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IMAGES OF HIGHER-ORDER DIFFERENTIAL OPERATORS OF POLYNOMIAL ALGEBRAS

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Abstract

We investigate images of higher-order differential operators of polynomial algebras over a field k. We show that, when $\operatorname{char} k > 0$, the image of the set of differential operators $\{\xi_i - \tau_i \mid i = 1, 2, \ldots, n\}$ of the polynomial algebra $k[\xi_1, \ldots, \xi_n, z_1, \ldots, z_n]$ is a Mathieu subspace, where $\tau_i \in k[\partial_{z_1}, \ldots, \partial_{z_n}]$ for $i = 1, 2, \ldots, n$. We also show that, when $\operatorname{char} k = 0$, the same conclusion holds for $n = 1$. The problem concerning images of differential operators arose from the study of the Jacobian conjecture.

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1. Introduction

Throughout the paper, k denotes a field. The Jacobian conjecture, a long-standing open problem in affine algebraic geometry, asserts that, when $\operatorname{char} k = 0$, a polynomial map $F : k^n \to k^n$ is invertible if its Jacobian determinant is a nonzero constant (see [2, 3]).

The study of images of differential operators of polynomial algebras is closely related to the Jacobian conjecture. On the one hand, van den Essen *et al.* [5] showed that the two-dimensional Jacobian conjecture is equivalent to saying that the image, $\operatorname{Im} \delta$, of every k-derivation δ of the polynomial algebra $k[x, y]$ with $1 \in \operatorname{Im} \delta$ and divergence zero, is equal to $k[x, y]$. On the other hand, Zhao [9] showed that if the following image conjecture IC(n) (or its special case SIC(n)) holds for all $n \geq 1$, then the Jacobian conjecture has an affirmative answer for all $n \geq 1$.

IMAGE CONJECTURE (IC(n)). *Let $A[z] = A[z_1, \ldots, z_n]$ be the polynomial algebra in n variables over a commutative k-algebra A. Let $a_1, a_2, \ldots, a_n \in A$ be a regular sequence of A and let D be the set of differential operators*

$$a_1 - \partial_{z_1}, a_2 - \partial_{z_2}, \ldots, a_n - \partial_{z_n}.$$

Then $\operatorname{Im} D := \sum_{i=1}^{n} (a_i - \partial_{z_i}) A[z]$ *is a Mathieu subspace of* $A[z]$.

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Recall that $a_1, a_2, \ldots, a_n \in A$ is a regular sequence if the ideal $(a_1, a_2, \ldots, a_n)$ is not equal to A and each a_i is a nonzero divisor of $A/(a_1, a_2, \ldots, a_{i-1})$. A subspace M of a commutative k-algebra B is a Mathieu subspace if the following property holds. If $f \in B$ is such that $f^m \in M$ for all $m \geq 1$, then, for every $g \in B$, there exists a positive integer m_g such that $gf^m \in M$ for all $m \geq m_g$. Mathieu subspaces are a natural generalisation of ideals and named after a conjecture of Mathieu in [8]. They were first proposed by Zhao in [10] and further studied in [7, 11].

Remark 1.1. The image conjecture here is taken from [4, 6]. The original version in [9] is a little more general. It asserts that when $\operatorname{char} k = 0$, the same conclusion holds for n commuting differential operators of order one with constant leading coefficients, that is, differential operators of the form $\partial_{z_i}(q) - \partial_{z_i}$, $i = 1, 2, \ldots, n$, where $q \in A[z]$.

Let $\xi = (\xi_1, \ldots, \xi_n)$ be n new variables. Taking $A = k[\xi] = k[\xi_1, \ldots, \xi_n]$ and $a_i = \xi_i$ in IC(n) gives the so-called special image conjecture.

