ON CAYLEY'S PARAMETERIZATION

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1. Introduction. A matrix P with elements from an arbitrary field $\mathfrak{F}$ is called a *cogredient automorph* (c.a.) of a symmetric matrix A if $P'AP = A$, where P' is the transpose of P. A fundamental theorem concerning cogredient automorphs is:

THEOREM (Cayley). *If A is a non-singular symmetric matrix and if Q is a skew-symmetric matrix such that $A + Q$ is non-singular, then*

$$P = (A + Q)^{-1} (A - Q) \tag{1}$$

is a c.a. *of A and $I + P$ is non-singular.*

Conversely, if P is a c.a. *of A such that $I + P$ is non-singular, then there exists a unique skew-symmetric matrix Q such that P can be expressed by means of equation* (1).

The main purpose of this paper is to demonstrate the following generalization of Cayley's theorem as applied to the real field. (Henceforth all matrices are assumed to be real unless otherwise stated.)

THEOREM 1. *If A is a (not necessarily non-singular) symmetric matrix and if Q is a skew-symmetric matrix such that $A + Q$ is non-singular, then equation* (1) *defines a* c.a. *P of A whose determinant is $+ 1$ and having the property that A and $I + P$ span the same row space.*

Conversely, if P is a c.a. *of A whose determinant is $+ 1$ and if P has the property that $I + P$ and A span the same row space, then there exists a skew-symmetric matrix Q such that P is given by equation* (1).

The matrix Q is not unique. However, the size of the family of matrices Q which yield a particular c.a. P of A will be found and a set of necessary and sufficient conditions for two skew-symmetric matrices to yield the same c.a. will be given. A simple example will be included to show that Theorem 1 is false over a field of characteristic two.

2. Proof of the theorem. The first part of the theorem is immediate. Let $A + Q = U$, $A - Q = V$. Then (see **2)** $A = \frac{1}{2}(U + V)$ and

$$P'AP = \tfrac{1}{2}\, UV^{-1}\, (U + V)U^{-1}\, V = A.$$

Received February 6, 1957; presented to the American Mathematical Society, October 22, 1955. This paper constitutes a portion of a thesis submitted to the University of Wisconsin for the Ph.D. degree. The author wishes to thank Professor C. C. MacDuffee for his advice and encouragement.

The author also wishes to thank the referee for suggestions which shortened the proof of Theorem 4.

Furthermore $|P| = |(A+Q)^{-1}||(A-Q)| = |(A+Q)^{-1}||(A+Q)'| = +1$ and $I + P = (A+Q)^{-1}(2A)$. Thus $I+P$ and A span the same row spaces.

Proof of the converse. In order to facilitate the construction of a skew-symmetric matrix Q satisfying equation (1), we shall first simplify the forms of P and A. This can be done by repeated application of the following lemma.

LEMMA 1. *Let U be an orthogonal matrix. Then $U'PU$ is a* c.a. *of $U'AU$ if and only if P is a* c.a. *of A. Equation* (1) *holds if and only if*

$$U'PU = (U'AU + U'QU)^{-1}(U'AU - U'QU). \tag{2}$$

Moreover, $|P| = |U'PU|$ and $I + P$ spans the same row space as A if and only if $I + U'PU$ spans the same row space as $U'AU$.

When it is convenient to do so, we shall specify U and replace P, A and Q by $U'PU$, $U'AU$ and $U'QU$ respectively. In an effort to keep the notation as simple as possible, we shall refer to $U'PU$, $U'AU$ and $U'QU$ simply as P, A and Q whenever it is clear from the context what these symbols mean.

Let A be an arbitrary symmetric matrix of order n and rank r. Since there is an orthogonal matrix U such that

$$U'AU = \begin{bmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \lambda_r & & & \\ & & & \ddots & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix} \qquad (\lambda_i \neq 0),$$

and all the remaining elements are 0, we shall apply Lemma 1 and assume that A is in this form.

