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On three-Engel groups

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The following conditions for a group G are investigated:

(i) maximal class n subgroups are normal,

(ii) normal closures of elements have nilpotency class n at most,

(iii) normal closures are n-Engel groups,

(iv) G is an $(n+1)$-Engel group.

Each of these conditions is a consequence of the preceding one. The second author has shown previously that all conditions are equivalent for $n = 1$. Here the question is settled for $n = 2$ as follows: conditions (ii), (iii) and (iv) are equivalent. The class of groups defined by (i) is not closed under homomorphisms, and hence (i) does not follow from the other conditions.

1. Introduction

Engel groups are certain generalized nilpotent groups which have received considerable attention in recent years. Introducing commutator notation, $[y, x] = [y, {}_1x] = y^{-1}x^{-1}yx$, $[y, {}_{n+1}x] = [[y, {}_nx], x]$ and $[x_1, \ldots, x_n, x_{n+1}] = [[x_1, \ldots, x_n], x_{n+1}]$, nilpotency of class n at most is defined by

$$[x_1, \ldots, x_{n+1}] = 1 \text{ for all } x_1, \ldots, x_{n+1} \in G .$$

A group G is called an n-Engel group if it satisfies the seemingly

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weaker condition

$$[y, {}_n x] = 1 \quad \text{for all} \quad x, y \in G .$$

Research on Engel groups has centered mainly on the question whether n-Engel groups are nilpotent or locally nilpotent and on finding restrictions on the class of nilpotent n-Engel groups. See, for example, [1], [2], [3], [5], [8], [9].

Engel conditions also occur quite naturally if a group is covered by a system of normal subgroups of given class or, more generally, by suitable normal Engel subgroups. This observation motivates the investigation of groups defined by one of the following conditions:

(i) maximal class n subgroups of G are normal;

(ii) $\langle x^G \rangle$ has class n at most for $x \in G$;

(iii) $\langle x^G \rangle$ is an n-Engel group for all $x \in G$;

(iv) G is an $(n+1)$-Engel group.

It is easily seen that for arbitrary n each of the conditions listed is a consequence of the preceding one. Moreover, for $n = 1$, Satz II of [7] shows that (iv) in turn implies (i), so that for $n = 1$ all conditions are equivalent. For other values of n the interrelations of these conditions seem to be unknown at present.

In this paper the question will be settled for $n = 2$. We will show that conditions (ii), (iii), (iv) are equivalent, and that the class of groups defined by (i) is not closed under homomorphisms. Thus (i) is not a consequence of the other conditions. Moreover, examples show that (i) does not imply obvious restrictions on the commutator or power structure of such groups.

It should be noted that for $n = 2$ our result also shows that the variety given by (ii), while obviously defined by the 3-variable law $[x^y, x^z, x]$, may be alternatively described by a 2-variable law $[y, {}_3 x]$.

As a corollary we observe that an n-generator 3-Engel group has class $2n$ at most. In view of the results of Heineken [3], this only has nontrivial implications if the group contains elements of 2-power or 5-power order. The result had been previously established in [6] for 3-Engel groups of exponent 4 .

Notations

$\langle X \rangle$ = subgroup generated by the set X ;

$\langle x^G \rangle$ = normal closure of x in $G = \langle g^{-1}xg \mid g \in G \rangle$;

$[X, Y] = \langle [x, y] \mid x \in X, y \in Y \rangle$;

$A \times B$ = direct product of groups A and B ;

$Z(G)$ = center of G ;

$c(G)$ = nilpotency class of G ;

$G_n = [G_{n-1}, G]$, $G_1 = G$, $G' = [G, G]$ and $G'' = [G', G']$;

$\mho_{p,k}(G) = \left\langle g^{p^k} \mid g \in G \right\rangle$ for a prime power p^k .

The following commutator identities are used without further reference:

$$[ab, c] = [a, c]^b[b, c] = [a, c][a, c, b][b, c] ,$$

$$[a, bc] = [a, c][a, b]^c = [a, c][a, b][a, b, c] ,$$

$$[a^{-1}, b]^a[a, b] = 1 , \quad [a, b][b, a] = 1 .$$

2. Normal closures in 3-Engel groups

The following preliminary results are useful in evaluating certain commutator relations.