Special image conjecture (SIC(n)). *Let $k[\xi, z] = k[\xi_1, \ldots, \xi_n, z_1, \ldots, z_n]$ be the polynomial algebra in $2n$ variables over k and let*

$$D = \{\xi_1 - \partial_{z_1}, \xi_2 - \partial_{z_2}, \ldots, \xi_n - \partial_{z_n}\}.$$

Then $\operatorname{Im} D := \sum_{i=1}^n (\xi_i - \partial_{z_i}) k[\xi, z]$ *is a Mathieu subspace of* $k[\xi, z]$.

Several special cases of IC(n) have been proved: if $\operatorname{char} k > 0$, then IC($n$) holds for all $n \geq 1$ [6, Theorem 2.2]; if $\operatorname{char} k = 0$ and Aa_1 is a radical ideal, then IC(1) holds [6, Theorem 2.8]; and if $\operatorname{char} k = 0$ and A is a UFD, then IC(1) holds [7, Theorem 5.1]. In particular, SIC(n) holds for all $n \geq 1$ when $\operatorname{char} k > 0$ and holds for $n = 1$ when $\operatorname{char} k = 0$.

Images of differential operators (including derivations) of polynomial algebras are far from being well understood and only a few results are known. The IC(n) involves differential operators of order one. The purpose of this paper is to investigate images of higher-order differential operators. More precisely, we propose the following conjecture.

Higher order image conjecture (HIC(n)). *Let $k[\xi, z] := k[\xi_1, \ldots, \xi_n, z_1, \ldots, z_n]$ be the polynomial algebra in $2n$ variables over k and let*

$$D = \{\xi_1 - \tau_1, \xi_2 - \tau_2, \ldots, \xi_n - \tau_n\},$$

where $\tau_i \in k[\partial_{z_1}, \ldots, \partial_{z_n}]$ for $i = 1, 2, \ldots, n$. Then $\operatorname{Im} D := \sum_{i=1}^n (\xi_i - \tau_i) k[\xi, z]$ *is a Mathieu subspace of* $k[\xi, z]$.

Our original idea was to find counterexamples to HIC(n) so as to understand the case of order one better. However, it seems that the higher-order case behaves similarly to the case of order one. In fact, we show in Section 2 that HIC(n) holds for all $n \geq 1$ when $\operatorname{char} k > 0$ and it holds for $n = 1$ when $\operatorname{char} k = 0$. The theory of $\mathfrak{D}$−modules plays an important role in our proof. This method was proposed in [4] to deal with the case of order one and characteristic zero, and we develop it to deal with higher order and arbitrary characteristic.

2. Main results

We begin with the basic properties of Weyl algebras and $\mathfrak{D}$−modules. The Weyl algebra of rank n over k, denoted by $A_n(k)$, is the algebra of differential operators (with polynomial coefficients) on the polynomial ring $k[x_1, \dots, x_n]$. When $\operatorname{char} k = 0$, $A_n(k)$ is isomorphic to the associative k-algebra $k[\partial_1, \dots, \partial_n, t_1, \dots, t_n]$ generated by free generators $\partial_1, \dots, \partial_n, t_1, \dots, t_n$ with relations

$$\partial_i\partial_j = \partial_j\partial_i, \quad t_it_j = t_jt_i, \quad \partial_it_j - t_j\partial_i = \delta_{ij}, \quad 1 \le i, j \le n. \tag{2.1}$$

When $\operatorname{char} k = p > 0$, the relations $\partial_i^p = 0, 1 \le i \le n$, should be added to (2.1). A $\mathfrak{D}$−module means a (left) $A_n(k)$-module.

The following result is well known (see, for example, [4, Proposition 3.2]). For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we write $|\alpha| = \sum_{i=1}^n \alpha_i$ and $t^\alpha = t_1^{\alpha_1}t_2^{\alpha_2}\cdots t_n^{\alpha_n}$.

