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A discriminant and an upper bound for ω^2 for hyperelliptic arithmetic surfaces

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Abstract. We define a natural discriminant for a hyperelliptic curve X of genus g over a field K as a canonical element of the $(8g+4)$th tensor power of the maximal exterior product of the vectorspace of global differential forms on X. If v is a discrete valuation on K and X has semistable reduction at v, we compute the order of vanishing of the discriminant at v in terms of the geometry of the reduction of X over v. As an application, we find an upper bound for the Arakelov self-intersection of the relative dualizing sheaf on a semistable hyperelliptic arithmetic surface.

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0. Introduction

In the present paper we introduce a natural discriminant for hyperelliptic curves of genus $g \geqslant 2$. It will be defined as a canonical element of the $(8g+4)$th tensor power of the maximal exterior product of the vectorspace of global differential forms on the curve. To fix ideas, let us first consider the analogous case of an elliptic curve E over a field K, given, say, by the Weierstraß equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

It is well known that while its discriminant $\Delta \in K$ depends on the special equation, the element $\Lambda_{E/K} = \Delta(\mathrm{d}x/(2y + a_1x + a_3))^{\otimes 12} \in (H^0(E, \Omega^1_{E/K}))^{\otimes 12}$ is a genuine invariant of E. Moreover, if v is a discrete valuation of K and $\mathcal{O}_v \subset K$ is the associated discrete valuation ring, we may regard $\Lambda_{E/K}$ as a rational section $\Lambda_{\mathcal{E}/S}$ of the bundle $(\pi_*\omega_{\mathcal{E}/S})^{\otimes 12}$ on $S = \operatorname{Spec} \mathcal{O}_v$, where $\pi\colon \mathcal{E} \to S$ is the minimal regular model of E over S and $\omega_{\mathcal{E}/S}$ is the relative dualizing sheaf. Assuming semistable reduction at v, there is a simple geometric interpretation of the order of vanishing of $\Lambda_{\mathcal{E}/S}$ at v: It is the number of singular points of the geometric special fibre of $\mathcal{E} \to S$. Our main Theorem (3.1) is a generalization of this fact to the case of hyperelliptic curves. Given a semistable curve $\mathcal{X}/S$ with smooth hyperelliptic

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generic fibre X/K, it gives a formula for the order of vanishing of the discriminant $\Lambda_{X/S} \in (\det H^0(X, \Omega^1_{X/K}))^{\otimes(8g+4)}$ of X at v in terms of the geometry of the special fibre. On the way to this formula we give an explicit construction of a regular model of X over a finite extension of S as the ramified double covering of a pointed prestable curve of genus zero. As an application, we find an upper bound for the Arakelov self-intersection ω^2 of the relative dualizing sheaf on semistable hyperelliptic arithmetic surfaces (Corollary 7.8). The search for an upper bound for ω^2 has been initiated by A. N. Parshin [Par1]. He observed that in the geometric case such a bound follows from the Bogomolov–Miyaoka–Yau inequality between Chern-classes of an algebraic surface and that in the arithmetic case it would lead to the positive answer of various diophantine questions, as for instance the abc-conjecture. The most naive arithmetic analogue of the Bogomolov–Miyaoka–Yau inequality seems to be wrong: There are curves of genus two which provide counterexamples (cf. [BMM]). In [MB], Moret-Bailly therefore formulates as a hypothesis a shape of an upper bound for ω^2 which still implies the same arithmetic consequences as given by Parshin. Our bound has the shape of Moret-Bailly's hypothesis. Unfortunately, it has two shortcomings. Firstly, it involves the choice of a metric on the relative dualizing sheaf of the universal curve over the moduli space of curves of genus g, which we cannot make explicit. Secondly (and this is more serious), it is very special to the hyperelliptic situation. In particular it does not involve the discriminant of the number field.

This work owes most to Chapter 4 of [C-H], where Cornalba and Harris describe the structure of the boundary of the moduli space of hyperelliptic curves of genus g and give an expression of the $(8g+4)$th power of the Hodge bundle in terms of the boundary components. The existence of a canonical element in $(\det H^0(X, \Omega^1_{X/K}))^{\otimes(8g+4)}$ for a hyperelliptic curve of genus g seems to be well known (see for example [U] in the case of $g = 2$), though I don't know any reference in the general case. Our construction of a regular model for a hyperelliptic curve has been inspired by work of E. Horikawa [Hor] and U. Persson [Per]. An upper bound for ω^2 of the form of our Corollary 7.8 had been previously established in the case $g = 2$ and with respect to the Arakelov metric by J.-B. Bost in a letter to B. Mazur [Bost2], using explicit formulas given in [Bost1]. In fact it was this letter and Mazur's answer to it [Maz], which was the starting point of our investigation.

1. Hyperelliptic semistable curves

By a *graph* we understand a triple $(\mathcal{V}, \mathcal{E}, c)$, where $\mathcal{V}$ and $\mathcal{E}$ are disjoint sets (the set of vertices and the set of edges respectively) and c (the coincidence relation) is a rule that to each edge E associates a subset $c(E) \subset \mathcal{V}$ consisting of one or two vertices. For example, for every $n \in \mathbb{N}$ we define the *cyclic graph of length* n

$$C_n = (\{V_i\}_{i\in\mathbb{Z}/n\mathbb{Z}}, \{E_i\}_{i\in\mathbb{Z}/n\mathbb{Z}}, c(E_i) = \{V_i, V_{i+1}\} \text{ for } i \in \mathbb{Z}/n\mathbb{Z}).$$

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We adopt the usual notions from graph theory such as finiteness, connectedness, morphism of graphs, subgraph, quotient of a graph by a group action. A connected graph that has no cyclic subgraph will be called a *tree*.

DEFINITION 1.1. A *marked graph* is a graph together with a distinguished subset of its set of vertices (the set of 'marked vertices'). A marked graph is called *semistable* (*stable*), if it is connected and if from every unmarked vertex there start at least two (three) edges.

For example, the (unmarked) cyclic graph C_1 is semistable (the edge which starts and ends at the unique vertex, is counted twice).

DEFINITION 1.2. Let $\Gamma = (\mathcal{V}, \mathcal{E}, c)$ be a graph and τ an automorphism of Γ. An edge E of Γ will be called *direction-reversing* with respect to τ, if $\tau(E) = E$, $c(E) = \{V_1, V_2\}$ and $\tau(V_1) = V_2$.

For example, the unique edge of the cyclic graph C_1 is direction-reversing with respect to the identity. Let Γ be a (marked) graph and τ an involution (i.e. an automorphism of order $\leqslant 2$) of Γ. We denote by Γ_τ the (marked) graph that is obtained from Γ by omitting all edges which are direction-reversing with respect to τ.

DEFINITION 1.3. An involution τ of a marked graph Γ is called *hyperelliptic*, if it leaves marked vertices fixed and if the quotient $\Gamma_\tau/(\tau)$ is a tree. A marked graph is called *hyperelliptic*, if it admits a hyperelliptic involution.

PROPOSITION 1.4. *Let Γ be a finite semistable marked graph which is not an unmarked cyclic graph. Then there is at most one hyperelliptic involution on Γ.*

Proof. Let τ and τ' be two hyperelliptic involutions on Γ. Let V be a vertex and assume first that $\tau(V) \neq V$. It is then easy to see that there exists a cyclic subgraph C containing V and that C is mapped onto itself by any hyperelliptic involution. Now C contains a vertex V_0 which is marked or from which there starts an edge not belonging to C. In both cases it is easy to see that necessarily $\tau'(V_0) = \tau(V_0)$. But the action of τ (and τ') on C is completely determined by its action on one of its vertices, so we have $\tau(V) = \tau'(V)$. Now let us assume $\tau(V) = V$. Then $\tau'(V) = V$, because otherwise, interchanging the role of τ and τ' in the previous argument, we would obtain a contradiction. This shows that τ and τ' act identically on the set of vertices of Γ. It is now easy to see that τ and τ' act identically also on the set of edges of Γ. $\square$

Recall (cf. [Kn]) that an *n-pointed prestable curve* of genus g over a scheme S consists of a proper flat morphism $\pi\colon X \to S$ and n sections $s_i\colon S \to X$ of π such that for each geometric point s of S the following holds:

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(1) The geometric fibre X_s is a reduced curve and has only ordinary double points as singularities.
(2) The points $P_i = s_i(s)$ are distinct regular points of X_s.
(3) $\dim H^1(X_s, \mathcal{O}_{X_s}) = g$.

An n-pointed prestable curve $(X, (P_i)_{i=1,\ldots,n})$ over an algebraically closed field is called *stable* (*semistable*), if it is connected and if on each smooth rational component of X the number of points that are double points of X or are among the P_i is at least three (two). More generally, an n-pointed prestable curve over an arbitrary scheme is called stable (semistable), if each of its geometric fibres is stable (semistable).

To every prestable curve X over an algebraically closed field we associate in the usual way a graph Γ_X, whose vertices correspond to the irreducible components and whose edges correspond to the double points of X. We provide Γ_X with a canonical marking by requiring that a vertex be marked if and only if the corresponding irreducible component of X has genus $\geqslant 1$. It is then clear that the marked graph associated to a semistable (stable) curve is semistable (stable) in the above sense.

Let X be an n-pointed prestable curve over an algebraically closed field. Let $p \in X$ be a double point. Let $\tilde{X} \to X$ be the partial normalization of X at p and $p_1, p_2 \in \tilde{X}$ the points of the preimage of p. Let $Z \simeq \mathbb{P}^1_k$ and choose three points $z_1, z_2, z_3 \in Z$. The $(n+1)$-pointed prestable curve, obtained by taking the union of $\tilde{X}$ and Z, identifying p_i with z_i $(i = 1, 2)$ and taking as marked points z_3 and those coming from X, will be called the *modification of* X *at* p.

Let $(Y, (P_1, \ldots, P_n))$ be an n-pointed prestable curve over an algebraically closed field k of characteristic $\neq 2$. Recall (cf. [H-M], Section 4) that a morphism $\pi\colon X \to Y$ is called an *admissible double covering*, if the following holds

(1) X is a prestable curve and π is a finite morphism of degree two.
(2) π is ramified over every point P_i $(i = 1, \ldots, n)$ and étale over any other smooth point of Y.
(3) If $q \in Y$ is a double point, then either π is étale over q or $\pi^{-1}(q) = \{p\}$ and there are isomorphisms

$$\widehat{\mathcal{O}_{X,p}} \simeq k[[x, y]]/(x \cdot y),$$

$$\widehat{\mathcal{O}_{Y,q}} \simeq k[[u, v]]/(u \cdot v),$$

such that π induces the morphism $k[[u, v]]/(u \cdot v) \to k[[x, y]]/(x, y)$ which maps u to x^2 and v to y^2.

We can now define the central notion of this paper.

DEFINITION 1.5. Let k be a field of characteristic $\neq 2$ and let X/k be a semistable curve.

(1) Assume first that k is algebraically closed. Then we call X/k *hyperelliptic*, if the following holds

 (a) The associated marked graph Γ_X is hyperelliptic.

 (b) Let X' be the (pointed) prestable curve that is the modification of X at each of its double points which corresponds to an edge of Γ_X which is direction-reversing with respect to the hyperelliptic involution of Γ_X. Then there exists an admissible double covering $X' \to Y$ onto a pointed prestable curve Y of genus 0 that carries marked points into marked points.

(2) In general, we call X/k *hyperelliptic*, if $\overline{X}/\overline{k}$ is hyperelliptic in the sense of 1, where $\overline{X} = X \otimes_k \overline{k}$ and $\overline{k}$ is some algebraic closure of k.

Let X be a hyperelliptic semistable curve. The hyperelliptic involution τ of Γ_X induces a canonical involution on the set of components and on the set of double points of X. A double point of X is called *direction-reversing*, if the corresponding edge of Γ_X is direction-reversing with respect to τ.

LEMMA 1.6. *Let k be an algebraically closed field of characteristic $\neq 2$. Let X/k be a prestable curve of genus g and Y/k a connected n-pointed prestable curve Y of genus 0. Let $\pi : X \to Y$ be an admissible double covering. Then all components of X are smooth and X is connected if and only if $n \geqslant 1$ in which case we have $n = 2g + 2$. If, in addition, X is semistable and $g \geqslant 2$, then it is hyperelliptic without direction-reversing double points.*

Proof. It is clear that the components of X are smooth, since by definition of admissible double coverings, X has singular points only when two different components meet. If $n = 0$ then $X \to Y$ is étale and it follows that X is isomorphic to $Y \amalg Y$. Let $n \geqslant 1$. Then, by induction on the number of components of Y, it is easy to see that X is connected of genus g with $2g + 2 = n$. The last statement of the lemma is clear. □

Assume k algebraically closed with $\operatorname{char} k \neq 2$ and let X/k be a hyperelliptic semistable curve of genus $g \geqslant 2$. Denote by τ the canonical involution of its set of double points. Let p be a non-direction-reversing double point. Then the partial normalization of X at $\{p,\ \tau(p)\}$ is the disjoint sum of two connected components X_1, X_2. Let g_i be the genus of X_i $(i = 1, 2)$. There are two cases:

- $\tau(p) \neq p$. Then $g = g_1 + g_2 + 1$ and we call p *to be of type* α_l, where $l := \min\{g_1, g_2\} \in \{0, \ldots, [(g - 1)/2]\}$.
- $\tau(p) = p$. Then $g = g_1 + g_2$ and we call p *to be of type* β_l, where $l := \min\{g_1, g_2\} \in \{1, \ldots, [g/2]\}$.

Direction-reversing points will also be called *to be of type* α'_0. If k is not algebraically closed and $\overline{k}$ an algebraic closure, then all points of $X \otimes \overline{k}$ lying over a fixed singular point p of X are of the same type. We call p to be of the corresponding type.

