THE WIDTH OF A MODULE

MICHAEL WICHMAN

Introduction. An R-module N is said to have *finite width* n if n is the smallest integer such that for any set of $n + 1$ elements of N, at least one of the elements is in the submodule generated by the remaining n. The width of N over R will be denoted by $W(R, N)$.

The notion of width was introduced by Brameret [**2**, p. 3605]. However, Cohen [**3**] investigated rings of finite rank, which, in the case that R is a local Noetherian domain, is equivalent to width (Proposition 1.6). He showed that finite width of R was both equivalent to R having Krull dimension one, and to R having the restricted minimum condition (Theorem 1.12).

In § 1, some general properties of modules of finite width will be established. Among these is a partial reduction of the theory to the local case (Theorem 1.9). It is shown that $W(R, N) \leqq \sum_m W(R_m, N_m)$, where m ranges over the maximal ideals of R, equality holding if R is Noetherian. Also, every module of finite width over a Noetherian ring is shown to be countably generated (Corollary 1.16).

The motivation of § 2 was the question as to whether the existence of a faithful torsion R-module of finite width implied that R had to have finite width. It turns out that if R is a local Noetherian domain with maximal ideal m, and if R^* is the m-adic completion of R, then there exists a faithful torsion R-module of finite width if and only if there exists a prime ideal P of R^* with $P \cap R = 0$ and Krull dimension $R^*/P = 1$ (Theorem 2.7).

Finally, in § 3, a faithful torsion R-module of width one over a Noetherian local ring is shown to be S-isomorphic to Q/S, where S is a complete valuation ring dominating R, and Q is the quotient field of S. It follows that R must be a domain.

Throughout this paper all rings will be commutative with unit, all modules will be unitary, and local rings will not necessarily be Noetherian. If N is an R-module and $x_1, \ldots, x_n$ are elements of N, the R-submodule of N generated by the x_i will be denoted by $(x_1, \ldots, x_n)$.

1. General properties.

The statements in the following proposition are immediate consequences of the definition of width.

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PROPOSITION 1.1. *Let N be an R-module.*

(1) *If $W(R, N) = 0$, then $N = 0$.*

(2) *If M is a submodule of N, then $W(R, M) \leqq W(R, N)$.*

(3) *If M is a homomorphic image of N, then $W(R, M) \leqq W(R, N)$.*

(4) *If $W(R, N) = n$, then every finitely generated submodule of N is generated by n elements; in fact, every finite set of generators of a submodule M contains a set of n generators for M.*

(5) *If R is a domain, then R is a valuation ring if and only if $W(R, R) = 1$.*

(6) *If N is a simple R-module, then $W(R, N) = 1$.*

(7) *If R' is an R-algebra and N is an R'-module, then $W(R', N) \leqq W(R, N)$.*

(8) *If $W(R, N) < \infty$, then there is a finitely generated submodule M of N with $W(R, M) = W(R, N)$.*

PROPOSITION 1.2. *Let N be a submodule of an R-module M. If $W(R, N) = n$, and $W(R, M/N) = l$, then $W(R, M) \leqq l + n$.*

Proof. Let $a_1, \ldots, a_{n+1}$ and $b_1, \ldots, b_l$ be $l + n + 1$ elements of M. By hypothesis, there exists l of these elements, say $b_1, \ldots, b_l$, whose images in M/N generate the submodule generated by all the images of $a_1, \ldots, a_{n+1}$ and $b_1, \ldots, b_l$ in M/N. Thus, $a_i = d_i + e_i$ for suitable d_i in N and e_i in $(b_1, \ldots, b_l)$.

Since $W(R, N) = n$, there exists n of the d_i, say $d_1, \ldots, d_n$, such that $d_{n+1} \in (d_1, \ldots, d_n)$. Now $d_i = a_i - e_i$ implies $d_{n+1} \in (b_1, \ldots, b_l, a_1, \ldots, a_n)$, and thus $a_{n+1} \in (b_1, \ldots, b_l, a_1, \ldots, a_n)$.

Definition. If N is an R-module of width n, then a set of elements $x_1, \ldots, x_n$ in N such that $x_i \notin (x_1, \ldots, \hat{x}_i, \ldots, x_n)$, $1 \leqq i \leqq n$, is said to be a set of *width determiners of N.*

COROLLARY 1.3. *If an R-module M is the direct sum of submodules M_i, $1 \leqq i \leqq k$, then $W(R, M) = W(R, M_1) + \ldots + W(R, M_k)$.*

Proof. We first show that

$$W(R, M_1 \oplus \ldots \oplus M_k) \geqq W(R, M_1) + \ldots + W(R, M_k).$$

Let $W(R, M_i) = n(i)$ and let $\{x_{i,1}, \ldots, x_{i,n(i)}\}$ be a set of width determiners for M_i, $1 \leqq i \leqq k$. Then no element $x_{i,j}$ of

$$A = \{x_{1,1}, \ldots, x_{1,n(i)}, \ldots, x_{k,1}, \ldots, x_{k,n(k)}\}$$

is a linear combination of the elements of $A - \{x_{i,j}\}$; for the existence of such an $x_{i,j}$ would contradict the choice of $x_{i,1}, \ldots, x_{i,n(i)}$.

The proof of the reverse inequality is by induction. If $k = 1$, then $M = M_1$ and thus $W(R, M) = W(R, M_1)$.