Proposition 2.1. *Let $A_n(k) = k[\partial_1, \dots, \partial_n, t_1, \dots, t_n]$. Let M be an $A_n(k)$-module and $f \in M$. Suppose that each ∂_i is nilpotent on f, that is, there exists some positive integer m_i such that $\partial_i^{m_i} f = 0$. Then f can be written uniquely as*

$$f = \sum_\alpha t^\alpha f_\alpha, \quad f_\alpha \in \bigcap_{i=1}^n \operatorname{Ann}\partial_i \quad (\textit{with } \alpha_i < p \textit{ if } \operatorname{char} k = p > 0).$$

Now we consider the image of the set of differential operators

$$D = \{\xi_1 - \tau_1, \xi_2 - \tau_2, \dots, \xi_n - \tau_n\},$$

on the polynomial algebra $k[\xi, z] = k[\xi_1, \dots, \xi_n, z_1, \dots, z_n]$, where $\tau_i \in k[\partial_{z_1}, \dots, \partial_{z_n}]$ for $i = 1, 2, \dots, n$. Let c_i be the constant term of τ_i. We always assume that $c_i = 0$ (by the coordinate transformation $\xi_i' = \xi_i - c_i$), which ensures that $\tau_i^p = 0$ if $\operatorname{char} k = p > 0$.

When $\operatorname{char} k = 0$ and $\tau_i = \partial_{z_i}$, Zhao [9] constructed a linear map $l : k[\xi, z] \to k[z]$ with $l(\xi^\alpha z^\beta) = \partial_z^\alpha(z^\beta)$, where we use the notation $\partial_z^\alpha = \partial_{z_1}^{\alpha_1}\partial_{z_2}^{\alpha_2}\cdots\partial_{z_n}^{\alpha_n}$, and showed that $\sum_{i=1}^n \operatorname{Im}(\xi_i - \partial_{z_i}) = \ker l$. We will show that a similar result holds for higher-order differential operators in arbitrary characteristic.

Definition 2.2. For $D = \{\xi_1 - \tau_1, \xi_2 - \tau_2, \dots, \xi_n - \tau_n\}$, where $\tau_i \in k[\partial_{z_1}, \dots, \partial_{z_n}]$ for $i = 1, 2, \dots, n$, define $L : k[\xi, z] \to k[z]$ to be the k-linear map such that

$$L(\xi^\alpha z^\beta) = (\tau_1^{\alpha_1}\tau_2^{\alpha_2}\cdots\tau_n^{\alpha_n})(z^\beta).$$

Proposition 2.3. *With the notation of Definition 2.2,* $\operatorname{Im} D = \ker L$.

Proof. Let $f \in \operatorname{Im} D$. Then $f = \sum_{i=1}^n (\xi_i - \tau_i)f_i$ for some $f_i \in k[\xi, z]$. By the definition of L and since $\tau_i \in k[\partial_{z_1}, \dots, \partial_{z_n}]$, we know that $L(\xi_if_i) = \tau_i(L(f_i))$ and $L(\tau_i(f_i)) = \tau_i(L(f_i))$. Thus

$$L(f) = \sum_{i=1}^n (L(\xi_if_i) - L(\tau_i(f_i))) = \sum_{i=1}^n \tau_i(L(f_i)) - \tau_i(L(f_i)) = 0.$$

So $\operatorname{Im} D \subseteq \ker L$.

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For the converse, set $A_{2n}(k) = k[\partial_{\xi_1}, \dots, \partial_{\xi_n}, \partial_{z_1}, \dots, \partial_{z_n}, \xi_1, \dots, \xi_n, z_1, \dots, z_n]$. Note that $k[\xi, z]$ has a natural $A_{2n}(k)$-module structure. Define a k-algebra morphism

$$\rho : A_n(k) = k[\partial_1, \dots, \partial_n, t_1, \dots, t_n] \to A_{2n}(k)$$

by $\rho(\partial_i) = \partial_{\xi_i}, \rho(t_i) = \xi_i - \tau_i$. The definition is reasonable, since

$$\begin{cases} \partial_{\xi_i}\partial_{\xi_j} = \partial_{\xi_j}\partial_{\xi_i}, \\ (\xi_i - \tau_i)(\xi_j - \tau_j) = (\xi_j - \tau_j)(\xi_i - \tau_i), \\ \partial_{\xi_i}(\xi_j - \tau_j) - (\xi_j - \tau_j)\partial_{\xi_i} = \delta_{ij}, \\ \partial^p_{\xi_i} = 0 \quad (\text{when } \operatorname{char} k = p > 0). \end{cases}$$