Equation (1) is equivalent to the two conditions

$$Q(I+P) = A(I-P) \tag{3'}$$
$$|A+Q| \neq 0. \tag{3''}$$

Let P and A be partitioned as follows:

$$P = \begin{bmatrix} B & E \\ C & F \end{bmatrix}, \quad A = \begin{bmatrix} d & 0 \\ 0 & 0 \end{bmatrix},$$

where B and d are of order r. Since $I + P$ spans the same row space as A, we must have $E = 0$, $F = -I_1$ (where I_1 is the identity matrix of order $n-r$) and the rank of the n by r array consisting of the first r columns of $I+P$ must be r. Furthermore, since P is a c.a. of A, $B'dB = d$; also $|P| = +1$ implies that $|B| = |F|$. Equation (3′) has become

$$Q \cdot \begin{bmatrix} I_2 + B & 0 \\ C & 0 \end{bmatrix} = \begin{bmatrix} d(I_2 - B) & 0 \\ 0 & 0 \end{bmatrix}, \tag{4}$$

where I_2 is the identity matrix of order r.

https://doi.org/10.4153/CJM-1957-063-9 Published online by Cambridge University Press

Let the rank of $I_2 + B$ be s. Since the rank of $I + P$ is r, there is a rearrangement of the rows of $I_2 + B$ and of C such that the matrix formed by the last s rows of $I_2 + B$ and the first $r - s$ rows of C is non-singular. This rearrangement can be carried out using Lemma 1 without disturbing the form of A, as there are orthogonal matrices u and v of orders r and $n - r$ respectively, whose rows are permutations of the rows of the identity matrices I_2 and I_1 and which effect the desired rearrangements when operating on $I_2 + B$ and C respectively on the left. Let $U = u \dot{+} v$. After applying Lemma 1 once again, and denoting $u'Bu$ by B, $u'du$ by d and $v'Cu$ by C, equations (3′), (3″) and (4) remain unchanged. It is to be noted here for subsequent use that the set of principal submatrices of $I_2 + B$ is invariant under a similarity transformation by $u \dot{+} v$.

Now partition $I + P$ into

$$\begin{bmatrix} I_2 + B & 0 \\ C & 0 \end{bmatrix} = \begin{bmatrix} G & 0 \\ H & 0 \\ G_1 & 0 \end{bmatrix},$$

where H is the non-singular matrix constructed above.

By two transformations similar to those described by Lemma 1, G and G_1 may be eliminated. It is possible to eliminate G_1 without disturbing the right side of equation (3′), for there is a matrix

$$V = \begin{bmatrix} I_3 & 0 \\ V_1 & I_4 \end{bmatrix},$$

where I_3 and I_4 are identity matrices of orders $2r - s$ and $n - 2r + s$ respectively, such that

$$V(I + P) = \begin{bmatrix} G & 0 \\ H & 0 \\ 0 & 0 \end{bmatrix}.$$

Clearly, $2r - s \geqslant r$ and hence $(V')^{-1} A = A$. Thus equation (3′) becomes

$$(V')^{-1} Q V^{-1} \cdot V(I + P) = (V')^{-1} A (I - P) = A(I - P).$$

This process is repeated once again to eliminate G and, at the same time, to replace H by an I_2 which is more conveniently positioned. Let I_5 denote the identity matrix of order $r - s$ and define

$$M = \begin{bmatrix} 0 & H^{-1} & 0 \\ I_5 & -GH^{-1} & 0 \\ 0 & 0 & I_4 \end{bmatrix}.$$

Then $MV(I + P) = I_2 \dot{+} 0$ and so we have

$$((MV)')^{-1} Q (MV)^{-1} \cdot (MV)(I + P) = Q_1(I_2 \dot{+} 0) = (M')^{-1} A (I - P) \tag{5}$$

where $Q_1 = ((MV)')^{-1} Q (MV)^{-1}$. A direct computation shows that

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$$(M')^{-1}A(I-P) = \begin{bmatrix} (I_2+B)'d(I_2-B) & 0 \\ K & 0 \\ 0 & 0 \end{bmatrix},$$

where the $r-s$ by r array K consists of the first $r-s$ rows of $d(I_2-B)$. Since B is a c.a. of d, $(I_2+B)'d(I_2-B)$ is skew-symmetric.