Now assume the result for $n < k$. By Proposition 1.2,

$$\begin{aligned} W(R, M_1 \oplus \ldots \oplus M_k) &\leqq W(R, M_1 \oplus \ldots \oplus M_{k-1}) + W(R, M_k) \\ &= W(R, M_1) + \ldots + W(R, M_{k-1}) + W(R, M_k). \end{aligned}$$

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PROPOSITION 1.4. *If N is an R-module and S is a multiplicatively closed subset of R, then $W(S^{-1}R, S^{-1}N) \leqq W(R, S^{-1}N) \leqq W(R, N)$.*

Proof. That $W(S^{-1}R, S^{-1}N) \leqq W(R, S^{-1}N)$ follows from Proposition 1.1 (7).

Suppose now that $W(R, N) = n$, and let $x_1 = m_1/s, \ldots, x_{n+1} = m_{n+1}/s$ be $n + 1$ elements of $S^{-1}N$, $s \in S$. There exists an index, say $n + 1$, and elements $r_i \in R$ such that $m_{n+1} = r_1m_1 + \ldots + r_nm_n$. Therefore,

$$m_{n+1}/s = r_1m_1/s + \ldots + r_nm_n/s,$$

and thus $W(S^{-1}R, S^{-1}N) \leqq n$.

COROLLARY 1.5. *If R is an integral domain and R' is a non-zero R-submodule of the quotient field of R, then $W(R, R) = W(R, R')$.*

Proof. Let $S = R - \{0\}$. Since R' contains an isomorphic copy of R and $R' \subset S^{-1}R$, we have $W(R, R) \leqq W(R, R') \leqq W(R, S^{-1}R)$. By Proposition 1.4, $W(R, S^{-1}R) \leqq W(R, R)$. Thus, $W(R, R) = W(R, R')$.

PROPOSITION 1.6. *If R is a local ring and N is an R-module, then $W(R, N)$ is the smallest integer n such that every finitely generated submodule of N is generated by n elements.*

Proof. If n is the smallest integer for which every finitely generated submodule of N is generated by at most n elements, then $W(R, N) \geqq n$. On the other hand, a minimal set of generators can be extracted from any set of generators of a finitely generated submodule [**8**, p. 14, Statement 5.3].

If R is a Noetherian ring, the question as to whether a given module has finite width can be reduced to the local case. To this end we first state Brameret's theorem [**2**, p. 3607, Theorem 3].

THEOREM 1.7. *Let R be a ring, N an R-module, and $P_1 \supset P_2 \supset \ldots \supset P_k \supset \ldots$ a chain of submodules of N. Suppose for every finitely generated submodule P of N that $\bigcap (P + P_i) = P$, and that the numbers $W(R, N/P_i)$ are bounded by n. Then $W(R, N) \leqq n$.*

COROLLARY 1.8. *If R is a Noetherian ring with Jacobson radical m, if N is a finitely generated R-module, and if N^* and R^* are the m-adic completions of N and R, respectively, then $W(R, N) = W(R^*, N^*)$.*

Proof. $N/m^sN = N^*/m^sR^*N^*$ is both an R and an R^* isomorphism for each integer s. Thus, $W(R, N/m^sN) = W(R^*, N^*/m^sR^*N^*)$ for every s. Since R is Noetherian and N is finitely generated, Theorem 1.7 applies.

THEOREM 1.9. *If N is an R-module, then $W(R, N) \leqq \sum W(R_m, N_m)$, where m ranges over the maximal ideals of R, and equality holds if R is Noetherian.*

Proof. (1) We first show that $W(R, N) \leqq \sum W(R_m, N_m)$. If $\sum W(R_m, N_m) = \infty$, our proof is complete. Thus suppose that $W(R_{m_i}, N_{m_i}) = n_i < \infty$ for $i = 1, \ldots, k$ and that $W(R_m, N_m) = 0$ for $m \notin \{m_i\}$. Let $u = n_1 + \ldots + n_k$.

If $x_1, \ldots, x_{u+1}$ are any $u+1$ elements of N, then n_i of the images of the x_j in N_{m_i} generate the R_{m_i}-submodule generated by all the images of the x_j in N_{m_i}. Since $n_1 + \ldots + n_k = u$, at least one of the x_j, say x_{u+1}, does not need to be used as a generator in any of the N_{m_i}, $1 \leqq i \leqq k$, and since $W(R_m, N_m) = 0$ implies $N_m = 0$ (by Proposition 1.1 (1)), we see that x_{u+1} is not needed for any N_m.

Therefore, for each maximal ideal m there exists $r_t(m) \in R$ and $s(m) \in R - m$ such that $s(m)x_{u+1} = \sum_{t=1}^{u} r_t(m)x_t$. Since $s(m) \notin m$ for each m, the ideal generated by all the $s(m)$ contains the identity. Thus, there is a finite number of the $q(m)$ in R such that $1 = \sum q(m)s(m)$.

But then $x_{u+1} \sum q(m)s(m) = \sum q(m) \sum r_t(m)x_t$ implies $W(R, N) \leqq u$, and thus $W(R, N) \leqq \sum W(R_m, N_m)$.

(2) We shall now show that equality holds when R is Noetherian.

Assume first that R is semi-local with maximal ideals m_i, and that N is finitely generated. Let J be the Jacobson radical of R, and let R^* be the J-adic completion of R. Let $R_{m_i}{}^*$ be the $m_iR_{m_i}$-adic completion of R_{m_i}.

