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INFINITARY HARMONIC NUMBERS

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The infinitary divisors of a natural number n are the products of its divisors of the form $p^{y_\alpha 2^\alpha}$, where p^y is an exact prime-power divisor of n and $\sum_\alpha y_\alpha 2^\alpha$ (where $y_\alpha = 0$ or 1) is the binary representation of y. Infinitary harmonic numbers are those for which the infinitary divisors have integer harmonic mean. One of the results in this paper is that the number of infinitary harmonic numbers not exceeding x is less than $2.2\,x^{1/2}\,2^{(1+\epsilon)\log x/\log\log x}$ for any $\epsilon > 0$ and $x > n_0(\epsilon)$. A corollary is that the set of infinitary perfect numbers (numbers n whose proper infinitary divisors sum to n) has density zero.

1. INTRODUCTION

Unless otherwise noted, in what follows lower-case letters will be used to denote natural numbers, with p and q always representing primes. If $\tau(n)$ and $\sigma(n)$ denote, respectively, the number and sum of the positive divisors of n, Ore [5] showed that the harmonic mean of the positive divisors of n is given by $H(n) = n\tau(n)/\sigma(n)$. We say that n is a harmonic number if $H(n)$ is an integer. It is easy to see that every perfect number is a harmonic number.

The unitary analogue of $H(n)$ was studied by Hagis and Lord [3]. Thus, if $\tau^*(n)$ and $\sigma^*(n)$ denote, respectively, the number and sum of the unitary divisors of n (see Definition 1, below), then the unitary harmonic mean of n is given by $H^*(n) = n\tau^*(n)/\sigma^*(n)$, and n is said to be a unitary harmonic number if $H^*(n)$ is an integer.

In [1], Cohen initiated the study of the infinitary divisors of a natural number. In the present paper we investigate $H_\infty(n)$, the harmonic mean of the infinitary divisors of n. Particular attention is paid to IH, the set of natural numbers n for which $H_\infty(n)$ is an integer.

2. INFINITARY DIVISORS

The following three definitions may be found in [1].

DEFINITION 1: If $d \mid n$, d is said to be a 0-ary divisor of n. A divisor d of n is called a 1-ary (or unitary) divisor of n if the greatest common divisor of d and n/d is

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1. In general, if $k \geqslant 1$ then d is called a k-ary divisor of n (and we write $d|_k n$) if $d \mid n$ and the greatest common $(k-1)$-ary divisor of d and n/d is 1.

It is immediate that for any n and k, $1|_k n$ and $n|_k n$. Also, $p^x|_k p^y$ if and only if $p^{y-x}|_k p^y$. If $d|_1 n$, we shall write $d \| n$.

DEFINITION 2: We say p^x is an infinitary divisor of p^y (and we write $p^x|_\infty p^y$) if $p^x|_{y-1} p^y$.

In [1] it is proved that if $p^x|_{y-1} p^y$ then $p^x|_k p^y$ for $k \geqslant y-1$.

DEFINITION 3: Suppose that $d \mid n$. We say that d is an infinitary divisor of n (and we write $d|_\infty n$) if $p^x \| d$ implies that if $p^y \| n$ then $p^x|_\infty p^y$. The only infinitary divisor of 1 is 1.

Now let P be the set of all primes and let

$$I = \{ p^{2^\alpha} \mid p \in \mathsf{P} \ \& \ \alpha \in \mathbb{N}_0 \}.$$

From the fundamental theorem of arithmetic and the fact that the binary representation of a natural number is unique, it follows that if $n > 1$ then n can be written in exactly one way (except for the order of the factors) as the product of distinct elements from I. We shall call each element of I in this product an I-component of n.

Let the number of I-components of n be denoted by $J(n)$. Then $J(1) = 0$ and, if $y = \sum_{i=0}^{\infty} y_i 2^i$ where $y_i = 0$ or 1, $J(p^y) = \sum y_i$. It is obvious that J is an additive function so that, if $n = \prod_{p^y \| n} p^y$, then $J(n) = \sum_{p^y \| n} J(p^y)$.

