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FINITE GROUPS WHOSE NONCENTRAL COMMUTING ELEMENTS HAVE CENTRALIZERS OF EQUAL SIZE

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Abstract

A finite group is called a **CH**-group if for every $x, y \in G \setminus \mathbf{Z}(G)$, $xy = yx$ implies that $\|\mathbf{C}_G(x)\| = \|\mathbf{C}_G(y)\|$. Applying results of Schmidt ['Zentralisatorverbände endlicher Gruppen', *Rend. Sem. Mat. Univ. Padova* **44** (1970), 97–131] and Rebmann ['**F**-Gruppen', *Arch. Math.* **22** (1971), 225–230] concerning **CA**-groups and **F**-groups, the structure of **CH**-groups is determined, up to that of **CH**-groups of prime-power order. Upper bounds are found for the derived length of nilpotent and solvable **CH**-groups.

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1. Introduction

In 1953, Ito published a paper [I] dealing with the class of **F**-groups, consisting of finite groups G in which for every $x, y \in G \setminus \mathbf{Z}(G)$, $\mathbf{C}_G(x) \leq \mathbf{C}_G(y)$ implies that $\mathbf{C}_G(x) = \mathbf{C}_G(y)$. An important subclass of **F**-groups is the class of **I**-groups, in which all centralizers of noncentral elements are of the same order. Ito proved in [I] that **I**-groups are nilpotent and are direct products of an abelian group and a group of prime-power order. Only 49 years later, Ishikawa showed in [Ish] that groups in **I** are of class at most 3. For a simpler proof, see the papers by Mann [M1] and Isaacs [Is]. The **F**-groups were investigated by Rebmann in [R]. He determined their structure, up to that of **F**-groups which are central extensions of groups of prime-power order.

Another important subclass of **F**-groups is the class of **CA**-groups, consisting of groups in which all centralizers of noncentral elements are abelian. The **CA**-groups (or rather the equivalent class of $\mathfrak{M}$-groups) were investigated by Schmidt in [S] (see also [S1, Theorem 9.3.12]). He determined their structure up to that of **CA**-groups which are central extensions of groups of prime-power order. It is very similar to the structure of **F**-groups.

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In this paper we investigate the class of **CH**-groups, consisting of finite groups in which noncentral commuting elements have centralizers of the same order. We consider the centralizers in **CH**-groups to be in some sense *homogeneous*. We show in Theorem 4.1 that the classes **F**, **CA** and **CH** satisfy the relation

$$\mathbf{CA} \subset \mathbf{CH} \subset \mathbf{F},$$

with both inclusions being proper.

Concerning **CH**-groups, our aim is to determine their structure, up to that of **CH**-groups of prime-power order. In order to achieve that aim, we first improve a little the results of Schmidt and Rebmann, so that now the structure of **CA**-groups and **F**-groups is determined up to **CA**-groups and **F**-groups of prime-power order, respectively. Their results, in the improved form, are presented together in Theorem A (see Section 3). Applying Theorem A, we determine in Theorem 4.2 the structure of **CH**-groups, up to that of **CH**-groups of prime-power order (see Section 4). Our results in Theorem 4.2 are analogous to those of Theorem A.

Theorem 4.2 enables us to describe the structure of nonsolvable **CH**-groups (see Corollary 4.3), which, in view of Theorem A, coincide with nonsolvable **CA**-groups and nonsolvable **F**-groups. We also determine the structure of nilpotent **CH**-groups, **CA**-groups and **F**-groups (see Corollary 4.4).

In Section 5, we find upper bounds for the derived length of nilpotent **CH**-groups (see Theorem 5.2) and of solvable **CH**-groups (see Theorem 5.3). In both cases, the bounds are the best possible.

In our final Section 6, we prove in Theorem 6.3 that if G is a **CH**-group of p-power order for some prime p, then G is either a **CA**-group or its nilpotency class is at most $2p$. Moreover, we show in Corollary 6.2 that the derived length of **F**-groups of 2-power order is at most 2.

2. Preliminary results

The aim of this section is twofold: firstly, we present a summary of the definitions which are relevant to our discussion; and secondly, we quote or prove three important lemmas, which will be used later.

In this paper all groups are finite. The basic definitions are:

- a group G is a **CH**-group (written $G \in \mathbf{CH}$) if for every $x, y \in G \setminus \mathbf{Z}(G)$, $xy = yx$ implies that $|\mathbf{C}_G(x)| = |\mathbf{C}_G(y)|$;
- a group G is a **CA**-group ($G \in \mathbf{CA}$) if $\mathbf{C}_G(x)$ is abelian for every $x \in G \setminus \mathbf{Z}(G)$;
- a group G is an **F**-group ($G \in \mathbf{F}$) if for every $x, y \in G \setminus \mathbf{Z}(G)$, $\mathbf{C}_G(x) \le \mathbf{C}_G(y)$ implies that $\mathbf{C}_G(x) = \mathbf{C}_G(y)$.

We now turn to the three lemmas. Our first lemma is well known (see [Ro1, Theorem 5.1.11]). Here $\mathbf{Z}_2(G)$ denotes the second center of G.

LEMMA 2.1. *For every group G, $[G', \mathbf{Z}_2(G)] = 1$.*

The second lemma is a recent nice result of Isaacs (see [Is, Lemma 1]).