2. The discriminant of a hyperelliptic curve

Throughout this section, we make the following assumptions: g is an integer $\geqslant 2$. K is a field of characteristic $\neq 2$ with at least $2g+2$ elements. X_K/K is a smooth hyperelliptic curve of genus g such that there exists a finite morphism of degree two from X_K onto the projective line $\mathbb{P}^1_K$.

PROPOSITION 2.1. (1) *The function field F of X_K has generators x and y and defining relation $y^2 = f(x)$, where $f(x) \in K[x]$ is a separable polynomial of degree $2g+2$.*

(2) *If $F = K(\tilde{x}, \tilde{y})$, where $\tilde{y}^2 = \tilde{f}(\tilde{x})$ and $\tilde{f}(\tilde{x}) \in K[\tilde{x}]$ is separable of degree $2g+2$, then there is an invertible matrix*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Gl}_2(K)$$

and an element $e \in K^\times$ such that $x = (a\tilde{x}+b)/(c\tilde{x}+d)$ and $y = e/(c\tilde{x}+d)^{g+1} \cdot \tilde{y}$.

Proof. (1) It is well known (cf. [Art1] Ch. 16.7) that the function field of X_K is $K(x)[y]/(y^2 \Leftrightarrow f(x))$ for some square-free polynomial $f(x) \in K[x]$ of degree $2g+1$ or $2g+2$. If $\deg f(x) = 2g+1$, choose an element $a \in K$ with $f(a) \neq 0$ (here we use the assumption that $\#K \geqslant 2g+2$) and make the transformation $x = 1/(\tilde{x} \Leftrightarrow a) + a$, $y = (\tilde{x} \Leftrightarrow a)^{-(g+1)}\tilde{y}$, to get an equation $\tilde{y}^2 = \tilde{f}(\tilde{x})$, where $\deg \tilde{f}(\tilde{x}) = 2g+2$. Thus we may assume that $f(x)$ has degree $2g+2$. Let $\overline{K}$ be an algebraic closure of K. Since X_K is smooth by hypothesis, the curve $X_{\overline{K}} := X_K \otimes \overline{K}$ is also regular and of genus g and its function field is described by the same equation $y^2 = f(x)$. Therefore, by loc. cit., $f(x)$ is square-free in $\overline{K}[x]$ and thus separable.

(2) It follows from the assumptions that $K(\tilde{x})$ is of index two in F and of genus zero. By loc. cit. we conclude that $K(\tilde{x}) = K(x)$ and therefore that $x = \frac{a\tilde{x}+b}{c\tilde{x}+d}$ for an invertible matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Gl}_2(K).$$

It is easy to see that then we have necessarily $y = (e/(c\tilde{x}+d)^{g+1})\tilde{y}$ for some $e \in K^\times$. $\square$

Recall that the discriminant $\Delta(f)$ of a nonzero polynomial $f(x) \in K[x]$ of degree d is defined as

$$\Delta(f) = A^{2d-2} \cdot \prod_{i \neq j} (a_i \Leftrightarrow a_j) \quad \in K^\times,$$

where $A \in K^\times$ is the leading coefficient and $a_1, \ldots, a_d$ are the zeroes of f in an algebraic closure of K. Let $y^2 = f(x)$ be an equation for X_K as in the above

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proposition. Recall further the well-known fact that the differentials $x^i\,\mathrm{d}x/y$ $(i = 0,\dots,g-1)$ form a basis of $H^0(X_K,\Omega_{X_K/K})$. We define an element $\Lambda = \Lambda_{X_K/K}$ of the one-dimensional vector-space $V = (\bigwedge^g H^0(X_K,\Omega^1_{X_K/K}))^{\otimes(8g+4)}$ by setting

$$\Lambda = D^g\left(\frac{\mathrm{d}x}{y}\wedge\cdots\wedge\frac{x^{g-1}\,\mathrm{d}x}{y}\right)^{\otimes(8g+4)},$$

where $D := 2^{-(4g+4)}\cdot\Delta(f)$. (The reason for the power of two in the definition of D will be given in Section 6.) By the following result, Λ is a canonical element of V

PROPOSITION 2.2. (1) *Λ is independent of the special choice of the equation $y^2 = f(x)$.*

(2) *Let K'/K be an arbitrary field-extension and $X_{K'} := X_K\otimes K'$. Then $\Lambda' = \Lambda_{X_{K'}/K'}$ is the image of Λ under the canonical mapping*

$$V \to V\bigotimes_K K' = (\bigwedge^g H^0(X_{K'},\Omega^1_{X_{K'}/K'}))^{\otimes(8g+4)}.$$

Proof. It is a straightforward calculation to show that Λ is invariant under transformations as described in Proposition 2.1 (2). This proves the first statement. As to the behavior under base change: This follows directly from the construction of Λ. $\square$

3. Statement of the main theorem, first reductions

In this section, R denotes a discrete valuation ring, K its quotient field, v the induced discrete valuation on K (normalized in the sense that $v(K) = \mathbb{Z}$), k the residue field of R. Let X/R be a prestable curve with smooth generic fibre X_K and let p be a singular point of the special fibre X_k of X. The *multiplicity* $\mu(p)$ of p is defined as the length of the R-module $\Sigma(X/R)_p := \mathcal{O}_{X,p}/F_1$, where F_1 denotes the first Fitting ideal of the $\mathcal{O}_{X,p}$-module $\Omega_{\mathcal{O}_{X,p}/R}$. The *relative dualizing sheaf* $\omega_{X/R}$ of X/R is the unique invertible subsheaf of $j_*\Omega^1_{X_K/K}$ (where $j_*\colon X_K\hookrightarrow X$ is the canonical immersion), which coincides with $\Omega^1_{X/R}$ on the smooth part of X/R.

The behaviour of these invariants under base change is as follows: Let R'/R be a discrete valuation ring dominating R and set $X' = X\otimes R'$. Then we have $\Sigma\mu(p_i) = e\mu(p)$, where the sum is over the points of the fibre over p of $X'\to X$ and e is the ramification index of R'/R. The relative dualizing sheaf $\omega_{X'/R'}$ is simply the pull back of $\omega_{X/R}$ under the projection $X'\to X$.

Now assume k to be algebraically closed and let $\tilde{X}/R$ be a second prestable curve and $f\colon\tilde{X}\to X$ a birational projective R-morphism. Then we have $\Sigma\mu(p_i) =$

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$\mu(p)$, where the sum is over the singular points of $\tilde{X}_k$ mapping onto p. Therefore, if $\tilde{X} \to X$ happens to be the minimal desingularization of X, then $\mu(p)$ is simply the number of double points of $\tilde{X}_k$ lying above p (cf. [MB], proof of Theorem 2.4). On the other hand, we have $f_*\omega_{\tilde{X}/R} = \omega_{X/R}$. This follows from [Art2], Corollary (3.4)(ii) and the fact that X has only rational singularities.

Assume char $k \neq 2$ and k not necessarily algebraically closed and that X_K is hyperelliptic and satisfies the assumptions made at the beginning of Section 2. Then we can consider $\Lambda = \Lambda_{X_K/K}$ as a rational section $\Lambda_{X/R}$ of the 'line bundle' $M = (\bigwedge^g H^0(X, \omega_{X/R}))^{\otimes(8g+4)}$ on Spec R and denote by $\mathrm{ord}_s\Lambda$ the order of vanishing of Λ in the closed point $s \in$ Spec R. In other words, we set

$$\mathrm{ord}_s\Lambda = v(a),$$

where $a \in K$, $\Lambda = \Lambda_0 \otimes a \in M \underset{R}{\otimes} K$ and Λ_0 is a generator of M.

After these preliminaries we can formulate our main theorem

THEOREM 3.1. *Let (R, K, k, v) be a discrete valuation ring with $2 \in R^\times$. Let X/R be a semistable curve with smooth hyperelliptic generic fibre X_K of genus $g \geqslant 2$ and assume that there exists a finite morphism of degree two from X_K onto the projective line $\mathbb{P}^1_K$. If X/R is either the minimal regular or the stable model of X_K over R, then the following holds*

(1) *The special fibre X_k of X/R is hyperelliptic in the sense of Section* 1.
(2) *Let τ be the hyperelliptic involution on the set of double points of X_k. Then in a given τ-orbit all double points have the same multiplicity, which we call the multiplicity of that τ-orbit.*
(3) *With the notation introduced at the end of Section* 1, *let*

$$A'_0 = \textit{the number of double points of type } \alpha'_0,$$
$$A_i = \textit{the number of } \tau\textit{-orbits of double points of type } \alpha_i,$$
$$B_j = \textit{the number of double points of type } \beta_j$$

$(i = 0, \ldots, [g \Leftrightarrow 1/2], j = 1, \ldots, [g/2])$, *where we count double points (τ-orbits) with multiplicities. Then we have the equation*

$$\mathrm{ord}_s(\Lambda) = g \cdot A'_0 + 2\textstyle\sum_{i=0}^{[g-1/2]} A_i(g \Leftrightarrow i)(i+1) + 4\sum_{j=1}^{[g/2]} B_j(g \Leftrightarrow j)j.$$

The proof of the theorem will be given in Section 5. In the present section we restrict ourselves to prove some first reductions.

LEMMA 3.2. (a) *It suffices to show the theorem under the additional assumption that the residue field k is algebraically closed.*

(b) *Assume that k is algebraically closed and let X/R be a stable R-curve satisfying the assumptions of the theorem. Denote by $\tilde{X}/R$ the minimal regular*

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model of X. Assume that the conclusions of the theorem hold true for $\tilde{X}/R$ and that $\tilde{X}_k$ has no direction-reversing double points. Then the conclusions of the theorem hold true also for X/R.

(c) *Assume k to be algebraically closed. Let R'/R be a finite extension of discrete valuation rings and let X' be the minimal regular model of $X \otimes R'$. If the conclusions of the theorem hold true for X' then also for X.*

Proof. Part (a) and (c) of the lemma are easy consequences of Proposition 2.2 and the remarks we made at the beginning of this section. To prove part (b), first observe that by assumption, there exists an admissible double covering $\tilde{X}_k \to Y_k$ onto a $(2g+2)$-pointed genus-0-curve Y_k. In what follows, a $\mathbb{P}^1$-chain (of length n) of $\tilde{X}_k$ is a closed connected subscheme Z of $\tilde{X}_k$ that is maximal with the property that all the irreducible components of Z are isomorphic to $\mathbb{P}^1_k$ and meet the rest of $\tilde{X}_k$ in exactly two points. The graph of Z is then linear. It is well known that X_k arises from $\tilde{X}_k$ by contracting all the $\mathbb{P}^1_k$-chains of $\tilde{X}_k$. Let Z be a $\mathbb{P}^1$-chain of $\tilde{X}_k$ and let $Z_1, \dots, Z_n$ be its successive components and $p_1, p_2 \in \tilde{X}_k$ the points where Z meets the rest of $\tilde{X}_k$. Denote by τ the hyperelliptic involution on the set of components and double points of $\tilde{X}_k$. We distinguish three cases

- Type-0-case: $\tau(p_1) = p_2$. We have then $\tau(Z_i) = Z_{n+1-i}$ for $i = 1, \dots, n$ and it follows that $n = 2m+1$ is odd because otherwise the double point $p = Z_{(1/2)n} \cap Z_{(1/2)n+1}$ would be direction-reversing. The middle component Z_{m+1} is the only one of Z which is ramified over Y_k.
- Type-α-case: $\tau(p_1) \notin \{p_1, p_2\}$. Then there is a $\mathbb{P}^1$-chain Z' of $\tilde{X}_k$ which is disjoint to Z and has successive components $Z'_1, \dots, Z'_n$ such that $\tau(Z_i) = Z'_i$. There exists an $l \in \{1, \dots, [g \Leftrightarrow 1/2]\}$ such that all the $n+1$ τ-orbits of double points of $\tilde{X}_k$ contained in $Z \cup Z'$, are of type α_l.
- Type-β-case: $\tau(p_1) = p_1$. Then all components of Z are fixed by τ. There exists an $l \in \{1, \dots, [g/2]\}$ such that all the $n+1$ double points of $\tilde{X}_k$ lying on Z, are of type β_l.

Let $\mathcal{Z}$ be the set of all components of $\tilde{X}_k$ which are isomorphic to $\mathbb{P}^1$ and meet the rest of $\tilde{X}_k$ in exactly two points, except those, which are the middle component of some type-0-chain. Let X'_k be the pointed prestable curve obtained from $\tilde{X}_k$ by contracting all components belonging to $\mathcal{Z}$ and marking one ramification point on each middle component of type-0-chains. On the other hand, let Y'_k be the $(2g+2)$-pointed prestable genus-0-curve which is obtained from Y_k by contracting all images of components belonging to $\mathcal{Z}$. It is clear that τ restricts to a hyperelliptic involution on Γ_{X_k}, that X'_k is the modification of X_k in all of its direction-reversing double points, and that $\tilde{X}_k \to Y_k$ induces an admissible double covering $X'_k \to Y'_k$ carrying marked points into marked points. This proves that X_k is hyperelliptic.

In the type α and type β cases a $\mathbb{P}^1$-chain Z is contracted to a (τ-orbit of) double point(s) of X_k, which is of multiplicity $n+1$ and is of the same type as the (τ-orbits of) double points lying on Z. If Z is of type 0, the $m+1$ τ-orbits of double points contained in Z, are contracted to a double point of X_k, which is of multiplicity

$n+1$ and of type α_0'. It follows that statements (2) and (3) of the theorem also hold for X/R. $\square$

4. Construction of minimal models

Let R, K, k, v be as in the assumptions at the beginning of the last section. Except for Lemma 4.1, we assume k to be algebraically closed. Assume furthermore that $2 \in R^\times$.