We shall say that d is an I-divisor of n if every I-component of d is also an I-component of n. (Thus, if $n = 2^3 3^4 5^6 = 2 \cdot 2^2 \cdot 3^4 \cdot 5^2 \cdot 5^4$ then $2^2 5^2$ is an I-divisor of n while $3^2 5^4$ is not.) If $\sigma_I(n)$ is the sum of the I-divisors of n, we see that $\sigma_I(1) = 1$ and $\sigma_I(p^y) = \prod_{y_i = 1} \left(1 + p^{2^i}\right)$ if $y = \sum y_i 2^i$. It is obvious that σ_I is a multiplicative function so that, if $n = \prod_{p^y \| n} p^y$ then $\sigma_I(n) = \prod_{p^y \| n} \prod_{y_i = 1} \left(1 + p^{2^i}\right)$. It follows that if $\tau_I(n)$ is the number of I-divisors of n then $\tau_I(n) = \prod_{p^y \| n} 2^{J(p^y)} = 2^{J(n)}$.

It is proved (implicitly) in the first four sections of [1] that the set of infinitary divisors of n is equal to the set of I-divisors of n. Therefore, if $\tau_\infty(n)$ and $\sigma_\infty(n)$ denote the number and sum, respectively, of the infinitary divisors of n, we have (see Theorem 13 in [1]):

PROPOSITION 1. *If* $n = \prod_{p^y \| n} p^y$ *and* $y = \sum y_i 2^i$, *then*

$$\tau_\infty(n) = \prod_{p^y \| n} 2^{J(p^y)} = 2^{J(n)},$$

where $J(n) = \sum_{p^y \| n} J(p^y) = \sum_{p^y \| n} \sum y_i$, and

$$\sigma_\infty(n) = \prod_{p^y \| n} \prod_{y_i = 1} (1 + p^{2^i}).$$

It is easy to show that the infinitary harmonic mean of n (the harmonic mean of the infinitary divisors of n) is given by

$$H_\infty(n) = \frac{n\tau_\infty(n)}{\sigma_\infty(n)} = 2^{J(n)} \prod_{p^y \| n} \prod_{y_i = 1} \frac{p^{2^i}}{1 + p^{2^i}}. \tag{1}$$

We shall say that n is an infinitary harmonic number if $H_\infty(n)$ is an integer and shall denote by IH the set of these numbers. A computer search was made for the elements of IH in the interval $[1, 10^6]$ and 38 were found. They are listed in Table 1 below.

Cohen [1] has defined n to be an infinitary perfect number if $\sigma_\infty(n) = 2n$ and has found fourteen such numbers. Since $J(n) \geqslant 1$ if $n > 1$, the following result is immediate from (1).

PROPOSITION 2. *The set of infinitary perfect numbers is a subset of IH.*

3. Some elementary results concerning $H_\infty(n)$ and IH

LEMMA 1. *Let $J(n) = J$. Then, if $n > 1$,*

$$\frac{2^{J+1}}{J+2} \leqslant H_\infty(n) < 2^J. \tag{2}$$

PROOF: Since $x/(x+1)$ is monotonic increasing and bounded above by 1 for positive values of x, it follows from (1) that

$$2^J > H_\infty(n) \geqslant 2^J \frac{2}{3} \frac{3}{4} \cdots \frac{J+1}{J+2} = \frac{2^{J+1}}{J+2}.$$

∎

NOTE. We have equality on the left in (2) if and only if $n = 2$ or $n = 2 \cdot 3$ or $n = 2^3 \cdot 3$ or $n = 2^3 \cdot 3 \cdot 5$. Also, using (2) it is clear that $H_\infty(n) = 1$ if and only if $n = 1$.

LEMMA 2. *Suppose that there are s zeros in the binary representation of y. Then*

$$\frac{\tau(p^y)}{\tau_\infty(p^y)} \geqslant \frac{2^s + 1}{2}.$$

PROOF: Set $y = \sum_{i=0}^{t} y_i 2^i$ where $y_t = 1$. Since s values of y_i are 0,

$$y \geqslant 1 + 2 + 2^2 + \cdots + 2^{t-s-1} + 2^t = 2^t + 2^{t-s} - 1.$$

Therefore,

$$\frac{\tau(p^y)}{\tau_\infty(p^y)} = \frac{y+1}{2^{\Sigma y_i}} \geqslant \frac{2^t + 2^{t-s}}{2^{t+1-s}} = \frac{2^s+1}{2}.$$

□

THEOREM 1. *For all n, $H^*(n) \leqslant H_\infty(n) \leqslant H(n)$. For $n > 1$, equality holds on the left if and only if $p^y \parallel n$ implies $y = 2^\alpha$, and on the right if and only if $p^y \parallel n$ implies $y = 2^\beta - 1$.*

PROOF: Since $H^*(1) = H_\infty(1) = H(1)$, we may suppose that $n > 1$.