The problem has now been reduced to the construction of a skew-symmetric matrix Q which satisfies the conditions (3″) and (5). Equation (5) uniquely defines the first r rows and the first r columns of Q_1 but places no further restrictions on it. Hence, if such a matrix Q_1 exists, it must be of the form

$$\begin{bmatrix} (I_2+B)'d(I_2-B) & -K' & 0 \\ K & X & -Y' \\ 0 & Y & Z \end{bmatrix}$$

and it only remains to find matrices X, Y and Z satisfying the two conditions:

(i) X and Z are skew-symmetric matrices of orders $r-s$ and $n-2r+s$ respectively,

(ii) $|A_1+Q_1| \neq 0$, where A_1 is defined to be $((MV)')^{-1}A(MV)^{-1}$. By simplifying A_1+Q_1, it will be shown that X and Y are completely arbitrary (except for the restriction that X is skew-symmetric) but that Z must also be non-singular. A computation shows that

$$A_1+Q_1 = \begin{bmatrix} 2(I_2+B)'d & K_1' & 0 \\ 2[\delta \quad 0] & \delta+X & -Y' \\ 0 & Y & Z \end{bmatrix},$$

where δ is the uppermost principal submatrix of order $r-s$ of d and $[\delta\ 0]$ is the $r-s$ by r array

$$\begin{bmatrix} \lambda_1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \lambda_2 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \lambda_{r-s} & \cdot & \cdot & 0 \end{bmatrix}$$

and where K_1 consists of the first $r-s$ rows of $2dB$, i.e., K_1' consists of the first $r-s$ columns of $2B'd$. By using a series of elementary transformations, it can be shown that A_1+Q_1 is equivalent to

$$\begin{bmatrix} 0 & L_1 & -2\delta & 0 \\ 0 & L' & 0 & 0 \\ 2\delta & 0 & -\delta+X & -Y' \\ 0 & 0 & Y & Z \end{bmatrix}$$

where L is the lower right-hand principal submatrix of $2d(I_2+B)$ of order s. It is not necessary to define L_1 explicity.

By the Laplace development of the determinant, we have $|A_1+Q_1| = \pm|2\delta|^2|L'||Z|$. It is clear that $|2\delta| \neq 0$ and hence the proof of the theorem will be complete when it is shown that

(6′) the order of Z is even,

(6″) $|L| \neq 0$.

Condition (6′) follows directly from

LEMMA 2. *Let B be a* c.a. *of the non-singular matrix d and let the multiplicity of* -1 *as a root of B be α. Then* $|B| = (-1)^{\alpha}$.

Since B is a c.a. of d, $B = d^{-1}(B')^{-1} d$ and $xI - B = d^{-1}(xI - (B')^{-1})d$, that is, B and $(B')^{-1}$ have the same characteristic equations and hence the same characteristic roots. Thus, the characteristic roots of B, other than $+1$ and -1 occur in reciprocal pairs. Since $|B|$ is a product of these roots, the lemma follows.

Let us return to condition (6′). The order of $F = -I_1$ is $n - r$ and $|F| = (-1)^{n-r}$. Furthermore, -1 appears as a root of B with multiplicity $r - s$ and hence, by Lemma 2, $|B| = (-1)^{r-s}$. Moreover, it has been shown that $|P| = |F|\cdot|B| = +1$ and so

$$(n - r) + (r - s) = n - s$$

is even. The order of Z is

$$n - 2r + s = n - s - 2(r - s)$$

and thus is also even. We have shown that a non-singular skew-symmetric matrix Z always exists.

It remains to show that condition (6″) is always satisfied and this will constitute the second part of the proof.