LEMMA 4.1. *Let X_K/K be a smooth hyperelliptic curve. There exists a discrete valuation ring R' (with quotient field K', residue field k', and normalized discrete valuation v'), finite over R and dominating R, such that $X_{K'} = X_K \otimes K'$ is a regular proper model of the function field $K'(x, y)$ associated to the equation*

$$y^2 = f(x) := A \cdot \prod_{i=1}^{2g+2} (x \Leftrightarrow a_i),$$

where $A \in (R')^\times, a_i \in R'$ for $i = 1, \ldots, 2g+2$, $a_i \neq a_j$ and $v'(a_i \Leftrightarrow a_j) \in 2\mathbb{N}_0$ for $i \neq j$ and $\#\{\overline{a_i} \,|\, i = 1, \ldots, 2g+2\} \geqslant 3$ ($\overline{a_i} \in k'$ being the residue class of a_i).

Proof. By Proposition 2.1 we may assume (after passing to some finite extension of K) that X_K belongs to an equation $y^2 = A \cdot \Pi_{i=1}^{2g+2}(x \Leftrightarrow a_i)$ for some $A \in K^\times$ and pairwise different $a_i \in K$. After a simple transformation we may even assume $a_i \in R$ for all i and $\#\{\overline{a_i}, \ldots, \overline{a_{2g+2}}\} \geqslant 2$. If the number of different $\overline{a_i}$ equals 2, we can write (after some renumbering): $\overline{a_1} = \cdots = \overline{a_{r-1}} \neq \overline{a_r} = \cdots = \overline{a_{2g+2}}$, where $r \geqslant 3$ and where $m := v(a_1 \Leftrightarrow a_2)$ is minimal among the values $v(a_1 \Leftrightarrow a_i)$ ($1 \leqslant i \leqslant r \Leftrightarrow 1$). Let $t \in R$ be a local parameter. We may assume that k has at least r elements. Therefore we find an element $b \in R$ such that $b \Leftrightarrow (a_i \Leftrightarrow a_1)/t^m$ is invertible in R for $i = 1, \ldots, r \Leftrightarrow 1$. After the transformation $x = t^m(b + 1/\tilde{x}) + a_1, y = (1/\tilde{x})^{g+1}\tilde{y}$ we can make the additional assumption that $\#\{\overline{a_1}, \ldots, \overline{a_{2g+2}}\} \geqslant 3$. Passing to the extension $K(\sqrt{t})/K$ we achieve finally $v(A) \in 2\mathbb{Z}$ and $v(a_i \Leftrightarrow a_j) \in 2\mathbb{N}_0$ for all $i \neq j$. After the transformation $x = \tilde{x}, y = t^{(1/2)v(A)}\tilde{y}$ the equation has all the required properties. $\square$

For the rest of this section let X_K/K be the regular proper model of the function field $K(x, y)$ associated to the equation $y^2 = f(x) := A \cdot \Pi_{i=1}^{2g+2}(x \Leftrightarrow a_i)$, where A and the a_i satisfy the conditions (with respect to v) listed in the above lemma. Our purpose is to give a very explicit description of the minimal regular model X/R of X_K and to show that it is semistable. First we associate a $(2g+2)$-pointed prestable curve Y to the polynomial $f(x) = \Pi_{i=1}^{2g+2}(x \Leftrightarrow a_i)$ as follows: Let $Y_0 := \mathbb{P}_R^1$ and let $P_\infty \subset Y_0$ the closure of the point $(1:0) \in \mathbb{P}_K^1$ in Y_0. The principal divisor on Y_0, defined by $f(x) \in K(Y_0)$ is $\mathrm{div}_{Y_0}(f(x)) = \left(\Sigma_{i=1}^{2g+2} P_i\right) \Leftrightarrow (2g+2)P_\infty$, where

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$P_i \subset Y_0$ is the closure of $(a_i:1) \in \mathbb{P}^1_K$ in Y_0 $(i = 1, \ldots, 2g+2)$. By definition, Y is obtained from Y_0 by successively blowing up closed points of the special fibre, where the P_i meet, until the strict transform of $\Sigma_i P_i$ becomes regular. By abuse of notation, we denote the strict transform of P_i in Y $(i \in \{1, \ldots, 2g+2, \infty\})$ by the same symbol P_i. It is then clear that $(Y, (P_1, \ldots, P_{2g+2}))$ is a $(2g+2)$-pointed prestable curve of genus zero with generic fibre $Y_K = \mathbb{P}^1_K$.

The graph Γ_{Y_k} associated to the special fibre Y_k of Y is a finite tree that can be described directly in terms of the elements $a_i \in R$: Let $\mathfrak{m} \subset R$ be the maximal ideal of R and for $n \in \mathbb{N}_0$ let $r_n \colon \{a_1, \ldots, a_{2g+2}\} \to R/\mathfrak{m}^n$ be the natural mapping that associates to a_i its residue class modulo $\mathfrak{m}^n$. We define a graph T as follows. The vertices of T are the elements of the set $\mathcal{V}(T) = \amalg_{n \geqslant 0} \mathcal{V}_n$, where $\mathcal{V}_n = \{V \in R/\mathfrak{m}^n \mid \# r_n^{-1}(V) \geqslant 2\}$. The set of edges of T consists of pairs (V, V'), where $V \in \mathcal{V}_n$, $V' \in \mathcal{V}_{n+1}$ for some $n \geqslant 0$ and $V' \mapsto V$ under the canonical map $\mathcal{V}_{n+1} \to \mathcal{V}_n$. The coincidence relation c, finally, is defined by $c((V, V')) = \{V, V'\}$. It is easily seen that T is a finite tree and that it is canonically isomorphic to Γ_{Y_k}.[1]

T has a canonical vertex V_0, the unique element of $\mathcal{V}_0$. It corresponds to the irreducible component of Y_k which is the strict transform of the special fibre of Y_0. We define a canonical partial ordering on the set of vertices of T by setting

$$\begin{aligned} V \leqslant V' \colon \Leftrightarrow\ & V \in \mathcal{V}_n, V' \in \mathcal{V}_m, \quad \text{where} \\ & m \geqslant n, \quad \text{and} \quad V' \mapsto V \quad \text{under } \mathcal{V}_m \to \mathcal{V}_n. \end{aligned}$$

The vertex V_0 is the absolute minimum with respect to this partial ordering. It will be convenient to associate to each vertex V of T the following list of numbers

$$\begin{aligned} n(V) &:= n, \quad \text{where } V \in \mathcal{V}_n, \\ \varphi(V) &:= \# r_n^{-1}(V), \\ \Phi(V) &:= \varphi(V) \Leftrightarrow \sum_{\substack{V' \in \mathcal{V}_{n+1} \\ V' \geqslant V}} \varphi(V'), \\ C(V) &:= \begin{cases} 1 & \text{if } n \text{ and } \varphi(V) \text{ are odd} \\ 0 & \text{otherwise} \end{cases}, \\ (f)(V) &:= \sum_{i=1}^{n} \varphi(V_i), \quad \text{where } V_0, V_1, \ldots, V_n = V \\ &\qquad \text{are the vertices of the linear subgraph of } T \\ &\qquad \text{that connects } V_0 \text{ and } V, \\ B(V) &:= \tfrac{1}{2}(C(V) \Leftrightarrow (f)(V)). \end{aligned}$$

If V is an irreducible component of Y_k, we denote by $n(V)$, $\varphi(V)$ etc. the numbers associated to the corresponding vertex of T. For example, $\Phi(V)$ is the number of

[1] Quing Liu has brought to my attention that the construction of this tree already appears in [Bosch].

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sections P_i $(i = 1, \ldots, 2g+2)$, meeting the irreducible component V of Y_k. The geometrical meaning of $B(V)$, $C(V)$ and $(f)(V)$ is revealed by the lemma below.

LEMMA 4.2. (1) *Let (f) denote the principal divisor on Y defined by $f(x) \in K(Y)$. Then we have $(f) = \Sigma_{i=1}^{2g+2}(P_i \Leftrightarrow P_\infty) + \Sigma_{V \subseteq Y_k \text{ irred.}}(f)(V)V$.*

(2) *The divisor $C := \Sigma_{i=1}^{2g+2} P_i + \Sigma_{V \subseteq Y_k \text{ irred.}} C(V)V$ is regular (as a subscheme of Y).*

(3) *$B(V)$ is integral for any V and, denoting by B the divisor $(g+1)P_\infty + \Sigma_V B(V)V$ on Y, we have $2B = C \Leftrightarrow (f)$. In particular, $\mathcal{L} := \mathcal{O}_Y(B)$ is a square root of $\mathcal{O}_Y(C)$ in the Picard group of Y.*

Proof. The statement for the horizontal part of (f) is obvious. Let $V \subseteq Y_k$ be an irreducible component and denote by v_V the corresponding valuation of $K(Y)$. It is easy to see (cf. proof of Lemma 5.1) that $v_V(x \Leftrightarrow a_i) = \min(n(V), v(a \Leftrightarrow a_i))$, where $a \in \{a_1, \ldots, a_{2g+2}\}$ represents the vertex of T that corresponds to V. From this, the statement about the vertical part of (f) follows immediately. Parts (2) and (3) of the lemma are now easy to verify. □

The proposition below describes a well-known construction of ramified double coverings and lists some of its properties.

PROPOSITION 4.3. *Let Y be a regular integral Noetherian separated scheme, C an effective divisor on Y, $\mathcal{L}$ an invertible $\mathcal{O}_Y$-module and F a global section of $\mathcal{L}^2$, whose divisor is C. F induces a morphism $\mathcal{L}^{-1} \otimes \mathcal{L}^{-1} \xrightarrow{\sim} \mathcal{O}_Y(\Leftrightarrow C) \hookrightarrow \mathcal{O}_Y$, which on the sheaf $\mathcal{A} := \mathcal{O}_Y \oplus \mathcal{L}^{-1}$ induces the structure of an $\mathcal{O}_Y$-algebra. Let $X :=$ Spec $\mathcal{A}$. Then the following holds*

(1) *The structure morphism $\pi\colon X \to Y$ is flat and finite of degree 2.*
(2) *If C is regular and all residue fields of Y have characteristic $\neq 2$, then X is regular.*
(3) *Let C be regular and V a connected component of C. Then $\pi^*V = 2W$ for a prime divisor W on X, which as a scheme is mapped isomorphically onto V by π.*
(4) *Let Y' be another regular integral Noetherian separated scheme and $f\colon Y' \to Y$ a morphism transversal to C (i.e. $f(Y') \nsubseteq C$). Let $C' := f^*C$, $\mathcal{L}' := f^*\mathcal{L}$. As above, the section $F' := f^*F$ of $\mathcal{L}'^2$ induces an $\mathcal{O}_{Y'}$-algebra structure on $\mathcal{A}' := \mathcal{O}_{Y'} \oplus \mathcal{L}'^{-1}$. We have* Spec $\mathcal{A}' = X \underset{Y}{\times} Y'$.

Proof. One can restrict to the affine case where the verification is straightforward. □

Returning to our special situation, let Y, C, $\mathcal{L}$ be as in Lemma 4.2 and define $F \in \Gamma(Y, \mathcal{L}^2)$ to be the image of the canonical section of $\mathcal{O}_Y(C)$ under the isomorphism $\mathcal{O}_Y(C) \to \mathcal{L}^{\otimes 2} = \mathcal{O}_Y(2B)$ induced by the relation $C = 2B + (f)$. The construction in the proposition gives us a scheme $X' := \mathrm{Spec}(\mathcal{O}_Y \oplus \mathcal{L}^{-1})$ together with a finite flat morphism $\pi\colon X' \to Y$ of degree two.

PROPOSITION 4.4. *X' is regular and X'/R is a proper flat R-curve whose special fibre is a normal crossing divisor and whose generic fibre is isomorphic to the hyperelliptic curve X_K. Let V be an irreducible component of the special fibre Y_k of Y. We set $W_V := \pi^* V$ and $g_V := 1/2(C.V) \Leftrightarrow 1$. Then g_V is integral and $\geqslant \Leftrightarrow 2$ and*

(1) *$g_V = \Leftrightarrow 2$ if and only if $C(V) = 1$. In this case, $W_V = 2 \cdot W'_V$ for an exceptional divisor W'_V of X' that meets the rest of the special fibre X'_k of X' in exactly two points.*

(2) *If $g_V = \Leftrightarrow 1$, then $W_V = W_{V,1} \amalg W_{V,2}$, where the $W_{V,i}$ are prime divisors isomorphic (as schemes) to $\mathbb{P}^1_k$ meeting the rest of X'_k in at least two points.*

(3) *If $g_V \geqslant 0$, then W_V is a prime divisor that (as a scheme) is a regular k-curve of genus g_V. If $g_V = 0$, then W_V meets the rest of X'_k in at least two points.*

Proof. It follows from Lemma 4.2 and Proposition 4.3, that X' is a regular proper flat R-curve. Using the local description of X' coming from the construction as a double covering of X, it is easy to see that the special fibre is a normal crossing divisor. By 4.3.4. the generic fibre of X' is integral with function field $K(x)[y]/(y^2 \Leftrightarrow f(x))$ and is therefore isomorphic to X_K. Now let V be an irreducible component of Y_k.

If $C(V) = 1$ then $n(V)$ is odd and from the description of the tree T it follows easily that V cuts the rest of Y_k in exactly two points. Therefore $g_V = \frac{1}{2}(V.V) \Leftrightarrow 1 = \Leftrightarrow 2$. By Proposition 4.3.3, $W_V = 2W'_V$ for a prime divisor W'_V which as a scheme is isomorphic to $V \cong \mathbb{P}^1_k$ and meets the rest of X'_k in exactly two points. It follows that $(W'_V.W'_V) = \Leftrightarrow 1$. By Castelnuovo's criterion, W'_V is exceptional.

