we can get an estimate of the Euclidean volume of the ball $\{v \in \mathbb{R}^n : F_{\mathcal{C}}(p, v) \leq 1\}$ and compute that $\mu(U) = +\infty$.) \square

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