If $p^y \parallel n$ implies that $y = 2^\alpha$ then, since $H^*\left(p^{2^\alpha}\right) = 2p^{2^\alpha}/\left(1 + p^{2^\alpha}\right) = H_\infty\left(p^{2^\alpha}\right)$ and since H^* and H_∞ are each multiplicative, it follows that $H^*(n) = H_\infty(n)$.

Now suppose that $p^y \parallel n$ and $y \neq 2^\alpha$. Then $y = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_u}$, where $a_1 > a_2 > \cdots > a_u \geqslant 0$ and $u \geqslant 2$. It follows that

$$\frac{H^*(p^y)}{H_\infty(p^y)} = \frac{2p^y}{1+p^y} \cdot \frac{\left(1 + p^{2^{a_1}}\right) \ldots \left(1 + p^{2^{a_u}}\right)}{2^u p^y} < \frac{1}{2^{u-1}} \cdot \frac{p^y + p^{y-1} + \cdots + p + 1}{p^y}$$
$$< \frac{1}{2^{u-1}} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) \leqslant \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right) = 1.$$

Therefore, $H^*(p^y) < H_\infty(p^y)$, so that $H^*(n) < H_\infty(n)$.

If $p^y \parallel n$ implies that $y = 2^\beta - 1$ then, since $H\left(p^{2^\beta - 1}\right) = p^{2^\beta - 1} 2^\beta (p-1)/\left(p^{2^\beta} - 1\right) = H_\infty\left(p^{2^\beta - 1}\right)$ and since H and H_∞ are each multiplicative, it follows that $H_\infty(n) = H(n)$.

Now suppose that $p^y \parallel n$ and $y \neq 2^\beta - 1$. We consider several cases.

Suppose first that, in Lemma 2, $s \geqslant 2$. Then

$$\frac{H(p^y)}{H_\infty(p^y)} = \frac{\tau(p^y)}{\tau_\infty(p^y)} \cdot \frac{\sigma_\infty(p^y)}{\sigma(p^y)} \geqslant \frac{5}{2} \cdot \frac{(p^y+1)(p-1)}{p^{y+1}-1} = 1 + \frac{3p^{y+1} - 5p^y + 5p - 3}{2(p^{y+1}-1)} > 1,$$

since $p \geqslant 2$.

Now suppose that y is odd. From Theorem 3 in [1], $p \mid_\infty p^y$ and hence $p^{y-1} \mid_\infty p^y$. Therefore, using Lemma 2 with $s \geqslant 1$,

$$\frac{H(p^y)}{H_\infty(p^y)} \geqslant \frac{3}{2} \cdot \frac{\sigma_\infty(p^y)}{\sigma(p^y)} \geqslant \frac{3}{2} \cdot \frac{(p^y + p^{y-1} + 1)(p-1)}{p^{y+1}-1} = 1 + \frac{p^{y+1} - 3p^{y-1} + 3p - 1}{2(p^{y+1}-1)} > 1,$$

since $p \geqslant 2$.

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One possibility remains: $s = 1$ and y is even. Then the binary representation of y has the form $11\ldots110$, so that $y = 2^\gamma - 2$, where $\gamma \geqslant 2$.

If $\gamma = 2$, then

$$\frac{H(p^y)}{H_\infty(p^y)} = \frac{H(p^2)}{H_\infty(p^2)} = \frac{3}{2} \cdot \frac{p^2+1}{p^2+p+1} = 1 + \frac{(p-1)^2}{2(p^2+p+1)} > 1,$$

since $p \geqslant 2$.