If $C(V) = 0$, then by Proposition 4.3, $W_V \cong X' \times_Y V$ (as a scheme) and $W_V \to V$ is finite of order two, ramified over exactly $(C.V)$ points of V. If $(C.V) > 0$, then by Hurwitz' formula $(C.V) = 2\cdot(\text{genus of } W_V) + 2$. In particular, g_V is integral.

One checks easily that if V corresponds to an extremal vertex of T, $\Phi(V)$ is at least two. Thus, if $(C.V) = 0$, then V does not correspond to an extremal vertex and cuts the rest of Y_k in at least two points. $\square$

The above proposition shows in particular that the exceptional divisors of X' are exactly the components of X'_k which dominate a component V of Y_k with $C(V) = 1$. Let $X' \to X$ be the blow-down of all exceptional divisors of X'. It is clear now that X is the minimal regular model of X_K and is semistable.

Let Z_k denote the $(2g+2)$-pointed curve that is obtained from Y_k by contracting all components V of Y_k with $C(V) = 1$ (observe that these V do not carry marked points).

COROLLARY 4.5. *The morphism $X_k \to Z_k$ induced by $X'_k \to Y_k$ is an admissible double covering and X_k is hyperelliptic without direction-reversing double points.*

Proof. By Lemma 1.6, it suffices to show the first part of the corollary. It is clear that $X_k \to Z_k$ has properties (1) and (2) of an admissible covering. Let $q \in Z_k$ be a double point and $V_1, V_2 \subset Z_k$ the two components meeting in q. If q is the image of a component V of Y_k under the blow-down map $Y_k \to Z_k$, then both the preimage of V_1 under $X_k \to Z_k$ and that of V_2 is ramified over q. Otherwise, $X_k \to Z_k$ is étale over q. Therefore $X_k \to Z_k$ satisfies also condition (3) of admissible coverings. □

5. Proof of Theorem 3.1

By the Lemmas 3.2 and 4.1, we may assume that k is algebraically closed and that X/R is the minimal regular model of the hyperelliptic curve X_K associated to an equation

$$y^2 = f(x) := A \cdot \prod_{i=1}^{2g+2} (x \Leftrightarrow a_i),$$

where A and the a_i satisfy the conditions listed in Lemma 4.1. We have already seen in Corollary 4.5 that the special fibre X_k is hyperelliptic. Since X is regular, all the double points of X_k have multiplicity one, so the second statement of the theorem is trivially fulfilled and the formula for $\mathrm{ord}_s(\Lambda)$ is all that remains to be shown. For this, some preparation is necessary.

We keep the notation of the previous section. Thus the objects X_K, Y, T, X', X etc., associated with the above equation, are defined. We will employ some abuse of notation to describe vertical divisors on Y, X' and X: If $V \in \mathcal{V}(T)$ is a vertex of T, we denote by the same symbol the corresponding prime divisor on Y. Its pull-back under the projection $\pi\colon X' \to Y$ will be denoted W_V, as in Proposition 4.4. If $C(V) \neq 1$, the image of W_V under the blow-down map $X' \to X$ will again be denoted by W_V. Finally, we will not distinguish between vertical divisors on Y and elements of $\mathbb{Z}^{\mathcal{V}(T)}$.

LEMMA 5.1. *For $F \in \mathcal{V}(T)$ let $D_F \in \mathbb{Z}^{\mathcal{V}(T)}$ be defined by $D_F(V) := n(\inf(F, V))$.*

(1) *For any $b \in R$ there exists an $F \in \mathcal{V}(T)$ such that the vertical part of the principal divisor of $(x \Leftrightarrow b) \in R[x] \subset K(Y)$ is precisely D_F. If, on the other hand, $F \in \mathcal{V}_{n(F)} \subset R/\mathfrak{m}^{n(F)}$ is represented by $b \in \{a_1, \ldots, a_{2g+2}\}$, then* $\mathrm{div}_{\mathrm{vert}}(x \Leftrightarrow b) \geqslant D_F$.

(2) *For any primitive polynomial $h \in R[x]$ of degree r there exist $b_1, \ldots, b_r \in \{a_1, \ldots a_{2g+2}\}$ with*

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$$\mathrm{div}_{\mathrm{vert}}(h) \leqslant \mathrm{div}_{\mathrm{vert}}\left(\prod_{i=1}^{r}(x \Leftrightarrow b_i)\right).$$

Proof. First we will introduce some notation. For a vertex V of T we let v_V be the valuation on $K(Y)$ that corresponds to the vertical prime divisor on Y associated to V. So we have $\mathrm{div}_{\mathrm{vert}}(h) = \Sigma_{V\in\mathcal{V}(T)} v_V(h)\cdot V$ for any $h \in K(Y)$. If $V = V_0$, we will omit the index V from v_V. There will be no confusion arising from this, since for a polynomial $h(x) = \Sigma_{i\geqslant 0} c_i x^i \in K[x] \subset K(Y)$ we have $v_{V_0}(h(x)) = \min_{i\geqslant 0} v(c_i)$.

From the construction of Y it follows that if V is an irreducible component of Y_k, a smooth point of V has an open affine neighborhood U that is R-isomorphic to some open part of $R[\xi]$ such that over U we have the equation $x = t^n\xi + a$, where $t \in R$ is a local parameter, $n = n(V)$, and $a \in \{a_1, \ldots, a_{2g+2}\}$ represents V as a vertex of T. Therefore for any $h(X) \in R[x]$ the equation $v_V(h(x)) = v(h(t^n x + a))$ holds.

We prove the first statement of the lemma. Given $b \in R$, we let $n \geqslant 0$ be maximal such that the image F of b under the mapping $R \to R/\mathfrak{m}^n$ lies in $\mathcal{V}_n$. We have to show that $v_V(x \Leftrightarrow b) = n(\inf(F, V))$ for any vertex V of T. For this, let $a \in \{a_1, \ldots, a_{2g+2}\}$ be a representative of V. By the above remark, we have $v_V(x \Leftrightarrow b) = v(t^{n(V)}x + a \Leftrightarrow b) = \min(n(V), v(a \Leftrightarrow b))$. Therefore the claim follows from the fact that $\inf(F, V) = \max\{V' \leqslant V \mid V' \text{ is represented by } b\}$.

On the other hand, given a vertex F of T let $b \in \{a_1, \ldots, a_{2g+2}\}$ be a representative. Clearly, for any vertex V with representative $a \in \{a_1, \ldots, a_{2g+2}\}$, we have $n(\inf(F, V)) \leqslant \min(n(V), v(a \Leftrightarrow b))$. Therefore $D_F \leqslant \mathrm{div}_{\mathrm{vert}}(x \Leftrightarrow b)$.

The proof of the second statement of the lemma is more involved. We need to introduce further notation.

If V and V' are two vertices of T with $n(V) = n(V') \Leftrightarrow 1$ and $V < V'$, then we will say that V is a *predecessor* of V' and that V' is a *successor* of V. Each vertex except V_0 has exactly one predecessor. The set of successors of V_0 is precisely $\mathcal{V}_1$. Let $h \in K(Y)$ and $V \neq V_0$ a vertex of T. Then we denote by $\Delta v_V(h)$ the difference $v_V(h) \Leftrightarrow v_{V'}(h)$, where V' is the predecessor of V.

I claim that for any primitive polynomial $h \in R[x]$ and any vertex V of T we have

$$\sum_{V'\in S(V)} \Delta v_{V'}(h) \leqslant \begin{cases} \deg(h) & \text{if } V = V_0 \\ \Delta v_V(h) & \text{otherwise,} \end{cases} \tag{$*$}$$

where $S(V)$ denotes the set of successors of V.

Admitting the claim for a moment, we conclude that given a primitive $h \in R[x]$ of degree r, there exists a family $(I_V)_{V\in\mathcal{V}(T)}$ of subsets $I_V \subset \{1, \ldots, r\}$ with

(i) $I_{V_0} = \{1, \ldots, r\}$.

(ii) $I_V \supseteq I_{V'}$ for $V \leqslant V'$.
(iii) $I_V = \dot{\bigcup}_{V' \in S(V)} I_{V'}$ for all $V \in \mathcal{V}(T)$ with $S(V) \neq \emptyset$.
(iv) $\# I_V \geqslant \Delta v_V(h)$ for all $V > V_0$.

From properties (i)–(iii) it follows that

$$\dot{\bigcup_{\substack{V \in \mathcal{V}(T) \\ V \text{ maximal}}}} I_V = \{1, \ldots, r\}.$$

We choose a representative $a_V \in \{a_1, \ldots, a_{2g+2}\}$ and set $b_i = a_V$ for all maximal vertices V of T and all $i \in I_V$.

It is easy to see then that for arbitrary $V > V_0$ we have $\Delta v_V(\Pi_{i=1}^r (x \Leftrightarrow b_i)) = \# I_V \geqslant \Delta v_V(h)$. Since $v_V = \Sigma_{j=1}^n \Delta v_{V_j}$, where $V_0 < V_1 < \cdots < V_n = V$ are successive vertices of the linear subgraph of T connecting V_0 and V, it follows that

$$v_V \left(\prod_{i=1}^r (x \Leftrightarrow b_i) \right) \geqslant v_V(h).$$

This proves the second statement of the lemma.

It remains the task to prove the claim $(*)$. Assume first that $V \neq V_0$. Let $a \in \{a_1, \ldots, a_{2g+2}\}$ a representative for V and $n = n(V)$. Let V' be the predecessor of V and set for abbreviation $m := v_V(h)$, $m' := v_{V'}(h)$. By the remark we made at the beginning of the proof, there are primitive polynomials $h_{V'}, h_V \in R[x]$ such that

$$h(t^{n-1}x + a) = t^{m'} h_{V'}(x),$$
$$h(t^n x + a) = t^m h_V(x).$$

Since obviously $t^{m'} h(tx) = t^m h_V(x)$, we have $\Delta v_V(h) = m \Leftrightarrow m' = v(h_{V'}(tx))$. Let $\mu \geqslant 0$ be maximal such that $h_{V'}(x) \in (t, x)^\mu$. Then $h_{V'}(x) = \Sigma_{i=0}^\mu k_i(x) t^i x^{\mu - i}$ for some polynomials $k_i(x) \in R[x]$ such that $k_{i_0}(x) \notin (x, t)$ for at least one $i_0 \in \{0, \ldots, \mu\}$. It follows that $h_{V'}(tx) = t^\mu \Sigma_{i=0}^\mu k_i(tx) x^{\mu - i}$ and that $\Sigma_{i=0}^\mu k_i(tx) x^{\mu - i}$ is primitive. Therefore $\Delta v_V(h) = \mu$ and $h_v(x) = \Sigma_{i=0}^\mu k_i(tx) x^{\mu - i}$. In particular, we have

$$\deg \overline{h}_V(x) \leqslant \Delta v_V(h), \tag{a}$$

where $\overline{h}(x) \in k[x]$ denotes the residue class of $h_V(x)$ modulo $tR[x]$.

Now let V be an arbitrary vertex of T and let $V_1, \ldots, V_s$ be the successors of V. For $i = 1, \ldots, s$ we choose elements $b_i \in \{a_1, \ldots, a_{2g+2}\}$ representing V_i and set $c_i := (b_i \Leftrightarrow a)/t^n$. Then $c_i \in R$ and $c_i \neq c_j \bmod \mathfrak{m}$ for $i \neq j$. Similarly as above, we have for $i = 1, \ldots, s$ the equations $\mu_i := \Delta v_{V_i}(h(x)) = v(h_V(tx + c_i))$ and

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$h_V(x) = \Sigma_{q=0}^{\mu_i} u_q^{(i)} t^q (x \Leftrightarrow c_i)^{\mu_i - q}$ for certain $u_q^{(i)} \in R[x]$. Therefore the residue class $\overline{c}_i$ of c_i in k is an at least μ_i-fold zero of $\overline{h}_V(x)$ and it follows

$$\sum_{i=1}^{s} \Delta v_{V_i}(h) \leqslant \deg \overline{h}_V(x). \tag{b}$$

Clearly, (a) and (b) together imply the claim.

LEMMA 5.2. *Define* $\mathcal{K} \in \mathbb{Z}^{\mathcal{V}(T)}$ *by* $\mathcal{K}(V) := B(V) + n(V)$. *We consider the global differential* $\mathrm{d}x/y$ *on* X_K *as a rational section of the dualizing sheaf* $\omega_{X/R}$. *Its divisor on* X *is*

$$\operatorname{div}\left(\frac{\mathrm{d}x}{y}\right) = (g \Leftrightarrow 1)\pi^* P_\infty + \sum_{\substack{V \in \mathcal{V}(T) \\ C(V) \neq 1}} \mathcal{K}(V) \cdot W_V.$$

Proof. Let $t \in R$ be a local parameter. Let $V \in \mathcal{V}_n$ be a vertex of T with $C(V) \neq 1$ and let $a \in \{a_1, \dots, a_{2g+2}\}$ be a representative for V. Let q be a smooth point of Y_k that lies on the irreducible component which corresponds to V. By construction of Y, there is an open affine neighborhood U of q which is R-isomorphic to an open part of Spec $R[\xi]$ such that in $A = \Gamma(U, \mathcal{O}_Y)$ we have the equation $x = t^n \xi + a$. By construction of X', we have

$$\Gamma(\pi^{-1}U, \mathcal{O}_{X'}) = \Gamma(U, \mathcal{O}_Y \oplus \mathcal{L}^{-1}) \simeq A[\eta]/(\eta^2 \Leftrightarrow h),$$

where $h \in R[\xi]$ is the primitive polynomial $t^{-(f)(V)} f(t^n \xi + a)$ and where over $U' = \pi^{-1}(U)$ we have the equation $y = t^{-B(V)}\eta$. It follows that $\mathrm{d}x/y = t^{B(V)+n(V)}\, \mathrm{d}\xi/\eta$. By choosing U small enough we may assume that A is generated by h and h' and we conclude (similarly as in the proof of Proposition 6.2 of Section 6) that $\mathrm{d}\xi/\eta$ is an invertible section of $\omega_{X'/R}|_{U'} = \Omega^1_{U'/R}$. The statement for the vertical part of div$(\mathrm{d}x/y)$ follows. For the horizontal part it is enough to look at the restriction of $\mathrm{d}x/y$ to the generic fibre of X/R. But there the result is classical (and easy to see). $\square$

Lemmas 5.3 and 5.4 below are of purely combinatorial nature. Nevertheless, Lemma 5.4 is the key to our proof of the theorem.