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LEMMA 2.2. *Let x be a noncentral element of G and let A be an abelian normal subgroup of G. If $a \in A$, then $|\mathbf{C}_G([a, x])| > |\mathbf{C}_G(x)|$.*

For our final lemma, we need some additional notation. If a group B acts on a group A, we denote by $[A, [B]_1]$ (or $[A, B]$) the commutator subgroup of A and B in the semidirect product AB. For an integer $n \geq 2$, we define $[A, [B]_n]$ by the recursion formula $[A, [B]_n] = [[A, [B]_{n-1}], B]$. Using this notation, the nilpotency class of a nilpotent group $G \neq 1$ is the smallest integer n such that $[G, [G]_n] = 1$.

LEMMA 2.3. *Let G be a p-group of exponent p and let A be a normal abelian subgroup of G. If $G/\mathbf{C}_G(A)$ is abelian, then $[A, [G]_{p-1}] = 1$.*

PROOF. Write $\bar{G} = G/\mathbf{C}_G(A)$. Then $\bar{G}$ is a group of automorphisms of exponent p of the elementary abelian p-group A. Since G has exponent p, for every $g \in G$ and $a \in A$ we have both $(g^{-1})^p = 1$ and $(ag^{-1})^p = 1$. Therefore,

$$a \cdot a^g \cdot a^{g^2} \cdots a^{g^{p-1}} = a \cdot a^g \cdot a^{g^2} \cdots a^{g^{p-1}} \cdot (g^{-1})^p = (ag^{-1})^p = 1.$$

Hence, viewing $\bar{G}$ embedded in the ring of the GF(p)-endomorphism of A, the minimal polynomial of every $g \in \bar{G}$ divides $q(X) = 1 + X + \cdots + X^{p-1}$. Since $\bar{G}$ is abelian, [Ro, Lemma 7.19] implies that $[A, [\bar{G}]_p] = [A, [\bar{G}]_{p-1}]$. But $\bar{G} = G/\mathbf{C}_G(A)$, so it follows that $[A, [G]_p] = [A, [G]_{p-1}]$. Thus $[B, G] = B$, where $B = [A, [G]_{p-1}]$ and hence $B = 1$, as G is a nilpotent group. □

3. The structure of CA-groups and F-groups

It is easy to see that the class of **CA**-groups coincides with the class of $\mathfrak{M}$-groups, consisting of groups in which all centralizers of noncentral subgroups are abelian. The $\mathfrak{M}$-groups were investigated by Schmidt in [S] (see also [S1, Theorem 9.3.12]). He determined their structure, up to that of $\mathfrak{M}$-groups which are central extensions of groups of prime-power order. We shall refer to his results in the language of **CA**-groups.

F-groups were investigated by Rebmann in [R]. He also determined their structure, up to that of **F**-groups which are central extensions of groups of prime-power order. Moreover, the structure of **F**-groups, as described by Rebmann, is very similar to the structure of **CA**-groups, as described by Schmidt.

The results of Schmidt and Rebmann will be stated together as our Theorem A. However, we state their condition (III) in a different form, in order to emphasize that as a matter of fact, their results determine the structure of **CA**-groups and **F**-groups up to that of **CA**-groups and **F**-groups of prime-power order, respectively. In the following Proposition 3.1, which precedes Theorem A, we show that the two forms of condition (III) are equivalent.

PROPOSITION 3.1. *Let G be a nonabelian group and write $Z = \mathbf{Z}(G)$. Suppose that G/Z is a Frobenius group with Frobenius kernel K/Z and Frobenius complement L/Z and let p denote a fixed prime. Then the following statements are equivalent.*

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(a) *K/Z is a p-group, $\mathbf{Z}(K) = Z$ and L is abelian.*
(b) *$K = PZ$, where P is a normal Sylow p-subgroup of G, $\mathbf{Z}(P) = P \cap Z$ and $L = HZ$, where H is an abelian p'-subgroup of G.*

Moreover, if (a) and (b) hold, then K is a **CA***-group (***F***-group,* **CH***-group) if and only if P is a* **CA***-group (***F***-group,* **CH***-group).*

PROOF. Suppose, first, that (a) holds. Since $(|L/Z|, |K/Z|) = 1$, K/Z is a normal Sylow p-subgroup of G/Z. Hence K is a normal nilpotent subgroup of G containing a Sylow p-subgroup P of G and $K = PZ = P \times Z_{p'}$, where $Z_{p'}$ is the p-complement in both K and Z. In particular, $P \trianglelefteq G$. Moreover, $\mathbf{Z}(P) = \mathbf{Z}(K) \cap P = Z \cap P$. Since $L \cap K = Z$, the Sylow p-subgroup Z_p of Z is a normal Sylow p-subgroup of L and there exists a p-complement H in L. Thus $L = HZ$ and since L is abelian, H is an abelian p'-subgroup of G. Hence (b) holds.

Conversely, suppose that (b) holds. Then K/Z is a p-group, L is abelian and $\mathbf{Z}(K) = \mathbf{Z}(P)Z = (P \cap Z)Z = Z$. Hence (a) holds.

Suppose, next, that (a) and (b) hold. We shall prove the following:

(i) K is a **CA**-group if and only if P is a **CA**-group;
(ii) K is an **F**-group if and only if P is an **F**-group;
(iii) K is a **CH**-group if and only if P is a **CH**-group.