3. Pr and CPr matrices. It is now possible to prove a corollary to the first half of the proof of the converse of Theorem 1 which will be used as a lemma to the second half.

The first application of Lemma 1 transformed A into $d \dotplus 0$. It is not necessary to determine what effect it had on B. However, the second application of Lemma 1, using $U = u \dotplus v$, has the property that it leaves the set of principal submatrices of $I_2 + B$ invariant. Thus, once A has been reduced to the form $d \dotplus 0$, the set of principal submatrices is fixed. We selected an arbitrary set of linearly independent rows of $I_2 + B$ and then showed that, for the given c.a. P of A, a skew-symmetric matrix Q satisfying conditions (3′) and (3″) can be found if and only if the principal submatrix of these rows, which has been denoted by L, is non-singular; that is, the non-singularity of L is independent of the particular set of rows of $I_2 + B$ selected. Furthermore, if B is a c.a. of d, there is some n for which a c.a. P of A exists which satisfies the hypotheses of the theorem and which is in the form

$$\begin{bmatrix} B & 0 \\ C & -I_1 \end{bmatrix};$$

that is, this discussion pertains to all B. Thus, we have proved part (a) of the

COROLLARY. (a) *Let B be a* c.a. *of a non-singular diagonal matrix d of order r. Let $I_2 + B$ have rank s and let $\mathbf{X}_1$ be a set of s linearly independent rows of $I_2 + B$ such that the principal submatrix of $I_2 + B$ determined by these rows is non-singular. If $\mathbf{X}_2$ is any set of s linearly independent rows of $I_2 + B$, then the principal submatrix of $I_2 + B$ determined by these rows is non-singular.*

(b) *Let b be a* c.a. *of d. If $\mathbf{Y}$ is a set of s linearly independent rows of $b^{-1}(I_2 + B)b$, then the principal submatrix determined by these rows is non-singular.*

To prove part (b), we define $B_1 = b \dotplus I$. The matrix P has a parameterization in the form of equation (1) if and only if $P_1 = B_1^{-1} PB_1$ has such a parameterization, for $P_1 = B_1^{-1} (A + Q)^{-1} (B_1')^{-1} (B_1') (A - Q) B_1 = (A + Q_1)^{-1} (A - Q_1)$, where $Q_1 = B_1'QB_1$. Moreover, $b^{-1}(I_2 + B)b = I_2+b^{-1}Bb$ in $I + P_1$ corresponds to $I_2 + B$ in $I + P$ and thus the principal submatrix of a set of s linearly independent rows is non-singular in one if and only if it is non-singular in the other.

Schwerdtfeger **(1)** has called a matrix of rank r which has a principal non-singular submatrix of order r a Pr matrix. We shall define a CPr matrix to be a matrix of rank r with the following property: whenever a set of s rows is linearly independent, then the set of the corresponding s columns is also linearly independent and conversely; that is, the same set of rows of the transpose of the matrix is linearly independent. Equivalently, a CPr matrix can be defined as a matrix of rank r such that the principal submatrix determined by any set of r linearly independent rows is non-singular. Clearly, a CPr matrix is always a Pr matrix. The preceding corollary asserts that if B is a c.a. of a non-singular diagonal matrix d and if $I_2 + B$ is a Pr matrix, then $I_2 + B$ is a CPr matrix. Theorem 1 will follow when we have proved

THEOREM 2. *If B is a* c.a. *of a non-singular diagonal matrix d, then $I_2 + B$ is a Pr matrix.*

It is sufficient to prove this theorem for the case where the non-zero elements of d are each $+ 1$ or $- 1$, since there is a non-singular diagonal matrix f such that fdf is a diagonal matrix whose diagonal elements are each $+ 1$ or $- 1$. Then $f^{-1} Bf$ is a c.a. of fdf and $I_2 + f^{-1} Bf$ is a Pr matrix if and only if $I_2 + B$ is. Hence, for the remainder of the proof we can assume that d is already in this form.