LEMMA 5.3. *Let I be a finite index-set. For any $i \in I$ let $f_i \colon \mathbb{N}_0 \to \mathbb{Z}$ be increasing mappings.*

(1) *There is a sequence $(x^n)_{n \in \mathbb{N}_0}$ of I-tupels $(x_i^n)_{i \in \mathbb{N}} \in \mathbb{N}_0^I$ with the properties*
   - (a) *for any $i \in I$ and any $n \in \mathbb{N}_0, x_i^n \leqslant x_i^{n+1}$;*
   - (b) *for any $n \in \mathbb{N}_0$ we have $\Sigma_{i \in I} x_i^n = n$;*

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(c) $$\min_{i\in I} f_i(x_i^n) = \max\left\{\min_{i\in I} f_i(y_i) \mid y_i \in \mathbb{N}_0, \sum_{i\in I} y_i = n\right\}.$$

(2) *If the f_i are strictly increasing and a sequence $((x_i^n)_{i\in I})_{n\in\mathbb{N}_0}$ in $\mathbb{N}_0^I$ satisfies the properties* (a)–(c) *of* (1), *then for any $n \in \mathbb{N}_0$ we have*

$$f_{i_n}(x_{i_n}^n) = \min_{i\in I} f_i(x_i^n),$$

where $i_n \in I$ is the unique index with $x_{i_n}^{n+1} = x_{i_n}^n + 1$.

The proof of Lemma 5.3 is straightforward, so we omit it. In the next lemma, for $D \in \mathbb{Z}^{\mathcal{V}(T)}$, we denote by $w(D)$ the number $\min_{V\in\mathcal{V}(T)} D(V)$.

LEMMA 5.4. *There exists a sequence $(F_i)_{i\geqslant 1}$ in $\mathcal{V}(T)$ with the following properties*

(1) *For any $m \in \mathbb{N}_0, w(\mathcal{K} + \Sigma_{i=1}^m D_{F_i})$ is maximal among the numbers $w(\mathcal{K} + \Sigma_{i=1}^m D_{G_i})$, where $G_1, \ldots, G_m$ runs through all m-tupels of elements of $\mathcal{V}(T)$.*
(2) *For $V \in \mathcal{V}(T)$ set*

$$\gamma(V) := \begin{cases} \varphi(V)/2 \Leftrightarrow 1, & \text{if } \varphi(V) \text{ is even,} \\ (\varphi(V) \Leftrightarrow 1)/2, & \text{otherwise.} \end{cases}$$

Then for all $V \neq V_0, \gamma(V)$ is the number of indices m with $F_m \geqslant V$. For $i \geqslant \gamma(V_0)$ we have $F_i = V_0$.
(3) *Let $m \in \mathbb{N}$. The mapping $\mathcal{K} + \Sigma_{i=1}^{m-1} D_{F_i} \in \mathbb{Z}^{\mathcal{V}(T)}$ takes its minimal value $w(\mathcal{K} + \Sigma_{i=1}^{m-1} D_{F_i})$ at the vertex F_m.*
(4) *For any $m \in \mathbb{N}$, the number $n(F_m)$ is even.*

Proof. By induction on $N = \max\{v(a_i \Leftrightarrow a_j) \mid i \neq j\}$. The case $N = 0$ is trivial. Let $V_1, \ldots, V_r$ be the elements of $\mathcal{V}_2$. For $i = 1, \ldots, r$ we denote by T^i the full subtree of T, whose vertices are greater or equal to V_i. T^i is isomorphic to the tree associated to the polynomial $\Pi_j (x \Leftrightarrow (a_j \Leftrightarrow b_i)/t^2) \in R[x]$, where t is a local parameter, $b_i \in \{a_1, \ldots, a_{2g+2}\}$ represents $V_i \in R/t^2R$ and where the index j runs through all values for which $a_j \equiv b_i \bmod t^2$. Let n^i, φ^i, $\mathcal{K}^i$ etc. be the objects associated to T^i, analogous to n, φ, $\mathcal{K}$ etc. For example, we have

$$\begin{aligned} n^i(V) &= n(V) \Leftrightarrow 2, \\ \mathcal{K}^i(V) &= \mathcal{K}(V) + \varphi(V_i) \Leftrightarrow 2, \\ D_F^i(V) &= D_F(V) \Leftrightarrow 2 \end{aligned}$$

for $V, F \in \mathcal{V}(T^i) \subset \mathcal{V}(T)$.

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By induction hypothesis, there are sequences $(F^i_j)_{j\in\mathbb{N}}$ in $\mathcal{V}(T^i)$ that have the properties 1–4 of the lemma. We define functions $f_i\colon \mathbb{N}_0 \to \mathbb{Z}$ by setting

$$f_i(x) = w^i\left(\mathcal{K} + \sum_{j=1}^{x} D_{F^i_j}\right) = \min_{V\in\mathcal{V}(T^i)}\left(\mathcal{K}(V) + \sum_{j=1}^{x} D_{F^i_j}(V)\right).$$

The f_i are strictly increasing ($f_i(x+1) \geqslant f_i(x)+2$) and we can apply Lemma 5.3 to obtain a sequence $(i_m)_{m\in\mathbb{N}_0}$ of elements in $I := \{1,\ldots,r\}$ with

$$f_{i_m}(x^m_{i_m}) = \min_{i\in I} f_i(x^m_i) \quad \text{for all } m \in \mathbb{N}_0,$$

where $x^m_i = \#\{0 \leqslant k \leqslant m \Leftrightarrow 1 \mid i_k = i\}$. Now we define for $m \geqslant 0$

$$F_{m+1} := \begin{cases} F^{i_m}_{x^m_{i_m}+1}, & \text{if } f_{i_m}(x^{i_m}_m) < 0,\\ V_0, & \text{otherwise.}\end{cases}$$

I contend that this sequence has the required properties.

For abbreviation, set $D_m := \mathcal{K} + \Sigma^m_{j=1} D_{F_j}$ and $n_0 = \Sigma_{i\in I}\gamma(V_i)$. Let V'_i be the unique element between V_0 and V_i. It is not difficult to see that the following statements hold true:

(i) $F_m = V_0$ if and only if $m \geqslant n_0 + 1$.
(ii) For all $m \leqslant n_0$ and $i = 1,\ldots,r$, we have $x^m_i = \#\{1 \leqslant j \leqslant m \mid F_j \geqslant V_i\}$.
(iii) For all $i = 1,\ldots,r$ we have $\gamma(V_i) = \#\{m \in \mathbb{N}_0 \mid F_m \geqslant V_i\}$.
(iv) If $m \leqslant n_0 \Leftrightarrow 1$, we have $D_m(V'_i) \geqslant w^i(D_m)$ for all i and $0 \geqslant D_m(V'_i) > w^i(D_m)$ for at least one i.
(v) If $m \geqslant n_0$ then for all i we have $0 \leqslant D_m(V'_i) \leqslant w^i(D_m)$.

Using these statements, properties (1)–(4) for the sequence (F_m) are easy to derive. □

With the help of the F_i we can now construct a basis of $H^0(X, \omega_{W/R})$.

PROPOSITION 5.5. *Let $(F_i)_{i\geqslant 1}$ be a sequence in $\mathcal{V}(T)$ satisfying the properties listed in Lemma* 5.4. *For every $i \in \mathbb{N}$ let $b_i \in \{a_1,\ldots,a_{2g+2}\}$ be an element representing the class $F_i \in \mathcal{V}_{n(F_i)} \subset R/\mathfrak{m}^{n(F_i)}$. Let $t \in \mathfrak{m}$ denote a prime element of R and set for $0 \leqslant i \leqslant g \Leftrightarrow 1$*

$$\omega_i := t^{e_i}\cdot\left(\prod_{j=1}^{i}(x \Leftrightarrow b_j)\right)\frac{\mathrm{d}x}{y} \in H^0(X_K, \Omega_{X_K/K}),$$

where $e_i := \Leftrightarrow w(\mathcal{K} + \Sigma^i_{j=1} D_{F_j})$ and w is defined as before Lemma 5.4. *Then*

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(1) $\omega_0, \ldots, \omega_{g-1}$ is an R-basis of $H^0(X, \omega_{X/R})$

(2) $$\sum_{i=0}^{g-1} e_i = \frac{1}{2} \sum_{\substack{V > V_0 \\ \varphi(V) \text{ even}}} \gamma(V)(\gamma(V)+1) + \frac{1}{2} \sum_{\substack{V > V_0 \\ \varphi(V) \text{ odd}}} \gamma(V)^2,$$

where $\gamma(V)$ *is defined as in Lemma* 5.4.

Proof. (1) By Lemmas 5.1 and 5.2 we have $\operatorname{div}(\omega_i) \geqslant 0$ and therefore $\omega_i \in H^0(X, \omega_{X/R})$ for $i = 0, \ldots, g \Leftrightarrow 1$. Since the ω_i form a K-Basis of $H^0(X_K, \Omega^1_{X_K/K})$, they are linearly independent. We show that they span $H^0(X, \omega_{X/R})$. For this let $\omega = t^e h(x)(\mathrm{d}x/y)$ (where $h(x) \in R[x]$ is primitive of degree $d \leqslant g \Leftrightarrow 1$) be an arbitrary element of $H^0(X, \omega_{X/R})$. By Lemma 5.1 and by construction of the F_i, we have $w(\mathcal{K} + \operatorname{div}_{\text{vert}}(h)) \leqslant \Leftrightarrow e_d$. Since $\operatorname{div}(\omega) \geqslant 0$, it follows $e_d \leqslant e$ and therefore there exists an element $c \in R$, such that $\omega \Leftrightarrow c \cdot \omega_d = h_1(x) \cdot \mathrm{d}x/y$ for a polynomial $h(x) \in R[x]$ of degree at most $d \Leftrightarrow 1$. We proceed by induction on d.

(2) By property 3 of the F_i, we have

$$e_i = \Leftrightarrow \mathcal{K}(F_{i+1}) \Leftrightarrow \sum_{j=1}^{i} D_{F_i}(F_{i+1}).$$

Since furthermore $F_i = V_0$ for $i \geqslant \gamma(V_0) = g$ and $\mathcal{K}(V_0) = D_{F_j}(V_0) = 0$, it follows

$$\sum_{i=0}^{g-1} e_i = \sum_{i \geqslant 1} \mathcal{K}(F_i) \Leftrightarrow \sum_{i \geqslant 1} \sum_{j=1}^{i-1} D_{F_j}(F_i). \tag{$*$}$$

For the first term of this equation, we have

$$\sum_{i \geqslant 1} \mathcal{K}(F_i) = \sum_{\substack{V \in \mathcal{V}(T) \\ n(V) \in 2\mathbb{N}}} \gamma(V)(2 \Leftrightarrow \varphi(V)).$$

This follows from the definition of $\mathcal{K}$ and property 2 of the F_i. Since $D_{F_i}(F_j) = D_{F_j}(F_i) = 2 \cdot \#\{V \in \mathcal{V}(T) \mid n(V) \in 2\mathbb{N}, V \leqslant F_i,\ V \leqslant F_j\}$, we have for the second term

$$\sum_{i \geqslant 1} \sum_{j=1}^{i-1} D_{F_j}(F_i) = \sum_{\substack{V \in \mathcal{V}(T) \\ n(V) \in 2\mathbb{N}}} \gamma(V)(\gamma(V) \Leftrightarrow 1),$$

where we have again made use of property 2 of the F_i. Plugging in these expressions into $(*)$, we get the required formula. □

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Let $D := 2^{-(4g+4)} \cdot A^{(4g+2)} \cdot \Pi_{i \neq j}(a_i \Leftrightarrow a_j)$ and $\omega_0, \ldots, \omega_{g-1}$ as in the above proposition. Then obviously

$$\Lambda_{X/R} = D^g \cdot (t^{-e}(\omega_0 \wedge \cdots \wedge \omega_{g-1}))^{(8g+4)},$$

where $t \in R$ is a prime element and $e = \Sigma_{i=0}^{g-1} e_i$. Therefore we have

$$\mathrm{ord}_s \Lambda_{X/R} = g \cdot v(D) \Leftrightarrow (8g+4) \cdot \sum_{i=0}^{g-1} e_i. \tag{1}$$

It is easily seen that

$$v(D) = \sum_{\substack{V > V_0 \\ \varphi(V) \text{ even}}} (2\gamma(V)+2)(2\gamma(V)+1) + \sum_{\substack{V > V_0 \\ \varphi(V) \text{ odd}}} (2\gamma(V)+1)2\gamma(V).$$

Plugging in this and the formula of Proposition 5.5.2 into (1), we get

$$\begin{aligned} \mathrm{ord}_s \Lambda_{X/R} \;=\; & 2 \sum_{\substack{V > V_0 \\ \varphi(V) \text{ even}}} (\gamma(V)+1)(g \Leftrightarrow \gamma(V)) \\ & +2 \sum_{\substack{V > V_0 \\ \varphi(V) \text{ odd}}} \gamma(V)(g \Leftrightarrow \gamma(V)). \end{aligned} \tag{2}$$