Thus $k[\xi, z]$ has an $A_n(k)$-module structure defined by

$$\begin{cases} t_i f = (\xi_i - \tau_i) f, \\ \partial_i f = \partial_{\xi_i}(f). \end{cases}$$

For any $f \in k[\xi, z]$, since ∂_i is nilpotent on f by Proposition 2.1, f can be written as

$$f = \sum_\alpha t^\alpha f_\alpha, \quad f_\alpha \in \bigcap_{i=1}^n \operatorname{Ann} \partial_i,$$

and thus

$$f = f_0 + \sum_{\alpha \neq 0} (\xi_1 - \tau_1)^{\alpha_1} \cdots (\xi_n - \tau_n)^{\alpha_n} f_\alpha, \quad f_0 \in \bigcap_{i=1}^n \operatorname{Ann} \partial_i.$$

When $\operatorname{char} k = 0$, $\bigcap_{i=1}^n \operatorname{Ann} \partial_i = \bigcap_{i=1}^n \ker \partial_{\xi_i} = k[z]$, so $f_0 \in k[z]$. When $\operatorname{char} k = p$,

$$\bigcap_{i=1}^n \operatorname{Ann} \partial_i = \bigcap_{i=1}^n \ker \partial_{\xi_i} = k[z, \xi_1^p, \dots, \xi_n^p],$$

and thus

$$f_0 = g_0 + \sum_{\beta \neq 0} (\xi_1^p)^{\beta_1} \cdots (\xi_n^p)^{\beta_n} h_\beta(z) = g_0 + \sum_{\beta \neq 0} (\xi_1 - \tau_1)^{p\beta_1} \cdots (\xi_n - \tau_n)^{p\beta_n} h_\beta(z),$$

where $g_0 \in k[z]$.

In conclusion, f can be written as $f = g + h$, where $g \in k[z]$ and $h \in \operatorname{Im} D \subseteq \ker L$. So $L(f) = L(g) + L(h) = g + 0 = g$. If $f \in \ker L$, then $g = L(f) = 0$ and consequently $f = g + h = h \in \operatorname{Im} D$. It follows that $\ker L \subseteq \operatorname{Im} D$. Therefore, $\ker L = \operatorname{Im} D$. □

In what follows, for $f \in k[\xi, z]$, we denote by f_i the homogeneous part of f with degree i in z. If $\deg_z f = d$, then $f = f_0 + f_1 + \cdots + f_d$, where $f_i = \sum_{|\alpha|=i} c_\alpha z^\alpha$, $c_\alpha \in k[\xi]$.

Lemma 2.4. *Let $g = \sum_\alpha c_\alpha z^\alpha \in \operatorname{Im} D$, where $c_\alpha \in k[\xi]$, and let $\deg_z g = d$. Then $c_\alpha \in I$ for all α with $|\alpha| = d$, where $I = (\xi_1, \dots, \xi_n)$ is the ideal of $k[\xi]$ generated by $\xi_1, \dots, \xi_n$.*

Proof. By Lemma 2.3, $g \in \ker L$, that is, $L(g) = 0$. Since $g = \sum_{|\alpha|=d} c_\alpha z^\alpha + \sum_{|\alpha|<d} c_\alpha z^\alpha$, we know that $0 = L(g) = \sum_{|\alpha|=d} c_\alpha(0) z^\alpha +$ lower order terms, and thus $c_\alpha(0) = 0$, that is, $c_\alpha \in I$ when $|\alpha| = d$. □

THEOREM 2.5. *Suppose that* $\operatorname{char} k = p > 0$. *Then* HIC($n$) *holds for all* $n \geq 1$, *that is,* $\operatorname{Im} D := \sum_{i=1}^{n}(\xi_i - \tau_i)k[\xi, z]$ *is a Mathieu subspace of* $k[\xi, z]$.