We start with some notation and with a few remarks, which will be used freely in the proofs below. We write $Z = Z_p \times Z_{p'}$, where Z_p is the Sylow p-subgroup of Z and $Z_{p'}$ is its p-complement. Since $K = PZ$, it follows that $K = P \times Z_{p'}$ and if $x \in K$, then $x = uz$, where $u \in P$ and $z \in Z_{p'}$. Moreover,

$$\mathbf{C}_K(x) = \mathbf{C}_P(u) \times Z_{p'}$$

and if $x \in K \setminus \mathbf{Z}(K)$, then $u \in P \setminus \mathbf{Z}(P)$. In particular, if $x \in P$, then $\mathbf{C}_K(x) = \mathbf{C}_P(x) \times Z_{p'}$.

(i) Suppose, first, that K is a **CA**-group and $x \in P \setminus \mathbf{Z}(P)$. Since $P \leq K$, it follows that $x \in K \setminus \mathbf{Z}(K)$ and $K \in \mathbf{CA}$ implies that $\mathbf{C}_K(x)$ is abelian. Thus $\mathbf{C}_P(x)$ is also abelian and it follows that P is a **CA**-group.

Conversely, suppose that P is a **CA**-group and let $x \in K \setminus \mathbf{Z}(K)$. Then $\mathbf{C}_K(x) = \mathbf{C}_P(u) \times Z_{p'}$ for a suitable $u \in P \setminus \mathbf{Z}(P)$ and $P \in \mathbf{CA}$ implies that $\mathbf{C}_P(u)$ is abelian. Thus $\mathbf{C}_K(x)$ is abelian and it follows that K is a **CA**-group.

(ii) Suppose, first, that K is an **F**-group and $x, y \in P \setminus \mathbf{Z}(P)$ satisfy $\mathbf{C}_P(x) \leq \mathbf{C}_P(y)$. Since $P \leq K$, it follows that $x, y \in K \setminus \mathbf{Z}(K)$ and

$$\mathbf{C}_K(x) = \mathbf{C}_P(x) \times Z_{p'} \leq \mathbf{C}_P(y) \times Z_{p'} = \mathbf{C}_K(y).$$

But $K \in \mathbf{F}$, so $\mathbf{C}_K(x) = \mathbf{C}_K(y)$ and hence $\mathbf{C}_P(x) = \mathbf{C}_P(y)$, implying that P is an **F**-group.

Conversely, suppose that P is an **F**-group and $x, y \in K \setminus \mathbf{Z}(K)$ satisfy $\mathbf{C}_K(x) \leq \mathbf{C}_K(y)$. Then

$$\mathbf{C}_K(x) = \mathbf{C}_P(u) \times Z_{p'} \leq \mathbf{C}_K(y) = \mathbf{C}_P(v) \times Z_{p'},$$

for suitable $u, v \in P \setminus \mathbf{Z}(P)$. Hence $\mathbf{C}_P(u) \leq \mathbf{C}_P(v)$, and $P \in \mathbf{F}$ implies that $\mathbf{C}_P(u) = \mathbf{C}_P(v)$. Thus $\mathbf{C}_K(x) = \mathbf{C}_K(y)$ and it follows that K is an $\mathbf{F}$-group.

(iii) Suppose, first, that K is a $\mathbf{CH}$-group and $x, y \in P \setminus \mathbf{Z}(P)$ satisfy $xy = yx$. Since $P \leq K$, it follows that $x, y \in K \setminus \mathbf{Z}(K)$ and $K \in \mathbf{CH}$ implies that $|\mathbf{C}_K(x)| = |\mathbf{C}_K(y)|$. Hence $|\mathbf{C}_P(x) \times Z_{p'}| = |\mathbf{C}_P(y) \times Z_{p'}|$, which implies that $|\mathbf{C}_P(x)| = |\mathbf{C}_P(y)|$. It follows that P is a $\mathbf{CH}$-group.

Conversely, suppose that P is a $\mathbf{CH}$-group and $x, y \in K \setminus \mathbf{Z}(K)$ satisfy $xy = yx$. As $K = PZ$, there exist $u, v \in P \setminus \mathbf{Z}(P)$ and $z_1, z_2 \in Z_{p'}$ such $x = uz_1$, $y = vz_2$ and $uv = vu$. Since $P \in \mathbf{CH}$, it follows that $|\mathbf{C}_P(u)| = |\mathbf{C}_P(v)|$ and consequently $|\mathbf{C}_K(x)| = |\mathbf{C}_P(u) \times Z_{p'}| = |\mathbf{C}_P(v) \times Z_{p'}| = |\mathbf{C}_K(y)|$, which implies that K is a $\mathbf{CH}$-group. □

We now state Theorem A, which presents the classification theorems of Schmidt [S] (see also [S1, Theorem 9.3.12]) and Rebmann [R] together. As mentioned above, we state condition (III) in a different form. Our form corresponds to condition (b) of Proposition 3.1, which by that proposition is equivalent to the statements of the above authors, which correspond to condition (a) of Proposition 3.1. In Theorem A, the structure of $\mathbf{CA}$-groups and $\mathbf{F}$-groups is determined, up to that of $\mathbf{CA}$-groups and $\mathbf{F}$-groups of prime-power order, respectively. We denote by $\pi(G)$ the set of primes dividing the order of the group G.