There is an obvious one-to-one correspondence between vertices $> V_0$ of T and its edges (to $V \in V_n$ $(n \geqslant 1)$ there corresponds the edge that connects V' and V, where V' is characterized by $V < V', V \in \mathcal{V}_{n-1}$), and therefore also a bijection $V \mapsto p_V$ from $\mathcal{V}(T) \Leftrightarrow \{V_0\}$ to the set of double points of Y_k. We set $\varphi(p_V) := \varphi(V)$ and $\gamma(p_V) := \gamma(V)$. Let Z_k be the $(2g+2)$-pointed prestable genus-0 curve obtained from Y_k by contracting all components V with $C(V) = 1$ of Y_k (cf. end of Section 4) and let r: $Y_k \to Z_k$ be the contraction map. Observe that for two double points p_1, p_2 of Y_k with $r(p_1) = r(p_2)$, we have $\varphi(p_1) = \varphi(p_2)$, so that φ and γ are defined on the set of double points of Z_k. Observe furthermore that for a double point q of Z_k the number of double points of Y_k lying above q is one, if $\varphi(q)$ is even and it is two, if $\varphi(q)$ is odd. So we can reformulate (2) as

$$\begin{aligned} \mathrm{ord}_s(\Lambda_{X/R}) \;=\; & 2 \sum_{\substack{q \in Z_k \text{ double point} \\ \varphi(q) \text{ even}}} (\gamma(q)+1)(g \Leftrightarrow \gamma(q)) \\ & +4 \sum_{\substack{q \in Z_k \text{ double point} \\ \varphi(q) \text{ odd}}} \gamma(q)(g \Leftrightarrow \gamma(q)). \end{aligned} \tag{3}$$

Let q be a double point of Z_k and let $Z_1, Z_2 \subset Z_k$ be the closed subsets of Z_k whose union is Z_k and which intersect precisely in $\{q\}$. Then one of these two subsets carries exactly $\varphi(q)$ marked points of Z_k, while the other carries $(2g+2) \Leftrightarrow \varphi(q)$ of these. The admissible double covering $X_k \to Z_k$ is ramified over q if and only if $\varphi(q)$ is odd. It follows from Lemma 1.6 that in this case the double points of X_k lying above q are of type β_l, where $l = \min\{\gamma(q), g \Leftrightarrow \gamma(q)\}$, whereas if $\varphi(q)$ is even, they are of type α_l, where $l = \min\{\gamma(q), g \Leftrightarrow 1 \Leftrightarrow \gamma(q)\}$. After these considerations it is clear that (3) implies the formula of the theorem.

Remark. From the above proof it is easy to see that there are hyperelliptic curves X/K, such that $\mathrm{ord}_s(\mathrm{d}x/y \wedge \cdots \wedge x^{g-1}\,\mathrm{d}x/y) < 0$ for any choice of equation $y^2 = f(x)$. This is in opposition to the case of elliptic curves, where $\mathrm{ord}_s(\mathrm{d}x/y) = 0$ if and only if the corresponding equation is minimal. Still one could think of calling an equation $y^2 = f(x)$ *minimal*, if $\mathrm{ord}_s\,\Delta(f)$ is minimal – or, what is the same, if $\mathrm{ord}_s(\mathrm{d}x/y \wedge \cdots \wedge x^{g-1}\,\mathrm{d}x/y)$ is maximal (cf. [Lock], where such an attempt has been made). But unlike in the case of elliptic curves, we will in general have $\mathrm{ord}_s\,\Delta_{\min} > (1/g)\,\mathrm{ord}_s\,\Lambda_{X/K}$, where $\Delta_{\min} = \Delta(f)$ for a minimal equation $y^2 = f(x)$ for the hyperelliptic curve X/K.

6. Good reduction and characteristic two

There should be an analogue of Theorem 3.1 in characteristic two. In this section, we prove this analogue only in the case of good reduction and generic characteristic $\neq 2$. More precisely, let (R, K, k, v) be a discrete valuation ring where $\mathrm{char}\,K \neq 2$ and k is perfect and let X/R be a smooth proper R-curve of genus g with hyperelliptic generic fibre X_K. Let $\Lambda \in M_K := (\wedge^g H^0(X_K, \Omega^1_{X_K/K}))^{\otimes(8g+4)}$ be defined as in Section 2. As in Section 3, $M := (\wedge^g H^0(X, \Omega^1_{X/R}))^{\otimes(8g+4)}$ defines an integral structure on M_K and it makes sense to speak of the order of vanishing $\mathrm{ord}_s\Lambda$ of Λ at the closed point $s \in \mathrm{Spec}\,R$. Our objective is to prove that regardless of $\mathrm{char}\,k$ we have $\mathrm{ord}_s\Lambda = 0$, thereby justifying the power of 2 involved in the definition of Λ. It is clear that it is enough to show this after a base extension R'/R, where R'/R is a discrete valuation ring dominating R. So the statement will follow from Lemma 6.1 and Propositions 6.2 and 6.3 below.

LEMMA 6.1. *After some base-change with a discrete valuation ring dominating R, there exists an open affine subset $U \subset X$ which is isomorphic to the* Spec *of*

$$B = A[y]/(y^2 + ay + b),$$

for some $a, b \in A := R[x]$ such that the polynomial $f(x) = a(x)^2 \Leftrightarrow 4b(x) \in K[x]$ is separable and of degree $2g+2$ and such that $\deg a(x) \leqslant g+1$, $\deg b(x) \leqslant 2g+2$. For the reduced polynomials $\overline{a}(x), \overline{b}(x) \in k[x]$ we have $\deg \overline{a}(x) = g+1$ or $\deg \overline{b}(x) \geqslant 2g+1$.

Proof. By [L-K] (cf. proof of their Proposition 5.14 and Theorem 4.12), there exists a faithfully flat finite $R =$ morphism $X \xrightarrow{\pi} Y$ of degree two onto a twisted $\mathbb{P}^1$ over Spec R. Passing, if necessary, to an étale surjective extension R'/R, we may assume that Y is isomorphic to $\mathbb{P}^1_R$ ([L-K] Corollary 3.4). We may choose this isomorphism such that $X_K \to \mathbb{P}^1_K$ is unramified over the point at infinity $P_{\infty,K}$ of $\mathbb{P}^1_K$. Let $U' := \mathbb{P}^1_R \Leftrightarrow P_\infty =$ Spec $R[x]$, where P_∞ is the closure of $P_{\infty,K}$ in $\mathbb{P}^1_R$. Then $U := \pi^{-1}(U') =$ Spec B for a finite flat $R[x]$-algebra B of degree two. By [Sesh], B is free as an $R[x]$-module and it follows

$$B \cong A[y]/(y^2 + ay + b), \tag{$*$}$$

for some $a, b \in A := R[x]$. We chose a representation $(*)$ such that a has minimal degree. Then it is easy to see that deg$(a) \leqslant g + 1$ and deg$(b) \leqslant 2g + 2$. Regarding y as an element of $k(X_k)$, we have div$(y) \geqslant \min\{\deg \overline{a}, \frac{1}{2}\deg \overline{b}\} \cdot \pi^*\mathfrak{p}_\infty$, where $\mathfrak{p}_\infty$ is the place at infinity of $Y_k = \mathbb{P}^1_k$. Since, by Riemann–Roch, for any $n \leqslant g$ the k-vectorspace of functions $\varphi \in k(X_k)$ with div $\varphi \geqslant \Leftrightarrow n \cdot \pi^*\mathfrak{p}_\infty$ is contained in the span of $1, \ldots, x^g$, the last statement of the lemma follows. $\square$

PROPOSITION 6.2. *In the situation of Lemma* 6.1, *the differential* $\mathrm{d}x/(2y + a)$ *is nowhere vanishing on* U *and the differentials* $x^i\,\mathrm{d}x/(2y + a)$ $(i = 0, \ldots, g \Leftrightarrow 1)$ *extend to regular global sections of* $\Omega^1_{X/R}$.

Proof. It is easy to see that the morphism φ: $\Omega^1_{B/R} = (B\,\mathrm{d}x \oplus B\,\mathrm{d}y)/(F_x\,\mathrm{d}x + F_y\,\mathrm{d}y) \to B$, defined by $\varphi(\mathrm{d}x) = F_y$, $\varphi(\mathrm{d}y) = \Leftrightarrow F_x$, is an isomorphism of B-modules This proves the first part of the proposition. As to the second part, it suffices to check that the restrictions of the differentials $x^i\,\mathrm{d}x/(2y + a)$ on the generic fibre U_K extend to regular global sections of $\Omega^1_{X_K/K}$. But this is well-known. $\square$

PROPOSITION 6.3. *Let* $f(x) \in R[x]$ *as in Lemma* 6.1 *and set* $D := 2^{-(4g+4)} \cdot \Delta(f)$, *where* $\Delta(f)$ *is the discriminant of* f *(cf. Section* 2*). Then* D *is a unit in* R.

Proof. If char$(k) \neq 2$, this is clear. So assume char$(k) = 2$. In what follows, if C is any ring and $P = \Sigma_{i=0}^n u_i T^i$, $Q = \Sigma_{i=0}^m v_i T^i$ are two polynomials in $C[T]$, we denote by $R_T^{n,m}(P, Q) \in C$ the resultant of P and Q (c.f. [vdW] Sections 34, 35). Let $F(x, y) := y^2 + a(x)y + b(x)$, $Q := R_y^{2,1}(F, F_x)$, $P := R_y^{2,1}(F, F_y)$ and $A \in R$ the leading coefficient of P. Then we have

$$R_x^{2g+2,4g+2}(P, Q) = (A \cdot D)^2.$$

We can read this equation as a formal identity between polynomials in the coefficients of $a(x)$ and $b(x)$ and conclude that $D \in R$ and $A^2 \mid R_x^{2g+2,4g+2}(P, Q)$. Let $R[x] \to k[x]$, $h \mapsto \overline{h}$ be the residue homomorphism. If $\overline{A} \neq 0$, then we have $\deg \overline{P} = 2g + 2$ and $\overline{D}^2 = \overline{A}^{-2} \cdot R_x^{2g+2,4g+2}(\overline{P}, \overline{Q})$. If to the contrary $\overline{A} = 0$, then it is easy to see that we may assume $\deg \overline{P} \leqslant 2g$ and $\deg \overline{Q} = 4g$ and that we have $\overline{D}^2 = R_x^{2g,4g}(\overline{P}, \overline{Q})$. Thus in any case $\overline{D}^2 \neq 0$ by smoothness of X_k. $\square$

7. Asymptotic of metrics

Let R be the ring of integers in a number field and let X/R be a semistable curve with smooth hyperelliptic generic fibre of genus $g \geqslant 2$. Assume that X/R is the minimal regular model of its generic fibre. Let s be a closed point of Spec R. Denote by $\delta_s(X/R)$ the number of singular points in the geometric fibre of X/R over s. By an elementary estimate of the coefficients of A'_0, A_i, B_j in the expression for $\mathrm{ord}_s(\Lambda_{X/R})$ in 3.1, we deduce the inequality

$$\mathrm{ord}_s(\Lambda_{X/R}) \leqslant g^2 \cdot \delta_s(X/R). \tag{1}$$

The purpose of this section is to globalize this inequality in the sense of Arakelov-geometry (Theorem 7.7) and to deduce from this an upper bound for the self-intersection of the relative dualizing sheaf on X/R (Corollary 7.8). To this end, we will establish an analogue of (1) at the infinite place (Theorem 7.1). We start with some definitions and notations.

Let $g \geqslant 2$ be an integer. Let $\pi\colon X \to S$ be an analytic stable curve of genus g. Then we denote by $\omega_{X/S}$ (or by ω_π) the relative dualizing sheaf of X/S. If $\pi\colon X \to S$ is generically smooth and hyperelliptic, we denote by $\Lambda_{X/S}$ the canonical section of $(\det \pi_*\omega_{X/S})^{\otimes(8g+4)}$, defined as in Section 2.

A *$\mathcal{C}^\infty$-metric* $\|\ \|_{\mathrm{Mod}}$ *on* $\omega_{\overline{\mathcal{X}}_g/\overline{\mathcal{M}}_g}$ is a rule that to each analytic stable curve X/S of genus g over a complex manifold S associates a continuous metric $\|\ \|$ on the line bundle $\omega_{X/S}$, such that the rule is compatible with any basechange and such that $\|\ \|$ is $\mathcal{C}^\infty$, if X/S happens to be the universal local deformation of a stable curve. It is easy to see that $\mathcal{C}^\infty$-metrics on $\omega_{\overline{\mathcal{X}}_g/\overline{\mathcal{M}}_g}$ exist (cf. [Bost3], 4.2.4, p. 249). It would be desirable to have a *canonical* choice of such a metric, but I do not know of any such construction. Because of its bad behavior in the neighborhood of degenerate fibres (cf. [Jor], [Wen]), the Arakelov-metric (cf. [Ara2], p. 1178 or [Sou], p. 338) unfortunately cannot be extended to give a $\mathcal{C}^\infty$-metric on $\omega_{\overline{\mathcal{X}}_g/\overline{\mathcal{M}}_g}$.

Let $C/\mathbb{C}$ be a smooth proper curve of genus g over $\mathbb{C}$. The hermitian scalar product $\langle s_1, s_2\rangle = (i/2)\int_{C(\mathbb{C})} s_1 \wedge \overline{s_2}$. induces a metric $\|\ \|_0$ on $H^0(X, \Omega^1_{C/\mathbb{C}})$, which we will call the *natural metric*. We will call *natural metric* and will denote by the same symbol also the metric induced by $\|\ \|_0$ on $\lambda_{C/\mathbb{C}} = \det H^0(X, \Omega^1_{C/\mathbb{C}})$, and on tensor powers of this vectorspace.