PROOF. Since $\xi_i^p h = (\xi_i - \tau_i)^p h$ for any $h \in k[\xi, z]$, It follows that $I^p k[\xi, z] \subseteq \operatorname{Im} D$, where we write $I = (\xi_1, \ldots, \xi_n)$ for the ideal of $k[\xi]$ generated by $\xi_1, \ldots, \xi_n$.

Let $f = \sum_\alpha c_\alpha z^\alpha \in k[\xi, z]$ be such that $f^p \in \operatorname{Im} D$, where $c_\alpha \in k[\xi]$.

Claim. $c_\alpha \in I$ for all α.

Suppose the claim is true. Then $f^p = (\sum_\alpha c_\alpha z^\alpha)^p = \sum_\alpha c_\alpha^p z^{\alpha p} \in I^p k[\xi, z]$. Thus, for any $g \in k[\xi, z]$, we have $g f^m = f^p(g f^{m-p}) \in I^p k[\xi, z] \subseteq \operatorname{Im} D$ for all $m \geq p$. It follows that $\operatorname{Im} D$ is a Mathieu subspace. It now remains to prove the claim.

Write $f = f_0 + f_1 + \cdots + f_d$, where $f_i = \sum_{|\alpha|=i} c_\alpha z^\alpha$. Since

$$f^p = f_0^p + f_1^p + \cdots + f_d^p \in \operatorname{Im} D,$$

by Lemma 2.4, all coefficients of $f_d^p = (\sum_{|\alpha|=d} c_\alpha z^\alpha)^p = \sum_{|\alpha|=d} c_\alpha^p z^{\alpha p}$ belong to I, that is, $c_\alpha^p \in I$ and so $c_\alpha \in I$ for all $|\alpha| = d$. It follows that $c_\alpha^p \in I^p$. So $f_d^p \in I^p k[\xi, z] \subseteq \operatorname{Im} D$. Then $(f_0 + f_1 + \cdots + f_{d-1})^p = f^p - f_d^p \in \operatorname{Im} D$. The claim follows by induction on $d = \deg f$. □

THEOREM 2.6. *Suppose* $\operatorname{char} k = 0$. *Then* HIC(1) *holds, that is, on the polynomial ring* $k[\xi, z]$ *in two variables,* $\operatorname{Im}(\xi - \tau)$ *is a Mathieu subspace, for* $\tau \in k[\partial_z]$.

PROOF. Recall that $L : k[\xi, z] \to k[z]$, $L(\xi^a z^b) = \tau^a(z^b)$. By Proposition 2.3, we have $\operatorname{Im}(\xi - \tau) = \ker L$. We may assume, without loss of generality, that

$$\tau = \partial_z^r + a_{r+1}\partial_z^{r+1} + \cdots + a_d\partial_z^d,$$

where $a_i \in k$. Define a linear map

$$L_0 : k[\xi, z] \to k[z], \quad L_0(\xi^a z^b) = (\partial_z^r)^a(z^b).$$

Consider the w-degree on $k[\xi, z]$, where $w = (-r, 1)$. Then $\deg_w \xi^a z^b = b - ar$. For any $h \in k[\xi, z]$, we denote by $\bar{h}$ the highest homogeneous part of h with respect to w-degree. Let $s = \deg_w h$. Then $h = \sum_{j-ir \leq s} c_{ij}\xi^i z^j$, $\bar{h} = \sum_{j-ir=s} c_{ij}\xi^i z^j$ and