THEOREM A. *Let G be a nonabelian group and write $Z = \mathbf{Z}(G)$. Then G is a* $\mathbf{CA}$-*group* ($\mathbf{F}$-*group*) *if and only if it is of one of the following types.*

(I) *G is nonabelian and has an abelian normal subgroup of prime index.*
(II) *G/Z is a Frobenius group with Frobenius kernel K/Z and Frobenius complement L/Z, where K and L are abelian.*
(III) *G/Z is a Frobenius group with Frobenius kernel K/Z and Frobenius complement L/Z, such that $K = PZ$, where P is a normal Sylow p-subgroup of G for some $p \in \pi(G)$, P is a* $\mathbf{CA}$-*group* ($\mathbf{F}$-*group*), *$\mathbf{Z}(P) = P \cap Z$ and $L = HZ$, where H is an abelian p'-subgroup of G.*
(IV) *$G/Z \simeq S_4$ and if V/Z is the Klein four group in G/Z, then V is nonabelian.*
(V) *$G = P \times A$, where P is a nonabelian* $\mathbf{CA}$-*group* ($\mathbf{F}$-*group*) *of prime-power order and A is abelian.*
(VI) *$G/Z \simeq \mathrm{PSL}(2, p^n)$ or $\mathrm{PGL}(2, p^n)$ and $G' \simeq \mathrm{SL}(2, p^n)$, where p is a prime and $p^n > 3$.*
(VII) *$G/Z \simeq \mathrm{PSL}(2, 9)$ or $\mathrm{PGL}(2, 9)$ and G' is isomorphic to the Schur cover of* $\mathrm{PSL}(2, 9)$.

We conclude this section with a proposition concerning $\mathbf{CA}$-groups. We denote the derived length of G by $\mathrm{dl}(G)$.

PROPOSITION 3.2. *The following statements hold.*

(a) *The group G is a* $\mathbf{CA}$-*group if and only if whenever $x, y \in G \setminus \mathbf{Z}(G)$ satisfy $xy = yx$, then $\mathbf{C}_G(x) = \mathbf{C}_G(y)$.*

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(b) *If G is a nilpotent* **CA***-group, then* $\mathrm{dl}(G) \le 2$ *and that bound is the best possible.*
(c) *If G is a solvable* **CA***-group, then* $\mathrm{dl}(G) \le 4$ *and that bound is the best possible.*

PROOF. (a) If G is a **CA**-group and $x, y \in G \setminus \mathbf{Z}(G)$ satisfy $xy = yx$, then $\mathbf{C}_G(x)$ is abelian and $y \in \mathbf{C}_G(x)$. Hence $\mathbf{C}_G(x) \le \mathbf{C}_G(y)$ and, by symmetry, also $\mathbf{C}_G(y) \le \mathbf{C}_G(x)$. Thus $\mathbf{C}_G(x) = \mathbf{C}_G(y)$, as claimed.

Conversely, suppose that $\mathbf{C}_G(x) = \mathbf{C}_G(y)$ whenever $x, y \in G \setminus \mathbf{Z}(G)$ and $xy = yx$. Let $z \in G \setminus \mathbf{Z}(G)$ and let $u \in \mathbf{C}_G(z) \setminus \mathbf{Z}(G)$. Since $uz = zu$, it follows from our assumptions that $\mathbf{C}_G(z) = \mathbf{C}_G(u)$ and hence $u \in \mathbf{Z}(\mathbf{C}_G(z))$. Thus $\mathbf{C}_G(z)$ is abelian, which implies that G is a **CA**-group.

(b) If G is a nilpotent **CA**-group, then either G is abelian or $\mathbf{Z}_2(G) > \mathbf{Z}(G)$. In the latter case, let $g \in \mathbf{Z}_2(G) \setminus \mathbf{Z}(G)$. Then, by Lemma 2.1, $G' \le \mathbf{C}_G(g)$ and hence G' is abelian. It follows that $\mathrm{dl}(G) \le 2$. On the other hand, each group of order p^3 for some prime p is a **CA**-group, so that bound is the best possible.

(c) Let G be a solvable **CA**-group. We may assume that G is nonabelian. Then, by Theorem A applied to **CA**-groups, G is of one of the types (I)–(V) of that theorem. If G is of type (I) or (II), then clearly $\mathrm{dl}(G) = 2$. If G is of type (III), then $G = PL$, where P is a nilpotent **CA**-group which is normal in G and L is abelian. It follows by (b) that $\mathrm{dl}(G) \le 3$. If G is of type (IV), then $\mathrm{dl}(G) \le 4$ since $\mathrm{dl}(S_4) = 3$. Finally, if G is of type (V), then it follows by (b) that $\mathrm{dl}(G) \le 2$. Thus in the solvable case we have $\mathrm{dl}(G) \le 4$. On the other hand, the group $\mathrm{GL}(2, 3)$ is a solvable **CA**-group of type (IV) and $\mathrm{dl}(\mathrm{GL}(2, 3)) = 4$, so that bound is the best possible. □

4. Main results

Our first main result determines the inclusion relations between the classes of groups dealt with in this paper.

THEOREM 4.1. *The following inclusion relations hold:*

$$\mathbf{CA} \subset \mathbf{CH} \subset \mathbf{F}.$$

PROOF. It follows by Proposition 3.2(a) that every **CA**-group is a **CH**-group.

We now prove that every **CH**-group is an **F**-group. Let G be a **CH**-group and let $x, y \in G \setminus \mathbf{Z}(G)$. Suppose that $\mathbf{C}_G(x) \le \mathbf{C}_G(y)$. Then $x \in \mathbf{C}_G(y)$ and since G is a **CH**-group, it follows that $|\mathbf{C}_G(x)| = |\mathbf{C}_G(y)|$. Thus $\mathbf{C}_G(x) = \mathbf{C}_G(y)$ and hence G is an **F**-group, as claimed.