Williamson **(3)** has called a c.a. of such a matrix d, a quasi-unitary matrix and he has given a comprehensive discussion of the problem of reducing a quasi-unitary matrix to a canonical form by a quasi-unitary similarity transformation. He has shown that, with at most an interchange of the rows and the corresponding columns, B can be made quasi-unitarily similar to a matrix of the form

$$A_0 \dotplus A_1 \dotplus \ldots \dotplus A_k \dotplus A_{k+1} \dotplus \ldots \dotplus A_{k+m},$$

where no root of A_0 is $- 1$, where $A_1, \ldots, A_k$ are each of odd order (say the order of A_h is $2a_h + 1$, $h = 1, 2, \ldots, k$) and A_h $(1 \leqslant h \leqslant k)$ has

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$$(\lambda + 1)^{2a_h+1}$$

as its only elementary divisor, and where $A_{k+1}, \ldots, A_{k+m}$ are each of order divisible by 4 (say the order of A_{k+h} is $4b_{k+h}$; $h = 1,2, \ldots, m$) and A_{k+h} $(1 \leqslant h \leqslant m)$ has

$$(\lambda + 1)^{2b_{k+h}}$$

as an elementary divisor of multiplicity two. By the above corollary, the property of being a *Pr* matrix is invariant under such a transformation. Let I_h be the identity matrix whose order is equal to that of A_h. This transformation does not effect the identity matrix and since $I_2 + B$ is a *Pr* matrix if and only if $I_h + A_h$ is a *Pr* matrix for each h, we may consider each $I_h + A_h$ separately.

Case I: $I_0 + A_0$. Since A_0 does not have -1 as a characteristic root, $I_0 + A_0$ is non-singular and hence is a *Pr* matrix.

Case II: $I_h + A_h$, $(1 \leqslant h \leqslant k)$. For convenience we shall drop the subscript h from I, A and a. Since $I + A$ has nullity 1, we wish to show that $I + A$ has a principal non-singular submatrix of order $a - 1$. Let W be the matrix of the same order as A which has 1's just above the main diagonal and zeros elsewhere, that is, $W = [\delta_{i,j-1}]$.[2] Then the Jordan form for A is $-I - W$. In particular, Williamson has shown that there exist matrices D and T such that

$$TDT' = e\left[\begin{array}{cc|c|cc} & & & & 1 \\ & & & -1 & \\ & & & \cdot^{\cdot^{\cdot}} & S \\ \hline & & (-1)^a & & \\ \hline & \cdot^{\cdot^{\cdot}} & & & \\ -1 & S' & & & \\ 1 & & & & \end{array}\right] = e\Delta \quad (e = \pm 1) \tag{7}$$

and $T^{-1}(-I - W)T = A$, where D is a matrix having the same form as d (a diagonal matrix whose diagonal elements are each $+1$ or -1) and where S represents a triangular array of terms which need not be specified here since it is soon to be eliminated. Furthermore, Williamson has shown that if T is any matrix satisfying equation (7) and if $\alpha = T^{-1}(-I - W)T$, then α is quasi-unitarily similar to A. Clearly α can be considered here instead of A.

Rewrite equation (7) as $T^{-1}(e\Delta)(T')^{-1} = D$. We shall construct a matrix T satisfying this form of equation (7) and then show that the resulting α is a *Pr* matrix. First, define H to be the matrix which has $+1$'s and -1's alternating along its skew-diagonal and zeros elsewhere, that is,

$$H = [(-1)^{i+1}\, \delta_{i,\, 2a-i+2}].$$

[2]δ is the Kronecker delta.