Since $R^* = \bigoplus R_{m_i}{}^*$ [**8**, p. 56, Theorem 17.7], $N^* = \bigoplus N_{m_i}{}^*$, and thus $W(R^*, N^*) = \sum W(R^*, N_{m_i}{}^*)$ by Corollary 1.3. Since $N_{m_i}{}^*$ viewed as an R^*-module is really an $R_{m_i}{}^*$-module, it follows that $\sum W(R^*, N_{m_i}{}^*) = \sum W(R_{m_i}{}^*, N_{m_i}{}^*)$. Thus, $W(R, N) = \sum W(R_{m_i}{}^*, N_{m_i}{}^*) = \sum W(R_{m_i}, N_{m_i})$ by Corollary 1.8.

Now assume that R is semi-local but that N is arbitrary. If $W(R, N) = \infty$, our proof is complete, by part (1). Thus assume that $W(R, N) < \infty$. Then since $W(R_m, N_m) < \infty$ by Proposition 1.4, and since R is semi-local, Proposition 1.1 (8) implies that we can choose a finitely generated submodule T of N for which $W(R, N) = W(R, T)$ and $W(R_m, N_m) = W(R_m, T_m)$ for every maximal ideal m. We then have by the finitely generated case that $W(R, N) = W(R, T) = \sum W(R_m, T_m) = \sum W(R_m, N_m)$.

Finally, let R be an arbitrary Noetherian ring. By part (1) $W(R, N) \leqq \sum W(R_m, N_m)$. It remains to demonstrate the reverse inequality.

If $\sum W(R_m, N_m) = \infty$, then given any integer k we can find a finite set of maximal ideals $m_1, \ldots, m_r$ such that $\sum_{i=1}^{r} W(R_{m_i}, N_{m_i}) > k$. If we set $S = R - \bigcup_{i=1}^{r} m_i$, then since $S^{-1}R$ is semi-local, the previous case implies that $\sum_{i=1}^{r} W(R_{m_i}, N_{m_i}) = W(S^{-1}R, S^{-1}N)$. By Proposition 1.4,

$$W(S^{-1}R, S^{-1}N) \leqq W(R, N).$$

Therefore, $W(R, N) = \infty$.

If $\sum W(R_m, N_m) < \infty$, then $N_m = 0$ except for finitely many maximal ideals. Assume that $m_1, \ldots, m_k$ are the only maximal ideals for which $N_{m_i} \neq 0$ and let $S = R - \bigcup_{i=1}^{k} m_i$. Then $\sum_{i=1}^{k} W(R_{m_i}, N_{m_i}) = W(S^{-1}R, S^{-1}N)$ by the semi-local case; $W(S^{-1}R, S^{-1}N) \leqq W(R, N)$ by Proposition 1.4;

$$W(R, N) \leqq \sum_m W(R_m, N_m)$$

by part (1). Thus, since $\sum_m W(R_m, N_m) = \sum_{i=1}^{k} W(R_{m_i}, N_{m_i})$, it follows that $W(R, N) = \sum_{i=1}^{k} W(R_{m_i}, N_{m_i}) = \sum_m W(R_m, N_m)$.

The following result is a partial generalization of Theorem 1.9.

COROLLARY 1.10. *An integrally closed domain R has finite width if and only if it is the intersection of a finite number of valuation rings, and then the width is equal to the number of maximal ideals of R.*

Proof. Let R_i be valuation rings such that $R_i \not\subseteq R_j$, $i \neq j$. Let $S = R_1 \cap \ldots \cap R_n$. If the maximal ideal of R_i is m_i, then an ideal of S is maximal if and only if it is of the form $m_i \cap S$ for some i, and $R_i = S_{m_i \cap S}$ [**8**, p. 38, Theorem 11.11].

Therefore, by Theorem 1.9 and Proposition 1.1 (5),

$$W(S, S) \leqq \sum W(S_{m_i \cap S}, S_{m_i \cap S}) = n.$$

That $W(S, S) \geqq n$ follows from the following result.

LEMMA 1.11. *If $I_1, \ldots, I_k$ are ideals of a ring R with $I_i \not\subseteq \bigcap_{j \neq i} I_j$, $1 \leqq i \leqq k$, then $W(R, R) \geqq k$.*

Proof. $R/\bigcap I_i$ is canonically isomorphic to a submodule of the direct sum of the R/I_i, and intersects each summand non-trivially. Thus, $W(R, R/\bigcap I_i) \geqq k$ by Corollary 1.3. The conclusion follows from Proposition 1.1 (3).

THEOREM 1.12. *A Noetherian ring has finite width if and only if it is semi-local of Krull dimension at most one.*

Proof. If R has finite width, then R is semi-local by Theorem 1.9; and if P is a prime of R, then R/P has Krull dimension at most one [**3**, pp. 28–29, Theorem 1 and Corollary 1 of Theorem 10].

Conversely, if R is semi-local of Krull dimension at most one, then by Theorem 1.9 we can assume that R is local.

Let I be the intersection of the minimal primes of R. We have $W(R, R/I) < \infty$ since R/I is canonically isomorphic to a submodule of a finite direct sum of local domains of Krull dimension at most one, each of which has finite width [**3**, p. 35, Theorems 1 and 9].

By Proposition 1.2, it will suffice to show that $W(R, I) < \infty$. Since I^i/I^{i+1} is a finitely generated R-module, it has finite width since it is a homomorphic image of a finite direct sum of copies of R/I. Therefore, since I is nilpotent, Proposition 1.2 implies that $W(R, I) < \infty$.

Definition. A module is said to be *faithful* if it has zero annihilator. From now on, Ann N = *Annihilator of* N.