If $\gamma \geqslant 3$, then

$$\begin{aligned}\frac{H(p^y)}{H_\infty(p^y)} &= \frac{H(p^{2^\gamma-2})}{H_\infty(p^{2^\gamma-2})} = \frac{2^\gamma-1}{2^{\gamma-1}} \cdot \frac{(1+p^2)(1+p^4)(1+p^8)\ldots\left(1+p^{2^{\gamma-1}}\right)(p-1)}{p^{2^\gamma-1}-1} \\ &= \frac{2^\gamma-1}{2^{\gamma-1}} \cdot \frac{\left(1+p^2+p^4+p^6+\cdots+p^{2^\gamma-2}\right)(p-1)}{p^{2^\gamma-1}-1} \\ &= \frac{2^\gamma-1}{2^{\gamma-1}} \cdot \frac{\left(p^{2^\gamma}-1\right)(p-1)}{(p^2-1)(p^{2^\gamma-1}-1)} > \frac{2^\gamma-1}{2^{\gamma-1}} \cdot \frac{p}{p+1} \geqslant 2 \cdot \frac{2^\gamma-1}{2^\gamma} \cdot \frac{2}{3} \\ &= \frac{4}{3}\left(1 - \frac{1}{2^\gamma}\right) \geqslant \frac{4}{3} \cdot \frac{7}{8} > 1.\end{aligned}$$

Therefore, $H(p^y) > H_\infty(p^y)$, and it follows that $H_\infty(n) < H(n)$. This completes the proof of Theorem 1. □

Since $2^\alpha = 2^\beta - 1$ if and only if $\alpha = 0$ and $\beta = 1$, it follows from Theorem 1 that $H^*(n) = H_\infty(n) = H(n)$ if and only if n is square-free (or $n = 1$). Ore [5] proved that 6 is the only square-free harmonic number (he did not count 1 in this context; nor shall we), so the following result is immediate.

COROLLARY 1.1. *The only square-free infinitary harmonic number is* 6.

Since $2 \mid \left(1 + p^{2^i}\right)$ if p is odd, and since $4 \mid (1+p)$ if $p = 4m+3$, the next two results follow from (1) and the fact that $p \,|_\infty\, p^y$ if and only if y is odd (Theorem 3 in [1]).

PROPOSITION 3. *If n is odd and $n \in IH$, then $H_\infty(n)$ is odd.*

PROPOSITION 4. *If n is odd, $n \in IH$, $p^y \parallel n$ and $p = 4m+3$, then y is even.*

PROPOSITION 5. *If $n \in IH$, $(p,n) = 1$ and $\sigma_\infty(p^y) \mid \tau_\infty(p^y) H_\infty(n)$, then $p^y n \in IH$.*

This follows from (1) and the fact that H_∞ is multiplicative.

As an example of Proposition 5, $409500 \in IH$ and $H_\infty(409500) = 30$; since $(29, 409500) = 1$ and $(1+29) \mid 2 \cdot 30$, it follows that $29 \cdot 409500 \in IH$. Other results like Proposition 5, but where $p \mid n$, are easily obtained. For example, it may be shown that if $n \in IH$, $3 \mid H_\infty(n)$ and $2^{2a} \parallel n$, then $2n \in IH$.

4. Two cardinality theorems

Theorem 2. *If S_c is the set of natural numbers n such that $H_\infty(n) = c$, then S_c is finite (or empty) for every real number c.*

Proof: Since $2^{J+1}/(J+2) \geqslant J$, it follows from Lemma 1 that if $H_\infty(n) = c$ then the number of I-components of n is bounded above by c. Assume that S_c is infinite. Then S_c must contain an infinite subset, say S_{cm}, each of whose elements has exactly m I-components. It follows that an infinite sequence n_1, n_2, n_3, ... of distinct integers exists with the following properties.

(i) $n_i \in S_{cm}$, so that $H_\infty(n_i) = c$ for $i = 1, 2, 3, \ldots$.

(ii) $n_i = p_1^{2^{a_1}} \cdots p_{s-1}^{2^{a_{s-1}}} \cdot p_{is}^{2^{a_{is}}} \cdots p_{im}^{2^{a_{im}}} = P \cdot \prod_{j=s}^{m} p_{ij}^{2^{a_{ij}}}$, where $p_1^{2^{a_1}} < \cdots < p_{s-1}^{2^{a_{s-1}}} < p_{is}^{2^{a_{is}}} < \cdots < p_{im}^{2^{a_{im}}}$ for $i = 1, 2, \ldots$. (The ps are primes which are not necessarily distinct; P may be an empty product, but $s-1 \neq m$.)