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Then the arrays S and S' may be eliminated as follows: there exists a matrix $T_1 = \tau \dotplus I_a$, where τ is a triangular matrix of order $a + 1$ which has 1's on its main diagonal and zeros above it and where I_a is the identity matrix of order a, such that $T_1^{-1} (e\Delta)(T_1')^{-1} = eH$. Now, let E be the matrix of order a which has 1's on its skew diagonal and zeros elsewhere, that is, $E = [\delta_{i,a-i+1}]$. If we now define T_2 to be

$$\begin{bmatrix} \frac{1}{2}\sqrt{2}\, I_a & 0 & -\frac{1}{2}\sqrt{2}\, E \\ 0 & 1 & 0 \\ \frac{1}{2}\sqrt{2}\, E & 0 & \frac{1}{2}\sqrt{2}\, I_a \end{bmatrix}$$

and define T to be T_1T_2, then $T^{-1} (e\Delta)(T')^{-1}$ is in the desired form, namely eD. Furthermore,

$$I + \alpha = I + T^{-1}(-I - W)T = -T^{-1}WT.$$

A computation will show that the first row of T is $[\frac{1}{2}\sqrt{2}\ 0 \ldots 0 - \frac{1}{2}\sqrt{2}]$ and the last row is $[\frac{1}{2}\sqrt{2}\ 0 \ldots 0\ \frac{1}{2}\sqrt{2}]$. Hence, if the first row and the last column of T are removed, leaving the matrix t_1 of order $2a$, then t_1 is non-singular since $|T| = \sqrt{2}|t_1| \neq 0$. Similarly, if the last row and the last column of T^{-1} are removed, leaving the matrix t_2 of order $2a$, then t_2 is non-singular since $|T^{-1}| = \sqrt{2}|t_2| \neq 0$. The principal submatrix of order $2a$ of $I + \alpha$ which is formed by removing the last row and the last column of $I + \alpha$ is $-t_2t_1$, which is non-singular. Thus, we have shown that $I + \alpha$ and hence $I + A$, are Pr matrices.

Case III: $I + A_{k+h}$, $(1 \leqslant h \leqslant m)$. As before, we shall drop the subscript $k + h$ from I, A and b. Let I_b denote the identity matrix of order $2b$. Williamson has shown that in this case there exist matrices D and T such that

$$TDT' = \begin{bmatrix} 0 & I_b \\ I_b & 0 \end{bmatrix} \tag{8}$$

and $T^{-1} ((-I_b - W) \dotplus (-I_b - W')^{-1})T = A$, where D is of the same form as in Case II. Again, if T satisfies equation (8) and if

$$\alpha = T^{-1}((-I_b - W) \dotplus (-I_b - W')^{-1})T,$$

then α is quasi-unitarily similar to A. Set $V = (-I_b - W')^{-1} + I_b$. It is easily seen that T may be taken as

$$\frac{1}{\sqrt{2}}\begin{bmatrix} I_b & I_b \\ I_b & -I_b \end{bmatrix},$$

in which case

$$I + \alpha = -\tfrac{1}{2}\begin{bmatrix} W - V & W + V \\ W + V & W - V \end{bmatrix}.$$

In order to show that $I + \alpha$ is a Pr matrix, consider the principal submatrix t formed by deleting the first and last rows and the first and last columns of

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$I + \alpha$. Partition t as $[t_{ij}]$, $i, j = 1, 2$, where the t_{ij} are square matrices of order $2b - 1$. A series of elementary transformations will show that t is non-singular. First, subtract the $(i+1)^{\text{st}}$ row of $[t_{21}\ t_{22}]$ from the i^{th} row of $[t_{11}\ t_{12}]$ $(i = 1,2,\ldots, 2b-2)$. The resulting t_{12} is non-singular. Now, add the

$(i+1)^{\text{st}}$ column of $\begin{bmatrix} t_{12} \\ t_{22} \end{bmatrix}$ to the i^{th} column of $\begin{bmatrix} t_{11} \\ t_{21} \end{bmatrix}$ $(i = 1, 2, \ldots, 2b-2)$.

The resulting t_{11} is zero and the resulting t_{21} is a non-singular diagonal matrix. Hence, $|t| \neq 0$, that is, $I + \alpha$ and hence $I + A$, are Pr matrices, which completes the proofs of Theorems 1 and 2.