PROPOSITION 1.13. *If N is a faithful finitely generated R-module of width n, then $W(R, R) \leqq n^2$.*

Proof. Since N is finitely generated, it is generated by n elements, say $x_1, \ldots, x_n$. If we set $A_i = \text{Ann}\, Rx_i$, then $W(R, R/A_i) \leqq W(R, N) = n$ by Proposition 1.1 (2).

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Now R is canonically isomorphic to a submodule of $R/A_1 \oplus \ldots \oplus R/A_n$, since N faithful implies $\bigcap A_i = 0$. Therefore, by Proposition 1.1 (2) and Corollary 1.3, $W(R, R) \leqq \sum W(R, R/A_i) \leqq n^2$.

Definition. If M is an R-module, $E_R(M)$ will denote the injective envelope of M.

Theorem and Definition 1.14. *If M is an R-module of finite width over a Noetherian ring R, then $E_R(M)$ is a finite direct sum of modules of the form $E_R(R/P)$, where $W(R, R/P) < \infty$. The primes in the decomposition will be called the primes belonging to M.*

Proof. $E_R(M) = \bigoplus E_R(R/P_i)$ for some collection of prime ideals P_i [**5**, pp. 516–518, Theorem 2.5 and Proposition 3.1].

Since $E_R(M)$ is an essential extension of M, $N_i = M \cap E_R(R/P_i) \neq 0$ for every i. Further, i ranges over a finite index set since

$$\sum_i 1 \leqq W\left(R, \sum N_i\right) \leqq W(R, M) < \infty.$$

By [**5**, pp. 520–521, Theorems 3.4 and 3.6], R/P_i is isomorphic to a submodule of N_i, and thus $W(R, R/P_i) < \infty$.

Definition. Let W_1 and W_2 be submodules of an R-module N and let I be an ideal of R. Then

$$(W_1:W_2) = \{a \in R \mid aW_2 \subseteq W_1\} \quad \text{and} \quad (W:I) = \{x \in N \mid Ix \subseteq W\}.$$

THEOREM 1.15. *If $W(R, M) < \infty$ for an R-module M over a Noetherian ring, then $E_R(M)$ is countably generated.*

Proof. By Theorem 1.14 we need only show that $E_R(R/P)$ is countably generated when $W(R, R/P) < \infty$.

Now $E_R(R/P) = \bigcup_i (0:P^i)$, and $(0:P^i)/(0:P^{i-1})$ is a finite-dimensional vector space over the quotient field Q of R/P [**5**, p. 524, Theorem 3.9].

Therefore, it suffices to show that Q is countably generated over R/P. By Theorems 1.14 and 1.12, R/P is semi-local and of Krull dimension at most one. Thus, R/P localized by the powers of a non-zero element in the intersection of the maximal ideals of R/P must be all of Q. This implies that Q is countably generated over R/P.

COROLLARY 1.16. *A module of finite width over a Noetherian ring is countably generated.*

Proof. Over a Noetherian ring, any submodule of a countably generated module is countably generated.

Definition. An R-module M is said to be a *torsion R-module* if $\operatorname{Ann} x \neq 0$ for all x in M.

The following proposition yields a class of faithful torsion modules of finite width.

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PROPOSITION 1.17. *If R is an integral domain and Q is the quotient field of R, then Q/R is a faithful torsion R-module and $W(R, R) = W(R, Q/R)$.*

Proof. That Q/R is a faithful torsion R-module is easy to check.

Now since $W(R, R) = W(R, Q)$ by Corollary 1.5, Proposition 1.1 (3) implies that $W(R, R) \geqq W(R, Q/R)$.

To show the reverse inequality, let $x_1, \ldots, x_n \in R$ be a set of width determiners of R. Let $a_k = \Pi_{j \neq k} x_j$ and suppose that $1/a_i = \sum_{j \neq i} r_j/a_j + c$, where $r_j, c \in R$. Then $x_i/\Pi x_k = \sum_{j \neq i} r_j x_j/\Pi x_k + c$ implies that

$$x_i \in (x_1, \ldots, \hat{x}_i, \ldots, x_n),$$

a contradiction. Hence, $W(R, R) \leqq W(R, Q/R)$.

PROPOSITION 1.18. *Suppose that R is a ring and N is an R-module of width $n < \infty$. Let $W = (x_1, \ldots, x_n)$, where $\{x_1, \ldots, x_n\}$ is a set of width determiners. Then if $0 \neq x \in N$, there exists $b \in R$ such that $0 \neq bx \in W$. In particular, N/W is a torsion R-module.*

Proof. If $x \in W$, take $b = 1$. If $x \notin W$, then, since $\{x_1, \ldots, x_n, x\}$ is a set of $n + 1$ elements, $W(R, N) = n$ implies that there exists i such that $x_i = \sum_{j \neq i} r_j x_j + bx$, $r_j, b \in R$. Further, $bx \neq 0$ since $x_i \notin (x_1, \ldots, \hat{x}_i, \ldots, x_n)$.

2. Thin modules.

Definition. If R is a domain, a non-zero R-module N will be called *divisible* if either of the following equivalent conditions are satisfied:

(i) if $r \in R$, then the endomorphism of N given by $x \to rx$ is an epimorphism;

(ii) if $W \neq N$ is a submodule of N, then Ann $N/W = 0$.

A divisible R-module N will be called *thin* if M divisible implies $W(R, M) \geqq W(R, N)$.

