FUNCTIONAL INTEGRALS IN STELLAR DYNAMICS

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Abstract: In the late 70s, the functional integral formulation equivalent to the Fokker-Planck equation was worked out (Graham). We apply this functional integral formulation to gravitationally interacting systems, whose dynamics may be analyzed by separating the forces operating on a particle into a mean field force and fluctuations due to random collisions at intermediate range (scattering at small angles). In this poster the formalism is presented for short periods (in the stochastical meaning) to systems with isotropic distribution background in velocity space (different spatial densities are possible). Later the functional integral for the local change for the distribution function is evaluated in the steepest descent approximation. In the end we point out the applicability of the method to slowly evolving globular star clusters near thermal equilibrium. In conditions of slow evolution we can express the evolution of the orbits in terms of local deviation from equilibrium.

I. The functional integral formalism for Nartonian gravity

R. Graham treated Fokker Planck Equation of the type:

$$(1)\quad \frac{\partial f}{\partial t} = \frac{\partial}{\partial q^\nu}\left(K^\nu(\vec{q})f\right) + \frac{1}{2}\frac{\partial^2}{\partial q^\nu \partial q^\mu}\left(Q^{\nu\mu}(\vec{q})f\right) + V(\vec{q})f$$

He had built functional integral solution for the density distribution function at later times for an initial distribution of the delta function type. The functional integral has the form:

$$(2)\quad f(\vec{q}\,t\,|\,\vec{q}_0\,0) = \int_{\vec{q}(0)=\vec{q}_0}^{\vec{q}(t)=\vec{q}} \mathcal{D}\mu([\vec{q}(\tau)]\, \exp\left(-\int_0^t L(\dot{\vec{q}}(\tau), \vec{q}(\tau))\,d\tau\right)$$

where $L(\dot{\vec{q}}, \vec{q})$ - the Lagrangian in the problem is given by:

J. Goodman and P. Hut (eds.), Dynamics of Star Clusters, 313–315.

$$(3)\quad L(\dot{\vec q},\vec q)=\tfrac{1}{2}Q^{-1}_{\nu\mu}(\dot q^{\nu}-h^{\nu})(\dot q^{\mu}-h^{\mu})+\tfrac{1}{2}[Q^{-1}]^{-\frac{1}{2}}\frac{\partial}{\partial q^{\nu}}\{[Q^{-1}]^{\frac{1}{2}}h^{\nu}\}-$$
$$-V(\vec q)+\frac{R}{n}$$

where

$Q_{\nu\mu}=(Q^{-1})^{\nu\mu}$ - the diffusion matrix

$Q^{-1}_{\nu\mu}$ - the inverse of the diffusion matrix

K^{ν} - the original drift

h^{ν} - the covariant drift: $h^{\nu}=K^{\nu}-\tfrac{1}{2}[Q^{-1}]^{-\frac{1}{2}}\frac{\partial}{\partial q^{\mu}}\{[Q^{-1}]^{\frac{1}{2}}\{Q^{-1}\}^{\nu\mu}\}$
$Q^{-1}=\det Q^{-1}_{\nu\mu}$
n - integer taking the values: 6, 8, or 12.

R - the scalar curvature, built on a metric which is the inverse of the diffusion matrix.

The Lagrangian has the character of a scalar, which is covariant to general coordinate transformations, with the role of the metric being played by the reciprocal diffusion matrix.