COROLLARY. *If B is a* c.a. *of a non-singular diagonal matrix d, then $I + B$ is a CPr matrix.*

We have already shown that if B is a c.a. of a non-singular diagonal matrix d and if $\mathbf{X}_1$ is a set of linearly independent rows of $I + B$, then the set $\mathbf{X}^1$ of the corresponding columns is also linearly independent. However, B' is a c.a. of d^{-1} and so linear independence amongst a set of columns of $I + B$ implies linear independence amongst the set of the corresponding rows.

We wish to characterize all of the skew-symmetric matrices q which yield the same c.a. P as the skew-symmetric matrix Q which has just been constructed. Certainly, necessary and sufficient conditions that q also yields P are

$$\text{(i)}\quad (q - Q)\,(I + P) = 0$$
$$\text{(ii)}\quad |A + q| \neq 0.$$

Theorem 3 will provide a simpler set of conditions.

THEOREM 3. *Let P be a* c.a. *of A, having a parameterization as defined by equation* (1). *Then necessary and sufficient conditions that the skew-symmetric matrix q also yields P are*

$$(9')\qquad \text{(i)}\quad (q - Q)\,(I + P) = 0,$$
$$(9'')\qquad \text{(ii)}\quad \textit{Rank of } q = \textit{Rank of } Q \qquad (= \textit{Rank of } I - P).$$

Let $P = (A + q)\,(A - q)$. Then $2q = (A + q)\,(I - P)$ proving $(9'')$. Furthermore, equation $(9')$ follows immediately from equation $(3')$.

Conversely, let q satisfy $(9')$ and $(9'')$. By $(9')$, q_1 (the analogue of Q_1, formed by applications of Lemma 1 and similar transformations on q) is given by (10) for some X, Y and Z. Let I_6 denote the identity matrix of order s (s is the rank of $I_2 + B$). Now, partition B as $[B_{ij}]$, $(i, j = 1,2)$, such that B_{22} is of order s and $I_6 + B_{22}$ is non-singular. Define R by $B_{21} = (I_6 + B_{22})R$. Then

$$(I_2 + B)\,[I_5 - R']' = 0,\ (I_2 - B)\,[I_5 - R']' = 2\,[I_5 - R']'.$$

Hence, if we set

$$S = \begin{bmatrix} I_2 & 0 & 0 \\ 0 & I_5 & 0 \\ \frac{1}{2}\,Y\delta^{-1}[I_5 - R'] & 0 & I_4 \end{bmatrix},$$

then

$$Sq_1S' = \begin{bmatrix} (I_2 + B')d(I_2 - B) & -K' & 0 \\ K & X & 0 \\ 0 & 0 & Z \end{bmatrix},$$

which has rank equal to that of Q_1 if and only if $|Z| \neq 0$. However, we have previously shown that $|Z| \neq 0$ if and only if $|A + q| \neq 0$.

By considering matrices whose elements are taken from an arbitrary field of characteristic two, we can exhibit a counterexample to Theorem 1. It is easily seen that the matrix

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

is a c.a. of the symmetric matrix $A = 1 \dotplus 0$ and that $I + P$ spans the same row space as A. Furthermore, $|P| = +1$. However, for any skew-symmetric matrix Q, $(A + Q)^{-1}\,(A - Q) = I$.

4. The complex case. Since the proof of the theorem, analogous to Theorem 1, in which the underlying field is the complex field and in which transpose is replaced by conjugate transpose, is slightly simpler but extremely similar to the proof of Theorem 1, we shall only state the theorem and not repeat the proof.

THEOREM 4. *If A is a (not necessarily non-singular) Hermitian matrix and if Q is a skew-Hermitian matrix such that $A + Q$ is non-singular, then equation* (1) *defines a* c.a. *P of A having the property that A and $I + P$ span the same row space.*

Conversely, if P is a c.a. *of A having the property that $I + P$ and A span the same row space, then there is a skew-Hermitian matrix Q such that P is given by equation* (1).

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