PROPOSITION 2.1. *If N is a divisible R-module, then N is faithful. If N is thin, then every non-zero homomorphic image N' of N is thin, and hence $W(R, N) = W(R, N')$.*

Proof. By the definition of a divisible R-module, $N \neq 0$ and Ann $N/0 = 0$; thus N is faithful.

If N is thin, then N is divisible, and thus any non-zero homomorphic image N' of N is divisible. Thus by the definition of thin, $W(R, N') \geqq W(R, N)$. On the other hand, $W(R, N) \geqq W(R, N')$ by Proposition 1.1 (3), and thus N' is thin.

PROPOSITION 2.2. *Let R be a ring and N a module of with n. If W is a submodule of N such that $W(R, N/W) = n$, then W is contained in a finitely generated submodule of N. In particular, if R is Noetherian, then W is finitely generated.*

Proof. Let $x_1, \ldots, x_n \in N$ such that their images in N/W are width determiners for N/W. Let $x \in W$ and consider $\{x, x_1, \ldots, x_n\}$. By hypothesis, there does not exist an index i and elements $r_j, a \in R$ such that

$$x_i = \sum_{j \neq i} r_j x_j + ax.$$

Therefore, $W(R, N) = n$ implies that there exist $r_i \in R$ with $x = \sum r_i x_i$, and thus $W \subset (x_1, \ldots, x_n)$.

PROPOSITION 2.3. *Let R be a local Noetherian domain with maximal ideal m, and let N be a thin torsion R-module of finite width. Then $N = \bigcup (0:m^i)$.*

Proof. Let $a \in R$ be a zero divisor of N and let $W = (0:Ra)$. Since N is faithful by Proposition 2.1, $W \neq N$. Since N is divisible, $\mathrm{Ann}(N/W) = 0$. Since N is thin, $W(R, N/W) = W(R, N)$ by Proposition 2.1, and thus W is finitely generated by Proposition 2.2. Thus, if s is any non-zero element of m, then by Nakayama's lemma, $sW \subset W$ and $sW \neq W$. Let $y \in W - sW$. Since $sN = N$, there is $z \in N - W$ with $sz = y$. Now $z \notin W$, thus $az \neq 0$, and then $sz \in W$ implies $asz = 0$.

Now, by Theorem 1.14 and Corollary 1.3, N thin implies $E_R(N) = E_R(R/P)$ for exactly one prime P. Therefore, since $E_R(R/P)$ is an R_P-module by [**5**, p. 521, Theorem 3.6], and since by [**5**, p. 518, Lemma 3.2] no element of $R - P$ is a zero divisor of $E_R(R/P)$, we have $P = m$.

THEOREM 2.4. *If R is a domain for which there exists a faithful torsion R-module of finite width, then any faithful torsion R-module of minimum width among all faithful torsion R-modules is thin.*

Proof. Let N be a faithful torsion R-module of finite width whose width is minimum among all faithful torsion R-modules. We will first show that N is divisible with a proof by contradiction.

If N is not divisible, there is a submodule $W \subset N$ such that $\mathrm{Ann}\, N/W \neq 0$. It follows that W is a faithful torsion R-module since

$$(\mathrm{Ann}\, W)(\mathrm{Ann}\, N/W) \subseteq \mathrm{Ann}\, N = 0$$

and $\mathrm{Ann}\, N/W \neq 0$ implies $\mathrm{Ann}\, W = 0$. Further, $W(R, W) \leqq W(R, N)$ by Proposition 1.1 (2), and $W(R, W) \geqq W(R, N)$ by the choice of N, implies $W(R, W) = W(R, N)$.

Now let $W(R, N) = n$, suppose that $x \notin W$, and let $\{x_1, \ldots, x_n\}$ be any n elements of W. Then $\{x_1, \ldots, x_n, x\}$ is a set of $n + 1$ elements of $W + Rx$. Now since $W \subseteq W + Rx \subseteq N$ and $W(R, W) = W(R, N) = n$, we have $W(R, W + Rx) = n$. Since $x \notin W$, we have $x \notin (x_1, \ldots, x_n)$. Therefore, $W(R, W + Rx) = n$ implies that there exists x_i with

$$x_i \in (x_1, \ldots, \hat{x}_i, \ldots, x_n, x).$$

Therefore, the image of x_i in $(W + Rx)/Rx$ is a linear combination of the images of $x_1, \ldots, \hat{x}_i, \ldots, x_n$ in $(W + Rx)/Rx$. Therefore, $x_1, \ldots, x_n$ arbitrary implies $W(R, (W + Rx)/Rx) < n$.

Thus, if we show that $(W + Rx)/Rx$ is a faithful torsion R-module of finite width, we will have a contradiction to the choice of N.

Now $(W + Rx)/Rx$ has finite width by Proposition 1.1 (3), and $(W + Rx)/Rx$ is a torsion R-module since $W + Rx$ is a submodule of the torsion R-module N. Finally, $(W + Rx)/Rx$ is a faithful R-module since $\text{Ann}((W + Rx)/Rx)\text{Ann}(Rx) \subseteq \text{Ann}(W + Rx) = 0$, and since $\text{Ann}\, Rx \neq 0$ for any element x of the torsion module N.

To complete the proof we need to eliminate the possibility that there exists a thin R-module which is not a torsion R-module and which has width less than that of N.

To this end, let M be a thin R-module with width determiners $x_1, \ldots, x_n$. We need only show that $M/(x_1, \ldots, x_n) \neq 0$, for this would then be a faithful torsion R-module of finite width by Propositions 1.18 and 2.1.