$$\begin{aligned} L(h) = L\Big(\sum_{j-ir\leq s} c_{ij}\xi^i z^j\Big) &= \sum_{j-ir\leq s} c_{ij}(\partial_z^r + a_{r+1}\partial_z^{r+1} + \cdots + a_d\partial_z^d)^i z^j \\ &= \sum_{j-ir=s} c_{ij}(\partial_z^r)^i z^j + \text{lower order terms in } z \\ &= L_0(\bar{h}) + \text{lower order terms in } z. \end{aligned}$$

So $\overline{L(h)} = L_0(\bar{h})$, where $\overline{L(h)}$ means the highest homogeneous part of $L(h) \in k[z]$ in terms of z.

Claim. If $f^m \in \operatorname{Im}(\xi - \tau) = \ker L$ for all $m \geq 1$, then $\deg_w f < 0$.

Suppose that $d := \deg_w f \geq 0$ and write $d = qr + r_0$, where $0 \leq r_0 < r$. Since $L((\xi^q f)^m) = \tau^{qm} L(f^m) = 0$, it follows that $(\xi^q f)^m \in \ker L$. In addition, $\deg_w \xi^q f = (-qr) + d = r_0$. Replacing f by $\xi^q f$, we may assume that $\deg_w f = r_0$ with $0 \leq r_0 < r$.

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So $\overline{f} = \sum_{j-ir=r_0} a_{ij}\xi^i z^j, a_{ij} \in k$. Write

$$\overline{f} = \sum_{n_0 \le i \le n_1} c_i \xi^i z^{ir+r_0} \quad \text{with } c_i \in k.$$

For any $m \ge 1$, $L_0(\overline{f}^m) = L_0(\overline{f^m}) = \overline{L(f^m)} = \overline{0}$.

Now we show that we may assume that $\overline{f} \in \overline{\mathbb{Q}}[\xi, z]$, that is, all the c_i belong to $\overline{\mathbb{Q}}$. Since $L_0(\xi^a z^{ar+b}) = (\partial_z^r)^a (z^{ar+b}) = ((ar+b)!/b!)z^b$,

$$\begin{aligned} 0 = L_0(\overline{f}^m) &= L_0\bigg(\Big(\sum_{n_0 \le i \le n_1} c_i \xi^i z^{ir+r_0}\Big)^m\bigg) \\ &= L_0\Big(\sum_{n_0 \le i_1, i_2, \ldots, i_m \le n_1} c_{i_1} c_{i_2} \cdots c_{i_m} \xi^{i_1+\cdots+i_m} z^{(i_1+\cdots+i_m)r+mr_0}\Big) \\ &= \sum_{n_0 \le i_1, i_2, \ldots, i_m \le n_1} c_{i_1} c_{i_2} \cdots c_{i_m} \frac{((i_1+\cdots+i_m)r+mr_0)!}{(mr_0)!} z^{mr_0}. \end{aligned}$$

Observe that $L_0(\overline{f}^m) = 0$ means that $(c_{n_0}, \ldots, c_{n_1})$ is a (nonzero) zero point of the homogeneous polynomial g_m in the variables $x_{n_0}, \ldots, x_{n_1}$, where

$$g_m := \sum_{n_0 \le i_1, i_2, \ldots, i_m \le n_1} x_{i_1} x_{i_2} \cdots x_{i_m} \frac{((i_1+\cdots+i_m)r+mr_0)!}{(mr_0)!}.$$

Since all these homogeneous polynomials g_m are over $\mathbb{Q}$ and have a common nonzero zero point $(c_{n_0}, \ldots, c_{n_1})$ in k, it is well known that they must have a common nonzero zero point $(\overline{c}_{n_0}, \ldots, \overline{c}_{n_1})$ in $\overline{\mathbb{Q}}$. Replacing $c_{n_0}, \ldots, c_{n_1}$ by $\overline{c}_{n_0}, \ldots, \overline{c}_{n_1}$, we may assume that $c_{n_0}, \ldots, c_{n_1} \in \overline{\mathbb{Q}}$, that is, $\overline{f} \in \overline{\mathbb{Q}}[\xi, z]$.