It will be shown in Examples A and B that these inclusions are proper. □

EXAMPLE A. There exist **CH**-groups which are not **CA**-groups. Consider the group $G = (C_2 \times D_8) : C_2$ which can be described by the following presentation:

$$G = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = (ac)^2 = (bd)^2 = (cd)^2 = (abd)^2 \\ = (abc)^2 = (bc)^4 = 1\rangle.$$

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Observe that $G = \texttt{SmallGroup(32,49)}$ in [GAP]'s small groups library. It can be checked using [GAP] that G is a **CH**-group, but not a **CA**-group.

We also refer to Example C for another example of a **CH**-group that is not a **CA**-group.

EXAMPLE B. There exist **F**-groups which are not **CH**-groups. Consider the group

$$\begin{aligned} G = \langle a, b, c, d, e, f \mid a^2 &= b^2 = c^2 = d^2 = e^2 = f^2 = (ab)^2 = (bc)^2 \\ &= (cd)^2 = (de)^2 = (ef)^2 = (fa)^2 = (ad)^2 = (be)^2 = (cf)^2 \\ &= (abde)^2 = (bcef)^2 = (cdfa)^2 = 1\rangle. \end{aligned}$$

One can check, using [GAP], that G is an **F**-group of exponent 4. However, G is not a **CH**-group. To see that, observe that $|G| = 2^9$, $|\mathbf{Z}(G)| = 8$ and the class sizes of G are $\{1, 4, 8\}$. Hence there exist $x, y \in G \setminus \mathbf{Z}(G)$ such that $|\mathbf{C}_G(x)| = 2^7$ and $|\mathbf{C}_G(y)| = 2^6$. Thus $\mathbf{C}_G(x) \cap \mathbf{C}_G(y) > \mathbf{Z}(G)$ and there exists $g \in G \setminus \mathbf{Z}(G)$ such that $x, y \in \mathbf{C}_G(g)$. So, either $|\mathbf{C}_G(g)| \neq |\mathbf{C}_G(x)|$ or $|\mathbf{C}_G(g)| \neq |\mathbf{C}_G(y)|$ and hence G is not a **CH**-group.

We now state and prove our second main result. This is Theorem 4.2, in which we describe the structure of **CH**-groups using Theorems 4.1 and A. Theorem 4.2 is analogous to Theorem A, and it determines the structure of **CH**-groups up to that of **CH**-groups of prime-power order.

THEOREM 4.2. *Let G be a nonabelian group and write $Z = \mathbf{Z}(G)$. Then G is a* **CH**-*group if and only if it is of one of the following types.*

(I) *G is nonabelian and has an abelian normal subgroup of prime index.*
(II) *G/Z is a Frobenius group with Frobenius kernel K/Z and Frobenius complement L/Z, where K and L are abelian.*
(III) *G/Z is a Frobenius group with Frobenius kernel K/Z and Frobenius complement L/Z, such that $K = PZ$, where P is a normal Sylow p-subgroup of G for some $p \in \pi(G)$, P is a* **CH**-*group, $\mathbf{Z}(P) = P \cap Z$ and $L = HZ$, where H is an abelian p'-subgroup of G.*
(IV) *$G/Z \simeq S_4$ and if V/Z is the Klein four group in G/Z, then V is nonabelian.*
(V) *$G = P \times A$, where P is a nonabelian* **CH**-*group of prime-power order and A is abelian.*
(VI) *$G/Z \simeq \mathrm{PSL}(2, p^n)$ or $\mathrm{PGL}(2, p^n)$ and $G' \simeq \mathrm{SL}(2, p^n)$, where p is a prime and $p^n > 3$.*
(VII) *$G/Z \simeq \mathrm{PSL}(2, 9)$ or $\mathrm{PGL}(2, 9)$ and G' is isomorphic to the Schur cover of* $\mathrm{PSL}(2, 9)$.

PROOF. Assume, first, that G is a nonabelian **CH**-group. We shall prove that then G is one of the groups described in (I)–(VII). By Theorem 4.1, we know that G is an **F**-group. Hence we may apply Theorem A, which determines the structure of **F**-groups. In the following, we shall denote by (1)–(7) the conditions (I)–(VII) of

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Theorem A applied to **F**-groups, respectively. So we may assume that G satisfies one of the conditions (1)–(7) applied to **F**-groups.

Observe, first, that if G satisfies (1), (2), (4), (6) and (7), then G satisfies (I), (II), (IV), (VI) and (VII), respectively.

Next, suppose that G satisfies (3). We wish to show that G satisfies (III). We need only prove that P is a **CH**-group. Let $x, y \in P \setminus \mathbf{Z}(P) = P \setminus (Z \cap P)$ satisfy $xy = yx$. Since PZ/Z is the Frobenius kernel of G/Z, it follows that $\mathbf{C}_G(x), \mathbf{C}_G(y) \leq PZ$. Hence $\mathbf{C}_G(x) = \mathbf{C}_P(x) \times Z_{p'}$ and $\mathbf{C}_G(y) = \mathbf{C}_P(y) \times Z_{p'}$, where $Z_{p'}$ denotes the p-complement of Z. But G is assumed to be a **CH**-group and $x, y \in G \setminus Z$, so $|\mathbf{C}_G(x)| = |\mathbf{C}_G(y)|$. Thus $|\mathbf{C}_P(x)| = |\mathbf{C}_P(y)|$ and it follows that P is a **CH**-group, as required.