However, if $M = (x_1, \ldots, x_n)$, then $R = M/(x_1, \ldots, x_{n-1})$, since $x_1, \ldots, x_n$ width determiners implies $x_n \notin (x_1, \ldots, x_{n-1})$, and since M divisible implies $\text{Ann}\, M/(x_1, \ldots, x_{n-1}) = 0$. This in turn implies that R is a divisible R-module and thus R must be a field, which is impossible since by hypothesis there exists a faithful torsion R-module.

THEOREM 2.5. *If R is a complete local Noetherian domain for which there exists a faithful torsion R-module of finite width, then $W(R, R) < \infty$. If, furthermore, R has maximal ideal m and quotient field Q, then both Q/R and $E_R(R/m)$ are thin torsion R-modules, and the width of any thin R-module equals $W(R, R)$.*

Proof. We first prove that Q/R is thin.

By Theorem 2.4, there exists a thin torsion R-module N. Thus, since Q/R is a faithful torsion R-module, and since $W(R, Q/R) = W(R, R)$ by Proposition 1.17, we need only show that $W(R, R) \leqq W(R, N)$.

To do this, let m be the maximal ideal of R and let $0 \neq a \in m$. Since N is faithful, there is an $x_0 \in N$ with $ax_0 \neq 0$, and since N is divisible, there exist $x_i \in N$ with $ax_i = x_{i-1}$, $ax_1 = x_0$. The set of all the x_i generate N. For $N = \bigcup (0{:}m^i)$ by Proposition 2.3; thus $a^i x_i = x_0 \neq 0$, for all i, implies the submodule generated by the x_i is not contained in $(0{:}m^i)$ for any i. However, any proper submodule of N is finitely generated by Propositions 2.1 and 2.2, and thus is contained in $(0{:}m^i)$ for some i.

Now let $I_i = \text{Ann}\, Rx_i$. Since $ax_i = x_{i-1}$, $I_{i-1} \supseteq I_i$, and since N is faithful, $\bigcap I_i = 0$. Therefore, there exist $s(i)$ tending to infinity with i so that $m^{s(i)} \supseteq I_i$ [**8**, p. 103, Theorem 30.1]. Since R/I_i is isomorphic to a submodule of N, and since $R/m^{s(i)}$ is a homomorphic image of R/I_i, Propositions 1.1 (2) and 1.1 (3) imply that $W(R, R/m^{s(i)}) \leqq W(R, R/I_i) \leqq W(R, N)$. Therefore, since $s(i)$ tends to infinity, $W(R, R) \leqq W(R, N)$ by Theorem 1.7.

Finally, we show that $E_R(R/m)$ is thin. We know that Q/R is a thin R-module. Thus, if we show that $E_R(R/m)$ is an R-homomorphic image of Q/R, we will see that $E_R(R/m)$ is a thin R-module by Proposition 2.1.

By [**7**, p. 571, Theorem 1], Q/R is Artinian. Thus, it is isomorphic to a submodule of a finite direct sum of copies of $E_R(R/m)$ [**6**, p. 497, Proposition 3]. Therefore, the projection onto one of the summands $E_R(R/m)$ restricted to Q/R is a thin R-submodule of $E_R(R/m)$ by Proposition 2.1. Since a thin module is divisible and $E_R(R/m)$ has no proper divisible submodules [**7**, p. 573, Proposition 2], the projection is onto.

LEMMA 2.6. *Let R be a local Noetherian domain, and let R^* be the completion of R. If P is a prime ideal of R^* with $P \cap R = 0$, if Krull dimension $R^*/P = 1$, and if Q is the quotient field of R^*/P, then $N = Q/(R^*/P)$ is a faithful torsion R-module of width $W(R^*/P, N)$. Further, R^*/P is complete and dominates R.*

Proof. N is a faithful torsion R^*/P-module of finite width by Theorem 1.12 and Proposition 1.17. Therefore, since R^*/P is complete, N is a thin R^*/P-module by Theorem 2.5. Since $P \cap R = 0$, we have $R \subseteq R^*/P$, and thus N is also a faithful R-module.

By Proposition 2.3, $N = \bigcup (0:m_1{}^i)$ where m_1 is the maximal ideal of R^*/P. If $M = (0:m_1{}^i)$, then M is a module over $(R^*/P)/m_1{}^i$, but $R \to R^* \to R^*/P$ induces $R/m^i \to R^*/m^iR^* \to (R^*/P)/m_1{}^i$, the first of these maps being an isomorphism, the second an epimorphism. Hence, M is an R/m^i-module, and, therefore, a torsion R-module. Further,

$$W(R, M) = W(R/m^i, M) = W((R^*/P)/m_1{}^i, M)$$
$$= W(R^*/P, M) \leqq W(R^*/P, N)$$

by Proposition 1.1 (2).

Thus, given more than $W(R^*/P, N)$ elements of N, they all lie in $M = (0:m_1{}^i)$ for suitably large i, and thus one is an R-linear combination of the others. This proves that $W(R, N) \leqq W(R^*/P, N)$.

Since $R \subset R^*/P$, we see that $W(R, N) \geqq W(R^*/P, N)$ by Proposition 1.1 (7). Therefore, $W(R, N) = W(R^*/P, N)$.

If R^*/P does not dominate R, then there is an element of the maximal ideal of R which is invertible in R^*/P. However, this is impossible, because of the sequence $R \to R^*/P \to R^*/mR^* \to R/m$, where m is the maximal ideal of R, since the second map is surjective.