Assume, without loss of generality, that $c_{n_0} = 1$. Then, for any prime number p,

$$\begin{aligned} 0 = L_0(\overline{f}^p) &= L_0\bigg(\Big(\xi^{n_0} z^{n_0 r + r_0} + \sum_{n_0 < i \le n_1} c_i \xi^i z^{ir+r_0}\Big)^p\bigg) \\ &= L_0\Big(\xi^{n_0 p} z^{n_0 rp + r_0 p} + \sum_{n_0 < i \le n_1} c_i^p \xi^{ip} z^{irp+r_0 p} + p \sum_{n_0 p < j < n_1 p} d_j \xi^j z^{jr+r_0 p}\Big) \\ &= \frac{(n_0 rp + r_0 p)!}{(r_0 p)!} z^{r_0 p} + \sum_{n_0 < i \le n_1} c_i^p \frac{(irp + r_0 p)!}{(r_0 p)!} z^{r_0 p} + p \sum_{n_0 p < j < n_1 p} d_j \frac{(jr + r_0 p)!}{(r_0 p)!} z^{r_0 p}, \end{aligned}$$

where $d_j \in \mathbb{Z}[c_{n_0+1}, \ldots, c_{n_1}]$. Thus

$$1 + \sum_{n_0 < i \le n_1} c_i^p \frac{(irp + r_0 p)!}{(n_0 rp + r_0 p)!} + p \sum_{n_0 p < j < n_1 p} d_j \frac{(jr + r_0 p)!}{(n_0 rp + r_0 p)!} = 0.$$

Since $i > n_0$, we see that $(irp + r_0 p)!/(n_0 rp + r_0 p)!$ is an integer divisible by p and, since $j > n_0 p$, also $(jr + r_0 p)!/(n_0 rp + r_0 p)!$ is an integer. So $p|1$ in $\mathbb{Z}[c_{n_0+1}, \ldots, c_{n_1}]$ for any prime number p. Note that $c_{n_0+1}, \ldots, c_{n_1} \in \overline{\mathbb{Q}}$, so there exists an l in $\mathbb{Z}$ such that

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$c_{n_0+1}, \ldots, c_{n_1}$ are integral in $\mathbb{Z}[1/l]$. So $\mathbb{Z}[c_{n_0+1}, \ldots, c_{n_1}, 1/l]$ is integral over $\mathbb{Z}[1/l]$. When $p \nmid l$, $p\mathbb{Z}[1/l]$ is a prime ideal of $\mathbb{Z}[1/l]$. There exists a prime ideal α of $\mathbb{Z}[c_{n_0+1}, \ldots, c_{n_1}, 1/l]$ such that $\alpha \cap \mathbb{Z}[1/l] = p\mathbb{Z}[1/l]$ (see, for example, [1, Theorem 5.10]). Since $p|1$ in $\mathbb{Z}[c_{n_0+1}, \ldots, c_{n_1}]$, $\alpha = \mathbb{Z}[c_{n_0+1}, \ldots, c_{n_1}, 1/l]$, which contradicts the fact that α is a prime ideal.

This proves the claim, that is, if $f^m \in \operatorname{Im}(\xi - \tau)$ for all $m \geq 1$, then $\deg_w f < 0$. Then, for any $g \in k[\xi, z]$, $\deg_w(gf^m) = \deg_w g + m \deg_w f < 0$ for all sufficiently large m, and thus $L(gf^m) = 0$ for such m. It follows that $\operatorname{Im}(\xi - \tau) = \ker L$ is a Mathieu subspace of $k[\xi, z]$. □

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