Let $\mathcal{L}$ be a line bundle on C and let $\|\ \|$ and $\|\ \|_{\mathcal{L}}$ be metrics on $\Omega^1_{C/\mathbb{C}}$ and $\mathcal{L}$, respectively. For two global $\mathcal{C}^\infty$-sections s_1, s_2 of $\mathcal{L}$ (of $\mathcal{L} \otimes \Omega^{0,1}_C$) their L^2 scalar product is defined by

$$\langle s_1, s_2\rangle_{L^2} := \frac{1}{\pi} \cdot \int_C \langle s_1(x), s_2(x)\rangle \,\mathrm{d}\nu(x).$$

Here the scalar product under the integral is induced by the metric $\|\ \|_{\mathcal{L}}$ (by the metrics $\|\ \|_{\mathcal{L}}$ and $\|\ \|$), and $\mathrm{d}\nu$ is the volume form on C locally defined by

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$\mathrm{d}\nu = (i/2)(\alpha \wedge \overline{\alpha}/||\alpha||^2)$, where α is an invertible local section of $\Omega^1_{C/\mathbb{C}}$. The vectorspace $H^0(C, \mathcal{L})$ can be identified with the kernel of the Dolbeault-complex

$$\mathcal{C}^\infty(C, \mathcal{L}) \stackrel{\overline{\delta}}{\to} \mathcal{C}^\infty(C, \mathcal{L} \otimes \Omega^{0,1})$$

and we have canonically $H^1(C, \mathcal{L}) = \operatorname{coker} \overline{\delta} = (\operatorname{im} \overline{\delta})^\perp = \ker \overline{\delta}^*$. Restriction of the L^2-norms of $\mathcal{C}^\infty(C, \mathcal{L})$ and $\mathcal{C}^\infty(C, \mathcal{L} \otimes \Omega^{0,1})$ gives metrics on $H^0(C, \mathcal{L})$ and $H^1(C, \mathcal{L})$ and by this we get a norm on $\lambda(\mathcal{L}) := \det H^0(X, \mathcal{L}) \otimes (\det H^1(X, \mathcal{L}))^{-1}$, which we denote by the symbol $||\ ||_{\mathcal{L},L^2}$. The *Quillen-metric* $||\ ||_Q = ||\ ||_{Q,\mathcal{L},||\ ||,||\,||_{\mathcal{L}}}$ on $\lambda(\mathcal{L})$ is defined as

$$||\ ||_Q := (\det{}' \Delta_{\overline{\delta}})^{-(1/2)} \cdot ||\ ||_{\mathcal{L},L^2},$$

where $(\det{}' \Delta_{\overline{\delta}})^{-(1/2)}$ is the regularized determinant of the Laplace-Operator $\Delta_{\overline{\delta}} = \overline{\delta}^* \overline{\delta}$ (cf. [S-A-B-K], Ch. V, for the definition of $\det{}'$).

Let $||\ ||$ still denote a smooth hermitian metric on the line bundle $\Omega^1_{C/\mathbb{C}}$. The *delta invariant* of C with respect to the metric $||\ ||$ is defined as the real number

$$\delta(C) := 12 \cdot \log\left(\frac{||\ ||_Q}{||\ ||_0}\right),$$

where $||\ ||_0$ is the natural metric on $\lambda_{C/\mathbb{C}}$ and $||\ ||_Q$ is the Quillen-metric on $\lambda_{C/\mathbb{C}}$ associated to $||\ ||$.

Now we have all necessary definitions at hand, to be able to formulate the analogue of inequality (1) 'at infinity'.

THEOREM 7.1. *Let $g \geqslant 2$ be an integer and fix a $\mathcal{C}^\infty$-metric $||\ ||_{\mathrm{Mod}}$ on $\omega_{\overline{\mathcal{X}}_g/\mathcal{M}_g b}$. Let $\varepsilon > 0$. There exists a constant $c \in \mathbb{R}$ such that for all smooth hyperelliptic curves $C/\mathbb{C}$ we have the inequality*

$$\Leftrightarrow\log ||\Lambda_{C/\mathbb{C}}||_0 \leqslant (g^2 + \varepsilon) \cdot \delta(C) + c,$$

where $\delta(C)$ is the delta-invariant of C with respect to the metric induced by $||\ ||_{\mathrm{Mod}}$ on C.

The proof of the theorem will be given after Propositions 7.3–7.6 below.

DEFINITION 7.2. (a) Let $Y/\mathbb{C}$ be a $(2g+2)$-pointed stable genus-zero curve and let $p \in Y$ be a singular point. The partial normalization of Y is the disjoint sum of two connected pointed curves. Let k be the number of marked points on one of its components and put $m := \min(k, 2g+2 \Leftrightarrow k)$. The following cases are possible

(1) $m = 2i + 2$ for some $i \in \{0, \ldots, [g \Leftrightarrow 1/2]\}$.

Then we call p to be *of type* α_i.
(2) $m = 2j+1$ for some $j \in \{1, \ldots, [g/2]\}$.
Then we call p to be *of type* β_j.

(b) Let $X/\mathbb{C}$ be a stable curve of genus $g \geqslant 2$ and let $p \in X$ be a singular point. Let $\tilde{X}$ be the partial normalization of X at p. The following cases are possible

(1) $\tilde{X}$ is connected. Then we call p to be *of type* δ_0.
(2) $\tilde{X}$ is the disjoint union of two prestable curves. Let $j \in \{1, \ldots, [g/2]\}$ be the minimum of their respective geni. Then we call p to be *of type* δ_j.

Let X_0 be a hyperelliptic stable curve of genus $g \geqslant 2$ over $\mathbb{C}$. In what follows, we review the construction of the universal local deformation $\mathcal{X}_{\text{hsc}} \to \mathcal{U}_{\text{hsc}}$ of X_0 as a hyperelliptic stable curve (cf. [H-M], Section 4, [C-H], Section 4). Let X_0' be the modification of X_0 in its direction-reversing double points. By definition, there exists an n-pointed prestable genus-zero curve Y_0 and an admissible double covering $X_0' \to Y_0$. By Lemma 1.6 we have $n = 2g+2$, and it is easy to see that stability of X_0 implies that Y_0 is in fact a stable pointed curve. Let $\mathcal{Y}_{\text{spc}} \to \mathcal{U}_{\text{spc}}$ be the universal local deformation of Y_0 as a stable pointed curve. The locus $\Delta_{\text{spc}} \subset \mathcal{U}_{\text{spc}}$ of singular curves is a normal-crossing divisor whose branches correspond to the singular points of Y_0. We may assume that $\mathcal{U}_{\text{spc}}$ is an open neighborhood of $\mathbb{C}^{2g-1}$ and that $t_1 \ldots t_d = 0$ is an equation of Δ_{spc}, where $t_1, \ldots, t_{2g-1}$ are the standard coordinates on $\mathbb{C}^{2g-1}$ and d is the number of double points of Y_0. Let I_i (resp. J_j) be the subset of indices $k \in \{1, \ldots, d\}$, such that $\{t_k = 0\}$ is the branch of Δ_{spc}, which corresponds to a double point of type α_i (resp. of type β_j) of Y_0 ($i = 0, \ldots, [g \Leftrightarrow 1/2]$, $j = 1, \ldots, [g/2]$). Choose some small neighborhood $\mathcal{U}_{\text{adc}}$ of the origin of $\mathbb{C}^{2g-1}$ and let $z_1, \ldots, z_{2g-1}$ be the standard coordinates on $\mathcal{U}_{\text{adc}}$. Consider the morphism from $\mathcal{U}_{\text{adc}}$ to $\mathcal{U}_{\text{spc}}$, given by the equations

$$t_i = z_i^2 \quad \text{for } i \in J,$$

$$t_i = z_i \quad \text{else.}$$

Let $\mathcal{Y}_{\text{adc}} \to \mathcal{U}_{\text{adc}}$ be the pull-back of $\mathcal{Y}_{\text{spc}}$ by $\mathcal{U}_{\text{adc}} \to \mathcal{U}_{\text{spc}}$. There exists a admissible double covering $\mathcal{X}_{\text{adc}} \to \mathcal{Y}_{\text{adc}}$ which is unique up to isomorphism. The fibre of $\mathcal{X}_{\text{adc}}$ over the center of $\mathcal{U}_{\text{adc}}$ is isomorphic to X_0'. The curve $\mathcal{X}_{\text{adc}} \to \mathcal{U}_{\text{adc}}$ is semistable. It is the blow-up of a hyperelliptic stable curve $\mathcal{X}'' \to \mathcal{U}_{\text{adc}}$ along the closed subvariety $D \subset \mathcal{X}''$ of direction-reversing double points. $\mathcal{X}'' \to \mathcal{U}_{\text{adc}}$ is unique up to isomorphism. The special fibre $\mathcal{X}_0''$ is isomorphic to X_0. Now consider the morphism of $\mathcal{U}_{\text{adc}}$ to $\mathbb{C}^{2g-1}$ defined by

$$u_i = z_i^2 \quad (i \in I),$$

$$u_i = z_i \quad \text{else,}$$

$u_1, \ldots, u_{2g-1}$ being the standard coordinates on $\mathbb{C}^{2g-1}$. The curve $\mathcal{X}'' \to \mathcal{U}_{\text{adc}}$ descends to a hyperelliptic stable curve $\pi\colon \mathcal{X}_{\text{hsc}} \to \mathcal{U}_{\text{hsc}}$ over a neighborhood $\mathcal{U}_{\text{hsc}}$

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of the origin of $\mathbb{C}^{2g-1}$ and $\pi\colon \mathcal{X}_{\text{hsc}} \to \mathcal{U}_{\text{hsc}}$ is the universal local deformation of X_0 as a hyperelliptic stable curve. In the following diagram, which gives an overview of the construction, the two extreme squares are cartesian

$$\begin{tikzcd}
\mathcal{X}_{\text{hsc}} \arrow[d, "\pi"'] & \mathcal{X}'' \arrow[l] \arrow[d] & \mathcal{X}_{\text{adc}} \arrow[l, "\text{blow-up}"'] \arrow[r, "\text{admiss.double}", "\text{covering}"'] \arrow[d] & \mathcal{Y}_{\text{adc}} \arrow[r] \arrow[d] & \mathcal{Y}_{\text{spc}} \arrow[d] \\
\mathcal{U}_{\text{hsc}} & \mathcal{U}_{\text{adc}} \arrow[l, "\text{ramified}", "\text{covering}"'] \arrow[r, equal] & \mathcal{U}_{\text{adc}} \arrow[r, equal] & \mathcal{U}_{\text{adc}} \arrow[r, "\text{ramified}", "\text{covering}"'] & \mathcal{U}_{\text{spc}}
\end{tikzcd}.$$

For $i \in \{0,\dots,[(g \Leftrightarrow 1)/2]\}$ (resp. for $j \in \{1,\dots,[g/2]\}$), let $\Delta_{\text{hsc}}^{\alpha_i} \subset \mathcal{U}_{\text{hsc}}$ (resp. $\Delta_{\text{hsc}}^{\beta_j} \subset \mathcal{U}_{\text{hsc}}$) be the locus of curves carrying a double point of type α_i (resp. of type β_j). It is the normal-crossing divisor given by the equation $\Pi_{k\in I_i} u_k = 0$ (resp. by $\Pi_{k\in J_j} u_k = 0$).

PROPOSITION 7.3. *Let Λ_π be the canonical section of the line bundle $(\det \pi_*\omega_\pi)^{\otimes(8g+4)}$. Then we have the following equality of divisors on $\mathcal{U}_{\text{hsc}}$*

$$\operatorname{div}(\Lambda_\pi) = g\cdot\Delta_{\text{hsc}}^{\alpha_0} + \sum_{i=1}^{[g-1/2]} 2(i+1)(g \Leftrightarrow i)\Delta_{\text{hsc}}^{\alpha_i} + \sum_{j=1}^{[g/2]} 4j(g \Leftrightarrow j)\Delta_{\text{hsc}}^{\beta_j}.$$

Proof. Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$. It suffices to prove that the stated equality holds for the pull-back of the divisors under any morphism $\mathbb{D} \to \mathcal{U}_{\text{hsc}}$. But this can be shown analogously to the proof of Theorem 3.1. Alternatively, one may derive the equality from Proposition 4.7 of [C-H] and the fact that over the moduli space of smooth hyperelliptic curves, our canonical section Λ provides a trivialization of the $(8g+4)$th power of the Hodge-bundle. $\square$

Now let $\phi\colon \mathcal{X}_{\text{sc}} \to \mathcal{U}_{\text{sc}}$ be the universal deformation of X_0 as a stable curve (cf. [Bi-Bo] pp. 19–21). For $j = 0,\dots,[g/2]$, the locus $\Delta_{\text{sc}}^{\delta_j} \subset \mathcal{U}_{\text{sc}}$ of curves carrying a double point of type δ_j is a normal crossing divisor. We may identify $\mathcal{U}_{\text{sc}}$ with an open neighborhood of $0 \in \mathbb{C}^{3g-3}$ such that the divisor $\Delta_{\text{sc}} := \Sigma_{j=0}^{[g/2]}\Delta_{\text{sc}}^{\delta_j}$ is given by $f = 0$, where $f := \Pi_{i=1}^{d} t_j$ (the functions $t_1,\dots t_{3g-3}$ being the standard coordinates on $\mathbb{C}^{3g-3}$ and d the number of singular points of X_0). By the universal property of $\mathcal{X}_{\text{sc}} \to \mathcal{U}_{\text{sc}}$, we have a canonical morphism ι from (a neighborhood of 0 in) $\mathcal{U}_{\text{hsc}}$ to $\mathcal{U}_{\text{sc}}$, such that $\mathcal{X}_{\text{hsc}}$ is the pull-back of $\mathcal{X}_{\text{sc}}$ via this morphism. The following proposition follows from the above construction of $\mathcal{X}_{\text{sc}} \to \mathcal{U}_{\text{sc}}$.