The next theorem is the main result of this section.

THEOREM 2.7. *Let R be a local Noetherian domain, and let R^* be the completion of R. Then there exists a faithful torsion R-module of finite width if and only if there exists a prime ideal P of R^* with $P \cap R = 0$ and Krull dimension $R^*/P = 1$.*

If N is a thin R-module of finite width, then there is a complete local Noetherian domain S of finite width dominating R such that N is a thin torsion S-module of finite width.

Proof. If there exists a prime ideal P of R^* with $P \cap R = 0$ and Krull dimension $R^*/P = 1$, then there exists a faithful torsion R-module of finite width by Lemma 2.6.

Conversely, if there is a faithful torsion R-module of finite width, then there is a thin torsion R-module N of finite width by Theorem 2.4. N is an R^*-module of finite width by Propositions 2.3 and 1.1 (7).

We next show that the annihilator P of N as an R^*-module is a prime ideal with $P \cap R = 0$, and thus, that N is a faithful torsion R^*/P-module of finite width.

Suppose that a and b are elements of R^* with $abN = 0$ and $aN \neq 0$. If $aN = N$, then $0 = baN = bN$ implies $b \in \text{Ann}_{R^*}N$.

If $aN \neq N$, then $W(R, N/aN) = W(R, N)$ by Proposition 2.1. Therefore, by Proposition 2.2, aN is a finitely generated submodule of N. But then, since R is a domain and N is a torsion R-module, there exists $0 \neq c$ in R with $c(aN) = 0$, a contradiction since N being R-divisible implies $0 = c(aN) = a(cN) = aN$.

Therefore, the annihilator of N as an R^*-module is a prime ideal P. Further, $P \cap R = 0$ since N is a faithful R-module.

Therefore, $W(R^*/P, R^*/P) < \infty$ by Theorem 2.5, and since R^*/P is not a field, Krull dimension $R^*/P = 1$ by Theorem 1.12.

By Lemma 2.6, we can let $S = R^*/P$.

Example 1. We can now show that there are rings R of infinite width for which there exist faithful torsion R-modules of finite width.

Let $R = C[x, y]_{(x,y)}$, where C is the field of complex numbers and x and y are indeterminants. The completion of R is $R^* = C[[x, y]]$, the ring of formal power series in x and y over C. Let $f(x)$ be a non-unit in $C[[x]]$ such that $f(x)$ is not algebraic over $C(x)$, the quotient field of $C[x]$. Let I be the ideal in $C[[x, y]]$ generated by $y - f(x)$. Now $I \cap R = 0$; for if there exists g in $C[[x, y]]$ such that $h(x, y) = (y - f(x))\, g \in C[x, y]$, then $h(x, f(x)) = 0$ implies $h = 0$; otherwise $f(x)$ would be algebraic over $C(x)$.

Since R has Krull dimension two, and since the maximal ideal of R^* intersected with R is the maximal ideal of R, there is a minimal prime P containing I such that $P \cap R = 0$ and Krull dimension $R^*/P = 1$. Therefore, by Theorem 2.7, there exists a faithful torsion R-module of finite width. Finally, since the Krull dimension of R is two, R has infinite width by Theorem 1.12.

Example 2. It is natural to ask whether a module of finite width over a local Noetherian domain is a direct sum of thin modules. We will show that this is not so.

Let R be a local Noetherian domain for which there exist two prime ideals P_1, P_2 of R^* with $P_i \cap R = 0$ and Krull dimension $R^*/P_i = 1$ (in Example 1, we could choose two elements of $C[[x]]$ not algebraic over $C(x)$ and generating distinct prime ideals). By Lemma 2.6, if Q_i is the quotient field of $S_i = R^*/P_i$, then Q_i/S_i is an R-module of finite width. Therefore, if E is the injective envelope of R^*/m^* where m^* is the maximal ideal of R^*, we see [**5**, pp. 520–521, remark following Theorem 3.4 and Theorem 3.5] that the submodules $(0:P_1)$ and $(0:P_2)$ of E are faithful torsion R-modules of finite width. By [**5**, p. 18, Proposition 3.1], $(0:P_1) \neq (0:P_2)$.

Since the submodule $(0:P_1) + (0:P_2)$ of E generated by $(0:P_1)$ and $(0:P_2)$ is a homomorphic image of $(0:P_1) + (0:P_2)$, it has finite width by Corollary 1.3 and Proposition 1.1 (3).

If $(0:P_1) + (0:P_2)$ were thin, then $(0:P_1) \neq (0:P_1) + (0:P_2)$ would imply that $(0:P_1)$ is finitely generated by Proposition 2.2, which is impossible since $(0:P_1)$ is a faithful torsion R-module and R is a domain.

Thus $(0:P_1) + (0:P_2)$ is not thin. On the other hand, it is indecomposable [**5**, p. 514, Proposition 2.2].

3. Torsion modules of width one. In this section we shall show that every faithful torsion module of width one over a Noetherian local ring R is of the form $Q(S)/S$, where S is a complete Noetherian valuation ring dominating R, and $Q(S)$ is the quotient field of S.

LEMMA 3.1. *If R is a Noetherian local ring and N is a faithful torsion R-module of width one, then $N = \bigcup (0:m^i)$, where m is the maximal ideal of R.*

Proof. First we show that if $a \in m^k - m^{k+1}$, $b \in R$, and $x \in N$ with $bx = 0$ and $ax \neq 0$, then $by \in m^{k+1}y$ for all $y \in N$.