Finally, we show that if G satisfies (5), then it also satisfies (V). By (5), $G = P \times A$, where P is a nonabelian group of prime-power order and A is abelian. Since G is a **CH**-group, it follows immediately that P is a **CH**-group and thus G satisfies (V). This completes the proof in one direction.

Suppose, now, that G is a nonabelian group satisfying one of the conditions (I)–(VII). We wish to show that then G is a **CH**-group.

It follows by Theorem A, that groups satisfying (I), (II), (IV), (VI) and (VII) are **CA**-groups and hence, by Theorem 4.1, they are **CH**-groups, as required.

Suppose, next, that G satisfies (III). Let $x, y \in G \setminus Z$ satisfy $xy = yx$. We need to show that $|\mathbf{C}_G(x)| = |\mathbf{C}_G(y)|$.

If $x \in G \setminus K$, then also $y \in G \setminus K$. Thus x and y are elements of conjugates of L, say $x \in L^u \setminus Z$ and $y \in L^v \setminus Z$ for suitable $u, v \in G$. It follows that $\mathbf{C}_G(x) \leq L^u$ and $\mathbf{C}_G(y) \leq L^v$ and since L^u, L^v are abelian, we may conclude that $\mathbf{C}_G(x) = L^u$ and $\mathbf{C}_G(y) = L^v$. But $|L^u| = |L^v|$, so $|\mathbf{C}_G(x)| = |\mathbf{C}_G(y)|$, as required.

So assume that $x \in K \setminus Z = PZ \setminus Z$ and hence also $y \in PZ \setminus Z$. Then $\mathbf{C}_G(x), \mathbf{C}_G(y) \leq PZ$ and if $x = uz_1$, $y \in vz_2$, where $u, v \in P$ and $z_1, z_2 \in Z$, then $\mathbf{C}_G(x) = \mathbf{C}_P(u) \times Z_{p'}$ and $\mathbf{C}_G(y) = \mathbf{C}_P(v) \times Z_{p'}$, where $Z_{p'}$ denotes the p-complement of Z. As $\mathbf{Z}(P) = P \cap Z$, it follows that $u, v \in P \setminus \mathbf{Z}(P)$ and as $xy = yx$, we have $uv = vu$. Since $P \in$ **CH**, it follows that $|\mathbf{C}_P(u)| = |\mathbf{C}_P(v)|$, which implies that $|\mathbf{C}_G(x)| = |\mathbf{C}_G(y)|$, as required. Thus, if G satisfies (III), then G is a **CH**-group.

Suppose, finally, that G satisfies (V). Then $G = P \times A$, where P is a nonabelian **CH**-group of prime-power order and A is abelian. It follows easily that G is a **CH**-group in this case too. □

The relations between nonsolvable **CH**-groups, **CA**-groups and **F**-groups are described in the following remark.

REMARK A. It is easy to see from Theorems A and 4.2 that the nonsolvable **CH**-groups coincide with the nonsolvable **CA**-groups and with the nonsolvable **F**-groups.

In particular, Theorem A and Theorem 4.2 yield the following corollary.

COROLLARY 4.3. *If G is a nonsolvable* **CH**-*group (***F**-*group,* **CA**-*group), then G is a central extension of either* $\mathrm{PSL}(2, q)$ *or* $\mathrm{PGL}(2, q)$*, for some prime-power $q > 3$.*

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On the other hand, it follows from Theorems A and 4.2 that nilpotent **CH**-groups, **CA**-groups and **F**-groups have the following similar structure.

COROLLARY 4.4. *If G is a nonabelian nilpotent* **CH**-*group* (**CA**-*group*, **F**-*group*), *then $G = P \times A$, where P is a nonabelian* **CH**-*group* (**CA**-*group*, **F**-*group*) *of prime-power order and A is abelian.*

PROOF. If G is a nonabelian nilpotent **CH**-group (**CA**-group, **F**-group), then by Theorem 4.2 (Theorem A) either G is nonabelian and has an abelian normal subgroup of prime index, or $G = P \times A$, where P is a nonabelian **CH**-group (**CA**-group, **F**-group) of prime-power order and A is abelian. It is clear that nilpotency implies that also in the former case, the structure of G is as described in the latter case. □

5. Nilpotent and solvable CH-groups

The main results in this section are upper bounds for the derived length of nilpotent **CH**-groups (see Theorem 5.2) and of solvable **CH**-groups (see Theorem 5.3).

We start with an auxiliary proposition, the proof of which uses Lemmas 2.1, 2.2 and arguments from [Is].

PROPOSITION 5.1. *Let G be a nilpotent* **CH**-*group. Then there exists a normal subgroup M of G such that $G' \le M$ and M' is abelian. Moreover, $[M', M] \le \mathbf{Z}(G)$.*

PROOF. We may clearly assume that G is nonabelian. Write $Z = \mathbf{Z}(G)$ and let $g \in \mathbf{Z}_2(G) \setminus Z$. Denote $|\mathbf{C}_G(g)| = n$. For every $h \in G$ we have $g^h = g[g, h]$ and $[g, h] \in Z$, which implies that $\langle g^G \rangle$ is an abelian normal subgroup of G. Let A be a maximal abelian normal subgroup of G containing $\langle g^G \rangle$. Clearly $Z \le A$. Since G is nilpotent, A satisfies $\mathbf{C}_G(A) = A$. Moreover, G being a **CH**-group implies that $|\mathbf{C}_G(y)| = n$ for every $y \in A \setminus Z$.