(iii) $p_{ij}^{2^{a_{ij}}} \to \infty$ as $i \to \infty$ for $j = s, \ldots, m$.

(That is, each n_i is composed of a fixed constant block of elements from I and a variable block of elements from I arranged monotonically within the block and such that each element of this variable block goes to infinity with i.)

From (i) and (ii) and (1) and the fact that H_∞ is multiplicative, we see that

$$\frac{c}{H_\infty(P)} = \prod_{j=s}^{m} H_\infty\left(p_{ij}^{2^{a_{ij}}}\right) = 2^{m-s+1} \cdot \prod_{j=s}^{m} \frac{p_{ij}^{2^{a_{ij}}}}{1 + p_{ij}^{2^{a_{ij}}}} < 2^{m-s+1}.$$

Therefore, there exists a fixed positive number v such that $\prod_{j=s}^{m} H_\infty\left(p_{ij}^{2^{a_{ij}}}\right) = 2^{m-s+1} - v$. But, from (iii), it follows that $\lim_{i\to\infty} H_\infty\left(p_{ij}^{2^{a_{ij}}}\right) = 2$ for $j = s, \ldots, m$. Therefore, for large i,

$$\prod_{j=s}^{m} H_\infty\left(p_{ij}^{2^{a_{ij}}}\right) > 2^{m-s+1} - v.$$

This contradiction completes the proof. ∎

Theorem 3. *There exist at most finitely many infinitary harmonic numbers with a specified number of I-components.*

Proof: Consider the elements of IH with precisely K I-components. There are only finitely many integers between $2^{K+1}/(K+2)$ and 2^K. From Theorem 2, if l is one of these integers then S_l is finite (or empty). ∎

Corollary 3.1. *There is at most a finite number of infinitary perfect numbers with a specified number of I-components.*

5. The distribution of the infinitary harmonic numbers

For each positive number x, we shall denote by $A(x)$ the number of integers n such that $n \leqslant x$ and $n \in IH$.

Theorem 4. *For any $\epsilon > 0$ and for all sufficiently large values of x,*

$$A(x) < 2.2\, x^{1/2}\, 2^{(1+\epsilon)\log x/\log\log x}.$$

Proof: A positive integer m is powerful if $p \mid m$ implies that $p^2 \mid m$. Every positive integer can be written uniquely as a product $N_P N_F$, where $(N_P, N_F) = 1$, N_P is powerful and N_F is square-free. (We consider 1 to be both powerful and square-free.) If $P(x)$ denotes the number of powerful numbers not exceeding x, it is proved in [2] that $P(x) \sim cx^{1/2}$, where $c = \zeta(3/2)/\zeta(3) = 2.173\ldots$. Therefore, $P(x) < 2.2\, x^{1/2}$ for all large values of x.

If N_P is a (fixed) powerful number, let $g(N_P, x)$ denote the number of square-free numbers N_F such that $(N_P, N_F) = 1$, $N_P N_F \leqslant x$ and $N_P N_F \in IH$. If $G(x) = \max\{g(N_P, x)\}$ for $N_P \leqslant x$, it follows that

$$A(x) < 2.2\, x^{1/2}\, G(x) \qquad \text{for large } x. \tag{3}$$

We now investigate the magnitude of $G(x)$. Let N_P be a powerful number for which distinct square-free numbers $m_1, m_2, \ldots, m_{G(x)}$ exist such that $(N_P, m_i) = 1$, $N_P m_i \leqslant x$ and $N_P m_i \in IH$ for $i = 1, 2, \ldots, G(x)$. Then $H_\infty(N_P m_i) = H_\infty(N_P) \cdot H_\infty(m_i) = Z_i$, where Z_i is an integer, for $i = 1, 2, \ldots, G(x)$. Suppose that $Z_j = Z_k$ where $j \neq k$. If $(m_j, m_k) = d$ then, of course, $H_\infty(M_j) = H_\infty(M_k)$ where $M_j = m_j/d$ and $M_k = m_k/d$. Since $M_j \neq M_k$, we cannot have $M_j = 1$, so we may suppose that $2 \leqslant M_j < M_k$. If $M_j = p_1 \ldots p_s$ and $M_k = q_1 \ldots q_t$, where $p_1 < \cdots < p_s$, $q_1 < \cdots < q_t$ and $p_u \neq q_v$, then from (1) it follows that

$$2^s p_1 \ldots p_s (1 + q_1) \ldots (1 + q_t) = 2^t q_1 \ldots q_t (1 + p_1) \ldots (1 + p_s).$$