For if $by \neq 0$, then $Rx \subseteq Ry$. Consider by and ay. Since N has width one, either $(ay) \subseteq (by)$ or $(by) \subseteq (ay)$. If $ay = cby$, then $(a - cb)y = 0$ implies $(a - cb)x = 0$, which implies $0 \neq ax = cbx = 0$, a contradiction. Thus, $by = cay$. Furthermore, c is a non-unit, for otherwise $c^{-1}by = ay$. Therefore, $by \in m^{k+1}y$.

Now suppose that $x \in N$ and $m^n x \neq 0$ for all n. There exists $0 \neq b \in R$ such that $bx = 0$. By the above, it follows that $by \in m^{n+1}y$ for all y in N. Since R is Noetherian, $\bigcap m^n y = 0$, and thus $by = 0$ for all $y \in N$. But this implies that $b = 0$, a contradiction.

LEMMA 3.2. *Let R be a Noetherian local ring and N a faithful torsion R-module of width one. If $W(R, R) = 1$, then R is a domain.*

Proof. If m is the maximal ideal of R, then $N = \bigcup (0:m^i)$ by Lemma 3.1. It follows that m is not nilpotent. For if $m^n = 0$, then N would be finitely generated by [**5**, p. 525, Theorem 3.11] and Theorem 1.14, hence, cyclic by Proposition 1.6. This contradicts the hypothesis that N is a faithful torsion R-module.

We now show that R is a domain.

Since R is Noetherian of width one, Proposition 1.6 implies that m is generated by one element, say b. But then, since every element in a Noetherian ring can be written as a product of irreducible elements, $W(R, R) = 1$ implies that every element of R is of the form ub^n, where u is a unit and $n > 0$.

Thus, if $cb = 0$, setting $c = u_1b^m$ and $b = u_2b^n$ implies that $u_1u_2b^{m+n} = 0$, which implies the contradiction that m is nilpotent.

THEOREM 3.3. *If R is a local Noetherian ring and N is a faithful torsion R-module of width one, then R is a domain and N is a thin R-module.*

Proof. If R^* is the completion of R, then N is an R^*-module by Lemma 3.1. If we let I be the annihilator of N as an R^*-module, then R^*/I is complete [**8**, p. 57, Corollary 17.9]. Further, $I \cap R = 0$ implies $R \subseteq R^*/I$, and thus N is a faithful torsion R^*/I-module. Therefore, $W(R^*/I, R^*/I) = 1$ by Theorem 2.5, and thus R^*/I is a domain by Lemma 3.2. Thus, $R \subseteq R^*/I$ implies R is a domain.

That N is thin follows from Theorem 2.4.

THEOREM 3.4. *If N is a faithful torsion module of width one over a local Noetherian ring R, then N is S-isomorphic to $Q(S)/S$, where S is a complete Noetherian valuation ring dominating R and $Q(S)$ is the quotient field of S.*

Proof. By Theorem 3.3, N is a thin R-module. Therefore, by Theorem 2.7, N is a thin torsion S-module of width one, where S is a complete local Noetherian domain dominating R, and thus $W(S, S) = 1$ by Theorem 2.5.

It follows from Corollary 1.3 that N is an indecomposable divisible torsion module over the discrete valuation ring S, and thus N is isomorphic to $S_{p(\infty)} = Q(S)/S$ (cf. [**4**, p. 10, Theorem 4]).

THEOREM 3.5. *Let R be a local Noetherian domain and N a faithful torsion R-module of width one. If R^* is the completion of R, then R is a discrete valuation ring if and only if the annihilator of N as an R^*-module is zero.*

Proof. First note that N is an R^*-module by Lemma 3.1.

If the annihilator of N as an R^*-module is zero, then $W(R^*, R^*) = 1$ by Theorem 2.5, and thus $W(R, R) = 1$ by Corollary 1.8. Thus, R is a discrete valuation ring.

Conversely, suppose that $W(R, R) = 1$ and that $a^*N = 0$, where $a^* \in R^*$. Let $\{a_i\}$ be a Cauchy sequence in R converging to a^*. Now, since $N = \bigcup (0:m^i)$ by Lemma 3.1, and since every ideal of R is of the form m^i, it follows that there exists $x_i \in N$ with $\operatorname{Ann} Rx_i = m^i$ for arbitrarily large i. But then $a^*x_i = 0$ implies that $a_j \in m^i$ for sufficiently large j. Thus, $a^* = 0$.

REFERENCES

1. N. Bourbaki, *Algèbre*, Chapitre 8, Actualités Sci. Indust., no. 1261 (Hermann, Paris, 1958).
2. M.-P. Brameret, *Anneaux et modules de largeur finie*, C. R. Acad. Sci. Paris *258* (1964), 3605–3608.
3. I. S. Cohen, *Commutative rings with restricted minimum condition*, Duke Math. J. *17* (1950), 27–42.

4. I. Kaplansky, *Infinite abelian groups*, rev. ed. (University of Michigan Press, Ann Arbor, 1969).
5. E. Matlis, *Injective modules over Noetherian rings*, Pacific J. Math. *8* (1958), 511–528.
6. ——— *Modules with D.C.C.*, Trans. Amer. Math. Soc. *97* (1960), 495–508.
7. ——— *Some properties of Noetherian domains of dimension* 1, Can. J. Math. *13* (1961), 569–586.
8. M. Nagata, *Local rings* (Interscience, New York, 1962).

DePaul University,
Chicago, Illinois