Consider now the set

$$S = \{x \in G : |\mathbf{C}_G(x)| = n\}$$

and define $M = \langle S \rangle$. If $x \in S$ and $a \in A$, then Lemma 2.2 implies that $|\mathbf{C}_G([a, x])| > |\mathbf{C}_G(x)| = n$ and hence $[a, x] \in Z$. It follows that $[A, M] \le Z$, so $[A, M, M] = 1$ and the three-subgroups lemma yields $[M', A] = 1$. Thus $M' \le \mathbf{C}_G(A) = A$, which implies that M' is abelian and $[M', M] \le [A, M] \le Z$.

It remains only to show that $G' \le M$. Since G is a **CH**-group, we know that $\mathbf{C}_G(g) \setminus Z \subseteq S$ and hence $\mathbf{C}_G(g) \subseteq M \cup Z$. As $g \notin Z$, it follows that $\mathbf{C}_G(g) \le M$ and Lemma 2.1 yields

$$G' \le \mathbf{C}_G(\mathbf{Z}_2(G)) \le \mathbf{C}_G(g) \le M,$$

as required. □

THEOREM 5.2. *If G is a nilpotent* **CH**-*group, then $\mathrm{dl}(G) \le 3$ and this bound is the best possible.*

PROOF. By Proposition 5.1 there exists $M \leq G$ such that $G' \leq M$ and $M'' = 1$. Hence $G''' = 1$, which implies that $\mathrm{dl}(G) \leq 3$. It will be shown in Example C that this bound is the best possible. $\square$

EXAMPLE C. There exist nilpotent **CH**-groups G such that $\mathrm{dl}(G) = 3$. Consider the group $G = \texttt{SmallGroup}(5^7, 348)$ in [GAP]'s small groups library. It is defined as

$$G = \langle a, b \mid a^5 = b^5 = [b, c] = [c, e] = [d, e] = [a, f] = [b, f] = [c, f] \\ = [d, f] = [e, f] = [a, g] = [b, g] = [c, g] = [d, g] = [e, g] = 1 \rangle,$$

where

$$c = [a, b], \quad d = [a, c], \quad e = [a, d], \quad f = [a, e], \quad g = [b, d].$$

It can be checked using [GAP] that G is a **CH**-group of order 5^7 and that $\mathrm{dl}(G) = 3$. Moreover, we observe by Proposition 3.2(b) that G is not a **CA**-group.

Theorems 4.2 and 5.2 yield an upper bound for the derived length of solvable **CH**-groups.

THEOREM 5.3. *If G is a solvable* **CH**-*group, then* $\mathrm{dl}(G) \leq 4$ *and that bound is the best possible.*

PROOF. We may assume that G is nonabelian. Thus, by Theorem 4.2, G is of one of the types (I)–(V) of that theorem. If G is of type (I) or (II), then clearly $\mathrm{dl}(G) = 2$. If G is of type (III), then $G = PL$, where P is a nilpotent **CH**-group that is normal in G and L is abelian. It follows by Theorem 5.2 that $\mathrm{dl}(G) \leq 4$. If G is of type (IV), then $\mathrm{dl}(G) \leq 4$ since $\mathrm{dl}(S_4) = 3$. Finally, if G is of type (V), then it follows by Theorem 5.2 that $\mathrm{dl}(G) \leq 3$. Thus in the solvable case we have $\mathrm{dl}(G) \leq 4$. On the other hand, the group $\mathrm{GL}(2, 3)$ is a solvable **CH**-group of type (IV) and $\mathrm{dl}(\mathrm{GL}(2, 3)) = 4$, so that bound is the best possible. $\square$

Concerning an upper bound for the nilpotency class of nilpotent **CH**-groups, we have the following (negative) remark.

REMARK B. There exists no upper bound for the nilpotency class of nilpotent **CA**-groups, and consequently no such bound exists for nilpotent **CH**-groups either. Indeed, more precisely, for any prime p, the nilpotency class of **CA**-groups of p-power order is unbounded. For instance, the wreath product $C_{p^n} \wr C_p$ is a **CA**-group, and its nilpotency class is $n(p-1)+1$, where n can be any positive integer.

6. CH-groups and F-groups of prime-power order

We conclude this paper with two results concerning **CH**-groups and **F**-groups of prime-power order. Here p denotes a prime number.

Our first result deals with the derived length of **F**-groups, whose order is a power of 2 (see Corollary 6.2). For this result, as well as for the next one concerning the

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nilpotency class of **CH**-groups, we need the following lemma, which relies upon a result of Mann in [M].

LEMMA 6.1. *Let G be a p-group. If G is an* **F***-group and*

$$\exp(G/\mathbf{Z}(G)) > p,$$

then G is a **CA***-group. Moreover, if G is nonabelian, then its class sizes are $\{1, p, p^a\}$ for some integer $a > 1$.*