For one component, gravitationally interacting systems we get the following expression for the distribution function after short time τ, where the initial distribution function was $f(\vec r', \vec v'; t)$

$$(4)\quad f(\vec r,\vec v,t+\tau)=\int\frac{1}{(2\pi\tau)^{3/2}D^{1/2}}\exp\Big[-\Big\{\frac{1}{2\tau}D^{-1}_{\nu\mu}(v^{\nu}-v^{\nu\prime}-\tau h^{\nu}-\tau g^{\nu})$$
$$\times(v^{\mu}-v^{\mu\prime}-\tau h^{\mu}-\tau g^{\mu})+\tfrac{1}{2}[D^{-1}]^{-\frac{1}{2}}\frac{\partial}{\partial v^{\mu}}\{[D^{-1}]^{\frac{1}{2}}h^{\nu}\}\tau+\frac{R}{n}\tau+$$
$$+\ln f(\vec r',\vec v',t=0)\Big\}\Big]\times\delta(\vec r-\vec r'-\vec v\tau)\,d\vec r'\,d\vec v'$$

where: $K^{\nu}=\langle\Delta v^{\nu}\rangle$, $D_{\nu\mu}=\langle\Delta v^{\nu}\Delta v^{\mu}\rangle$, $\vec g=\frac{\partial\phi}{\partial\vec r}$, the mean force.

In the case of locally isotropic background the transport coefficients in Cartesian coordinates take the form:

$$(5)\quad D_{ij}=\frac{D_{\parallel}(v)-D_{\perp}(v)}{v^2}v^i v^j+\delta^{ij}D_{\perp}(v);\quad D^{-1}_{ij}=\frac{D^{-1}_{\parallel}-D^{-1}_{\perp}}{v^2}v^i v^j+\delta^{ij}D^{-1}_{\perp}$$
$$K^i=\eta(v)\frac{v^i}{v}\,[D_{\perp}=\langle\Delta v_{\perp}\Delta v_{\perp}\rangle,\ D_{\parallel}=\langle\Delta v_{\parallel}\Delta v_{\parallel}\rangle,\ \eta(v)=\langle\Delta v_{\parallel}\rangle]$$

We can diagonalize the diffusion matrix into the form $D_{ij}=\begin{pmatrix}D_{\parallel} & 0 & 0\\ 0 & D_{\perp} & \\ 0 & & D_{\perp}\end{pmatrix}$ and then the christoffels are given by:

https://doi.org/10.1017/S0074180900147503 Published online by Cambridge University Press

$$\Gamma_{11}^{1} = -\frac{1}{2}(\ln D_{\parallel})', \quad \Gamma_{22}^{1} = \Gamma_{33}^{1} = -\frac{1}{2} D_{\parallel} (D_{\perp}^{-1})' + D_{\parallel} \frac{D_{\parallel}^{-1} - D_{\perp}^{-1}}{v}$$

$$\Gamma_{21}^{2} = \Gamma_{31}^{3} = -\frac{1}{2}(\ln D_{\perp})' \tag{6}$$

The transport coefficients for that case are given by Hénon.

II. The steepest descent evaluation near equilibrium states

For systems near thermodynamic equilibrium we use the steepest descent method to evaluate the integrals in question.

In the limit of small velocities the transport coefficients takes the asymptotic form:

$$D_{\perp} \to D_{\parallel} = Q_0 = \text{const}, \quad \eta(v) \to \eta_0(v) \, [\eta_0 = \text{const}] \tag{7}$$

In that limit the method gives exact results for maxwellian type distributions:

$$f(\vec{r}, \vec{v}, \tau) = A \left(\frac{1}{1 + \Delta S \tau}\right)^{3/2} e^{-\frac{\beta}{(1+\Delta S \tau)}\frac{v^2}{2} - \beta \phi(\vec{r})} \tag{8}$$

where A is a normalization factor and $\Delta S = \beta Q_0 - 2\eta_0$

In Equilibrium $\Delta S = 0$ and so the form of the distribution is conserved.

In Wooley model $S = -\frac{2}{3}C$, where $C = 16\pi^2 G^2 \ln N$ (defined in Hénon paper).

The local changes in the distribution function has no meaning in globular clusters. However, if the evolution is slow, we can calculate the change in the distribution of orbits by first solving for the local changes and then averaging on the orbits.

Relating to this matter, we have found that in the Woolley models, particles below the energy curoff, lose energy at constant rate.

References

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https://doi.org/10.1017/S0074180900147503 Published online by Cambridge University Press