PROPOSITION 7.4. *We have the following equalities between divisors on $\mathcal{U}_{\text{hsc}}$*

$$\iota^*\Delta_{\mathrm{sc}}^{\delta_0} = \Delta_{\mathrm{hsc}}^{\alpha_0} + 2\sum_{i=1}^{g-1/2} \Delta_{\mathrm{hsc}}^{\alpha_i},$$

$$\iota^*\Delta_{\mathrm{sc}}^{\delta_j} = \Delta_{\mathrm{hsc}}^{\beta_j} \quad \text{for } j = 1, \dots, [g/2].$$

A special case of Theorem 4.1 in [Fa-Wü], p. 17 is the following

PROPOSITION 7.5. *Let σ be an invertible section of* $\det(\phi_*\omega_\phi)$. *Then there exist constants $C_1, C_2 > 0$, and a neighborhood U of $0 \in \mathcal{U}_{\mathrm{sc}}$ such that on $U \Leftrightarrow \Delta_{\mathrm{sc}}$ we have the inequalities*

$$C_1^{-1}\left(\log\frac{1}{|f|}\right)^{-C_2} \leqslant ||\sigma||_0 \leqslant C_1\left(\log\frac{1}{|f|}\right)^{C_2}.$$

Remark. Using results of Y. Namikawa, we have shown in [Ka] the following more precise statement: Let σ be an invertible section of $\det(\phi_*\omega_\phi)$ and let $h\colon \mathcal{U}_{\mathrm{sc}} \Leftrightarrow \Delta_{\mathrm{sc}} \to \mathbb{R}_+^*$ be defined by

$$h := \max_{i=1,\dots,d} \log\frac{1}{|t_i|}.$$

Then there are constants $0 < C' < C''$ and a neighborhood U of $0 \in \mathcal{U}_{\mathrm{sc}}$ such that on $U \Leftrightarrow \Delta_{\mathrm{sc}}$ we have the inequalities

$$C'h^\gamma \leqslant ||\sigma||_0^2 \leqslant C''h^\gamma,$$

where γ is the rank of the first homology group of the graph associated to the stable curve X_0. In what follows, however, we only need the statement of Proposition 7.5 above.

One of the hardest ingredients of our proof of Theorem 7.1 is the following consequence of a result of Bismut and Bost ([Bi-Bo], Theorem 2.2) that describes the asymptotic of the Quillen-metric (observe that the bundle denoted by $\lambda(\omega)$ in [Bi-Bo] is inverse to our determinant bundle det $\phi_*\omega$).

PROPOSITION 7.6. *Let σ be an invertible section of* $\det(\phi_*\omega_\phi)$. *There exists a continuous function h_1 on $\mathcal{U}_{\mathrm{sc}}$, such that on $\mathcal{U}_{\mathrm{sc}} \Leftrightarrow \Delta_{\mathrm{sc}}$ we have the equality*

$$12 \cdot \log ||\sigma_U||_Q = \Leftrightarrow \sum_{i=1}^{d} \log |t_i| + h_1.$$

Proof of Theorem 7.1. Let $\mathcal{I}_g$ be the (coarse) moduli space of smooth hyperelliptic curves of genus g over $\mathbb{C}$. It is a closed subset of the moduli space $\mathcal{M}_g$ of all smooth curves. Let $\overline{\mathcal{I}_g}$ be the Zariski-closure of $\mathcal{I}_g$ in the moduli space $\overline{\mathcal{M}_g}$ of stable curves. The maps $[C] \mapsto ||\Lambda_{C/\mathbb{C}}||$ and $[C] \mapsto \delta(C)$ from $\mathcal{I}_g$ to $\mathbb{R}$ are continuous. Since $\overline{\mathcal{I}_g}$ is compact, it suffices to show that for each point of $\overline{\mathcal{I}_g} \Leftrightarrow \mathcal{I}_g$ there exists a neighborhood $V \subset \overline{\mathcal{I}_g}$ such that $\Leftrightarrow \log ||\Lambda_{C/\mathbb{C}}|| < (g+\varepsilon) \cdot \delta(C)$ over $V \cap \mathcal{I}_g$. Such a point is represented by a hyperelliptic stable curve X_0. Let $\pi\colon \mathcal{X}_{\text{hsc}} \to \mathcal{U}_{\text{hsc}}$ be the universal local deformation of X_0 as a hyperelliptic stable curve. Adopting the above notation for the various divisors on $\mathcal{U}_{\text{hsc}}$, let

$$a_i \in \Gamma(\mathcal{U}_{\text{hsc}}, \mathcal{O}_{\mathcal{U}_{\text{hsc}}}) \quad \text{an equation for } \Delta_{\text{hsc}}^{\alpha_i}, \quad i = 0, \ldots, [g \Leftrightarrow 1/2],$$

$$b_j \in \Gamma(\mathcal{U}_{\text{hsc}}, \mathcal{O}_{\mathcal{U}_{\text{hsc}}}) \quad \text{an equation for } \Delta_{\text{hsc}}^{\beta_j}, \quad j = 1, \ldots, [g/2],$$

$$\sigma \in \Gamma(\mathcal{U}_{\text{hsc}}, \det \pi_*\omega_\pi) \quad \text{an invertible section.}$$

By definition of the delta invariant, Proposition 7.4 and Proposition 7.6, there exists a continuous function φ_1 on $\mathcal{U}_{\text{hsc}}$, such that

$$\delta(\mathcal{X}_{\text{hsc}}/\mathcal{U}_{\text{hsc}}) = \Leftrightarrow \log|a_0| \Leftrightarrow 2\sum_{i=1}^{[g-1/2]} \log|a_i| \Leftrightarrow \sum_{j=1}^{[g/2]} \log|b_j| \Leftrightarrow 12\log||\sigma||_0 + \varphi_1.$$

On the other hand, from Proposition 7.3 we have

$$\begin{aligned}\log||\Lambda_{\mathcal{X}_{\text{hsc}}/\mathcal{U}_{\text{hsc}}}||_0 \;=\; & g \cdot \log|a_0| + \sum_{i=1}^{[g-1/2]} 2(i+1)(g \Leftrightarrow i)\log|a_i| \\ & + \sum_{j=1}^{[g/2]} 4j(g \Leftrightarrow j)\log|b_j| + (8g+4)\log||\sigma||_0 + \varphi_2,\end{aligned}$$

for some continuous function φ_2 on $\mathcal{U}_{\text{hsc}}$. From Proposition 7.5 it follows that in the neighborhood of $\Delta_{\text{hsc}} := \Sigma_i \Delta_{\text{hsc}}^{\alpha_i} + \Sigma_j \Delta_{\text{hsc}}^{\beta_j}$ the function $\log||\sigma||_0$ is negligible compared with $\log|a_i|$ and $\log|b_j|$. Therefore an elementary comparison of the coefficients of $\log|a_i|$ and $\log|b_j|$ in the above expressions for $\delta(\mathcal{X}_{\text{hsc}}/\mathcal{U}_{\text{hsc}})$ and $\log||\Lambda_{\mathcal{X}_{\text{hsc}}/\mathcal{U}_{\text{hsc}}}||_0$ proves the existence of an open neighborhood $U \subseteq \mathcal{U}_{\text{hsc}}$ of Δ_{hsc} such that over $U \cap (\mathcal{U}_{\text{hsc}} \Leftrightarrow \Delta_{\text{hsc}})$ the inequality

$$\Leftrightarrow \log||\Lambda_{\mathcal{X}_{\text{hsc}}/\mathcal{U}_{\text{hsc}}}||_0 < (g^2+\varepsilon)\delta(\mathcal{X}_{\text{hsc}}/\mathcal{U}_{\text{hsc}})$$

holds. The canonical map $\mathcal{U}_{\text{hsc}} \to \overline{\mathcal{I}_g}$ identifies an open neighborhood of $[X_0] \in \overline{\mathcal{I}_g}$ with the quotient of $\mathcal{U}_{\text{hsc}}$ by the natural action of the finite group $\operatorname{Aut}(X_0)$. In particular it is open, so we may take V as the image of U under this morphism. $\square$

In what follows, we assume that the reader is familiar with the basic notions of (low-dimensional) Arakelov-theory, as presented for instance in Soulé's Bourbaki talk [Sou]. Thus, in particular, if K be a number field, $S := \operatorname{Spec}(\mathcal{O}_K)$ and X/S

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an arithmetic surface, we have the notion of the degree of metrized line bundles on S and the intersection pairing between metrized line bundles on X.

THEOREM 7.7. *Fix a $\mathcal{C}^\infty$-metric $\|\ \|_{\mathrm{Mod}}$ on $\omega_{\overline{\mathcal{X}}_g/\overline{\mathcal{M}}_g}$ and an $\varepsilon > 0$. Let K be a number field. Let $\pi\colon X \to S = \mathrm{Spec}(\mathcal{O}_K)$ be a minimal regular arithmetic surface with smooth hyperelliptic generic fibre of genus $g \geqslant 2$. Assume good reduction at all places dividing (2) and semistable reduction at all other places. Let $\|\ \|_0$ be the natural metric on $\det \pi_*\omega_{X/S}$. For each embedding $\sigma\colon K \hookrightarrow \mathbb{C}$ let $\delta(X_\sigma)$ be the delta-invariant of the Riemann-surface X_σ with respect to the metric $\|\ \|_{\mathrm{Mod}}$. Then there exists a constant $c(\|\ \|_{\mathrm{Mod}}, \varepsilon)$, depending only on the metric $\|\ \|_{\mathrm{Mod}}$ and on ε, such that the following inequality holds*

$$\deg(\det\pi_*\omega_{X/S}, \|\ \|_0) \leqslant \frac{g^2+\varepsilon}{8g+4}\left(\sum_{\mathfrak{p}} \delta_{\mathfrak{p}} \log N\mathfrak{p} + \sum_{\sigma} \delta(X_\sigma)\right)$$
$$+\, c(\|\ \|_{\mathrm{Mod}}, \varepsilon) \cdot [K:\mathbb{Q}].$$

Proof. This is an immediate consequence of inequality (1), Theorem 6 and Theorem 7.1. □

COROLLARY 7.8. *Assume the conditions of the above theorem. Then there exists a constant $c'(\|\ \|_{\mathrm{Mod}}, \varepsilon)$, depending only on $\|\ \|_{\mathrm{Mod}}$ and ε, such that the following inequality holds*

$$(\omega_{X/S}.\omega_{X/S}) \leqslant \left(3\frac{g^2+\varepsilon}{2g+1} \Leftrightarrow 1\right)\left(\sum_{\mathfrak{p}\in|S|} \delta_{\mathfrak{p}} \log N\mathfrak{p} + \sum_{\sigma:K\hookrightarrow\mathbb{C}} \delta(X_\sigma)\right)$$
$$+\, c'(\|\ \|_{\mathrm{Mod}}, \varepsilon) \cdot [K:\mathbb{Q}],$$

where the intersection number $(\omega_{X/S}.\omega_{X/S})$ and the delta-invariants $\delta(X_\sigma)$ are understood with respect to the metric induced by $\|\ \|_{\mathrm{Mod}}$ on $\omega_{X/S}$.

Proof. By a theorem of Deligne ([Del], Theorem 11.4, cf. also [Sou], Théorème 4), we have

$$(\omega_{X/S}.\omega_{X/S}) = 12 \cdot \deg(\det \pi_*\omega_{X/S}, \|\ \|_Q) \Leftrightarrow \sum_{\mathfrak{p}\in S} \delta_{\mathfrak{p}} \log N(\mathfrak{p}) \Leftrightarrow [K:\mathbb{Q}] \cdot a(g),$$

for some constant $a(g)$, which depends only on the genus. With this, the corollary follows immediately from Theorem 7.7. □

It was first observed by A. N. Parshin that a certain upper bound for the self-intersection ω^2 of the relative dualizing sheaf on arithmetic surfaces would have interesting number-theoretical consequences as, for example, the abc-conjecture (cf. [Par1], [Par2], [Par3]).

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The shape of the upper bound as proposed by Parshin was suggested by the analogous geometric situation of a surface fibred over a curve, where it follows from the so-called Bogomolov–Miyaoka–Yau inequality between the Chern classes of the surface. In the sequel, L. Moret-Bailly formulated as a hypothesis (cf. [MB] 'Hypothèse BM' (3.1.2)) a more general shape of an upper bound of ω^2 which he showed would still imply e.g. the abc-conjecture. Observe that Corollary 7.8 gives an upper bound which has the form required in [MB]. One might object that in Moret-Bailly's hypothesis the self-intersection ω^2 and the delta-invariant $\delta(X_\sigma)$ are defined with respect to the Arakelov-metric. But it is easy to see that in the hypothesis one could take ω^2 and $\delta(X_\sigma)$ with respect to any $\mathcal{C}^\infty$-metric on $\omega_{\overline{\mathcal{X}}_g/\overline{\mathcal{M}}_g}$, and draw the same conclusions. Only the constants which appear in the estimates, would depend on the choice of the metric.

Nevertheless our result does not suffice to draw arithmetic consequences at least not along the lines of [MB]. In fact, recall the crucial argument in [MB] which shows that Hypothèse BM implies a version of Mordell's conjecture: Let B be a smooth proper curve over a number field K. After replacing K by some finite extension and B by a finite étale covering, we may assume, by Kodaira's construction, that over the base B there exists a non-isotrivial smooth family of curves $V \to B$. One gets an upper bound on the height of rational points of B by applying Hypothèse BM to models of the fibres of the family. In view of Corollary 7.8, it is therefore natural to ask whether there exists a proper smooth curve which parametrizes a non-isotrivial family of smooth hyperelliptic curves. Unfortunately, the answer to this question is negative. This fact seems to be well-known, but for sake of completeness we give a short proof: Let $\pi: V \to B$ be a smooth family of hyperelliptic curves. By Theorem 3.1, the degree of $\det \pi_*\omega_{V/B}$ vanishes. As has been shown by Arakelov, this implies the isotriviality of V/B ([Ara1], Lemma 1.4 and Corollary 1 to Theorem 1.1).

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