PROOF. We may assume that G is nonabelian. By [M, theorem on p. 82], G has an abelian normal subgroup A of index p. If $x \in A \setminus \mathbf{Z}(G)$, then $\mathbf{C}_G(x) = A$. So, if $y \in G \setminus A$, then $y^p \in \mathbf{Z}(G)$ and $\mathbf{C}_G(y) = \mathbf{Z}(G)\langle y\rangle$. Hence, $\mathbf{C}_G(g)$ is abelian for every $g \in G \setminus \mathbf{Z}(G)$ and G is a **CA**-group. Moreover, if $g \in G$, then

$$|\mathbf{C}_G(g)| \in \{|G|, |A|, p|\mathbf{Z}(G)|\}.$$

As $\exp(G/\mathbf{Z}(G)) > p$, we must have $p|\mathbf{Z}(G)| < |A|$. □

COROLLARY 6.2. *Let G be a 2-group. If G is an* **F***-group, then $\mathrm{dl}(G) \leq 2$. More generally, this inequality holds for nilpotent* **CA***-groups,* **CH***-groups and* **F***-groups G, with $G/\mathbf{Z}(G)$ of even order.*

PROOF. Let G be an **F**-group of a 2-power order. If $\exp(G/\mathbf{Z}(G)) > 2$ then, by Lemma 6.1, G is a **CA**-group and hence $\mathrm{dl}(G) \leq 2$ by Proposition 3.2. If $\exp(G/\mathbf{Z}(G)) \leq 2$, then $G/\mathbf{Z}(G)$ is abelian and again $\mathrm{dl}(G) \leq 2$.

It follows, by Theorem 4.1, that if G is either a **CA**-group or a **CH**-group of 2-power order, then $\mathrm{dl}(G) \leq 2$. Finally, if G is a nonabelian nilpotent **CA**-group, **CH**-group or **F**-group, with $G/\mathbf{Z}(G)$ of even order, then, by Corollary 4.4, $G = P \times A$, where P is a nonabelian 2-group which is a **CA**-group, **CH**-group or **F**-group, respectively, and A is abelian. Hence it follows from our opening results that $\mathrm{dl}(G) \leq 2$. □

Our last result deals with the nilpotency class of **CH**-groups of prime-power order. By Remark B, the nilpotency class of **CA**-groups of p-power order is unbounded. So it is quite surprising that the nilpotency class of **CH**-groups of p-power order, which are not **CA**-groups, is bounded by $2p$.

THEOREM 6.3. *Let G be a p-group. If G is a* **CH***-group, then one of the following holds:*

(a) *G is a* **CA***-group; or*
(b) *the nilpotency class of G is at most $2p$.*

PROOF. Suppose that G is a **CH**-group. We may clearly assume that $Z = \mathbf{Z}(G) < G$.

If $\exp(G/Z) > p$, then G is a **CA**-group by Lemma 6.1. Hence it suffices to prove that if $\exp(G/Z) = p$, then (b) holds.

So suppose that $\exp(G/Z) = p$. By Proposition 5.1, there exists a normal subgroup M of G such that $G' \leq M$, M' is abelian and $[M', M] \leq Z$. Write $\overline{G} = G/Z$ and

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use the bar convention. We first apply Lemma 2.3 to the factor group $\overline{G}/\overline{M'}$ with respect to the abelian normal subgroup $\overline{M}/\overline{M'}$, recalling that $\overline{G}' \leq \overline{M}$. It follows that $[\overline{M}/\overline{M'}, [\overline{G}/\overline{M'}]_{p-1}] = 1$, so $[\overline{M}, [\overline{G}]_{p-1}] \leq \overline{M'}$.

Next, we apply Lemma 2.3 to $\overline{G}$, with respect to the abelian normal subgroup $\overline{M'}$. Observe that $\overline{M}$ centralizes $\overline{M'}$ and hence $\overline{G}/\mathbf{C}_{\overline{G}}(\overline{M'})$ is abelian. By Lemma 2.3, $[\overline{M'}, [\overline{G}]_{p-1}] = 1$. As $[\overline{M}, [\overline{G}]_{p-1}] \leq \overline{M'}$, we obtain $[\overline{M}, [\overline{G}]_{2p-2}] = 1$ and hence $[M, [G]_{2p-2}] \leq Z$. Finally, recalling that $G' \leq M$, we conclude that

$$[G, [G]_{2p}] = [G', [G]_{2p-1}] \leq [[M, [G]_{2p-2}], G] = 1$$

and hence the nilpotency class of G is at most $2p$. □

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References

[GAP] GAP– Groups, Algorithms and Programming, Version 4.4.12 (2008), http://www.gap-system.org.

[Is] I. M. Isaacs, 'Subgroups generated by small classes in finite groups', *Proc. Amer. Math. Soc.* **136** (2008), 2299–2301.

[Ish] K. Ishikawa, 'On finite p-groups which have only two conjugacy lengths', *Israel J. Math.* **129** (2002), 119–123.

[I] N. Ito, 'On finite groups with given conjugate type, I', *Nagoya J. Math.* **6** (1953), 17–28.

[M] A. Mann, 'Conjugacy classes in finite groups', *Israel J. Math.* **31** (1978), 78–84.

[M1] A. Mann, 'Elements of minimal breadth in finite p-groups and Lie algebras', *J. Aust. Math. Soc.* **81** (2006), 209–214.

[R] J. Rebmann, '**F**-Gruppen', *Arch. Math.* **22** (1971), 225–230.

[Ro] D. J. S. Robinson, *Finiteness Conditions and General Soluble Groups, Part 2* (Springer, Berlin, 1972).

[Ro1] D. J. S. Robinson, *A Course in the Theory of Groups* (Springer, Berlin, 1982).

[S] R. Schmidt, 'Zentralisatorverbände endlicher Gruppen', *Rend. Sem. Mat. Univ. Padova* **44** (1970), 97–131.

[S1] R. Schmidt, *Subgroup Lattices of Groups* (De Gruyter, Berlin, 1994).

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