Then $q_t \mid (1 + q_r)$ for some r, $1 \leqslant r < t$. This implies that $q_t = 3$ and $q_r = q_1 = 2$, which is a contradiction since we require $1 < M_j < M_k$. Hence $Z_j \neq Z_k$, unless $j = k$. Therefore, without loss of generality, $Z_1 < Z_2 < \cdots < Z_{G(x)}$ so that $G(x) \leqslant Z_{G(x)} = H_\infty(N_P m_{G(x)}) < \tau_\infty(N_P m_{G(x)})$. Since $\tau_\infty(n) \leqslant \tau(n)$, and since $N_P m_{G(x)} \leqslant x$, and since it follows from Theorem 317 in [4] that $\tau(n) \leqslant 2^{(1+\epsilon)\log x/\log\log x}$ if $n \leqslant x$ and $x > n_0(\epsilon)$, we conclude that

$$G(x) < 2^{(1+\epsilon)\log x/\log\log x} \qquad \text{for all large } x. \tag{4}$$

Theorem 4 follows from (3) and (4). □

Corollary 4.1. *IH has density zero.*

Corollary 4.2. *The set of infinitary perfect numbers has density zero.*

TABLE 1. The infinitary harmonic numbers in $[1, 10^6]$

n	$H_\infty(n)$	n	$H_\infty(n)$
1	1	$95550 = 2 \cdot 3 \cdot 5^2 7^2 13$	14
$6 = 2 \cdot 3$	2	$136500 = 2^2 3 \cdot 5^3 7 \cdot 13$	25
$45 = 3^2 5$	3	$163800 = 2^3 3^2 5^2 7 \cdot 13$	24
$60 = 2^2 3 \cdot 5$	4	$172900 = 2^2 5^2 7 \cdot 13 \cdot 19$	19
$90 = 2 \cdot 3^2 5$	4	$204750 = 2 \cdot 3^2 5^3 7 \cdot 13$	25
$270 = 2 \cdot 3^3 5$	6	$232470 = 2 \cdot 3^4 5 \cdot 7 \cdot 41$	15
$420 = 2^2 3 \cdot 5 \cdot 7$	7	$245700 = 2^2 3^3 5^2 7 \cdot 13$	27
$630 = 2 \cdot 3^2 5 \cdot 7$	7	$257040 = 2^4 3^3 5 \cdot 7 \cdot 17$	28
$2970 = 2 \cdot 3^3 5 \cdot 11$	11	$409500 = 2^2 3^2 5^3 7 \cdot 13$	30
$5460 = 2^2 3 \cdot 5 \cdot 7 \cdot 13$	13	$464940 = 2^2 3^4 5 \cdot 7 \cdot 41$	18
$8190 = 2 \cdot 3^2 5 \cdot 7 \cdot 13$	13	$491400 = 2^3 3^3 5^2 7 \cdot 13$	36
$9100 = 2^2 5^2 7 \cdot 13$	10	$646425 = 3^2 5^2 13^2 17$	13
$15925 = 5^2 7^2 13$	7	$716625 = 3^2 5^3 7^2 13$	21
$27300 = 2^2 3 \cdot 5^2 7 \cdot 13$	15	$790398 = 2 \cdot 3^4 7 \cdot 17 \cdot 41$	17
$36720 = 2^4 3^3 5 \cdot 17$	16	$791700 = 2^2 3 \cdot 5^2 7 \cdot 13 \cdot 29$	29
$40950 = 2 \cdot 3^2 5^2 7 \cdot 13$	15	$819000 = 2^3 3^2 5^3 7 \cdot 13$	40
$46494 = 2 \cdot 3^4 7 \cdot 41$	9	$900900 = 2^2 3^2 5^2 7 \cdot 11 \cdot 13$	33
$54600 = 2^3 3 \cdot 5^2 7 \cdot 13$	20	$929880 = 2^3 3^4 5 \cdot 7 \cdot 41$	24
$81900 = 2^2 3^2 5^2 7 \cdot 13$	18	$955500 = 2^2 3 \cdot 5^3 7^2 13$	28

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