LEMMA 1. *Suppose* $a \in H$ *is such that* $c(\langle a^H \rangle) \le 3$ *and* $[h, {}_3a] = 1$ *for all* $h \in H$. *Then*

(1) $$[z, a, a] = 1 \text{ for all } z \in \langle a^H \rangle .$$

Proof. Let $z_1, z_2 \in \langle a^H \rangle$. Then

$$[z_1z_2, a, a] = [z_1, a, a][z_2, a, a] .$$

Hence the elements $z \in \langle a^H \rangle$ such that $[z, a, a] = 1$ form a subgroup V of $\langle a^H \rangle$. Clearly $a \in V$ and $[h, a] \in V$ by assumption. Thus $a^h \in V$ for all $h \in H$; consequently $\langle a^H \rangle = V$.

LEMMA 2. *Let $a, d \in H$ be such that*

(a) H is a 3-Engel group;

(b) $\langle g^H \rangle_3 \subseteq Z(H)$ for $g = a, d, ad$.

Then for $z \in \langle a^H \rangle$ we have

(2) $$[z, a, d, a] = [z, a, d, a^d],$$

(3) $$[z, d, a^d, a] = 1,$$

(4) $$[z, a, d, a]^2 = 1.$$

Proof. These results follow from expanding $[z, ad, ad, ad] = 1$. For later use we record some of the intermediate steps.

(5) $$[z, ad, a] \doteq [z, d, a][[z, a]^d, a],$$

(6) $$[z, ad, d] = [z, d, d]^{[z,a]^d}[[z, a], d]^d,$$

(7) $$[z, ad, ad] = [z, d, d]^{[z,a]^d}[z, a, d]^d[z, d, a]^d[[z, a]^d, a]^d.$$

Denote the factors on the right side of (7) by z_1, z_2, z_3 and z_4. By (b) and the observation that $\langle a^H \rangle' \subseteq \langle a^H \rangle_4 = 1$ we have

$$[z, ad, ad, t] = [z_1, t][z_2, t][z_3, t][z_4, t] \quad \text{for } t = a, d.$$

For the individual terms we find the following simplifications: $[z_4, a]^d = [z_4, d] = 1$, $[z_2, a]^d = [z, a, d, a][z, a, d, d, a]$ and $[z_1, a]^d = [z, d, d, a][z, d, d, a, d]$ by (b). Further $[z_1, d] = [z, d, d, d] = 1$ by (b) and (a), and $[z_2, d] = [z, a, d, d]$ by (a). Finally $[z_3, d] = [z, d, a, d]$ by (1), and

$[z_3, a]^d = [z, d, a, a][z, d, a, d, a] = [z, d, a, d, a]$ by *(b)* and (1).

Combining these results we have

(8) $\quad 1 = [z, ad, ad, ad] = f(z, a, d)g(z, a, d)$ for all $z \in \langle a^H \rangle$,

where

$$f(z, a, d) = [z, a, d, d][z, d, a, d][z, d, d, a][z, a, d, a] ,$$

$$g(z, a, d) = [z, d, d, a, d][z, d, a, d, a][z, a, d, d, a] .$$

If $y \in \langle a^H \rangle$ and $z = [y, a]$, then $f([y, a], a, d) = [y, a, d, d, a]$ and $g([y, a], a, d) = 1$ by *(b)*. Thus (8) yields

(9) $\quad [y, a, d, d, a] = 1$ for all $y \in \langle a^H \rangle$.

Therefore by (9) and *(a)*, $[z, a, d, d, a^d] = [z, a, d, d, a]^d = 1$ for $z \in \langle a^H \rangle$. Together with *(b)* this proves (2):

$$[z, a, d, a] = [[z, a]^d, d, a^d]$$
$$= [z, a, d, a^d][z, a, d, d, a^d] = [z, a, d, a^d] .$$

Similarly for $z = [y, d]$ and $y \in \langle a^H \rangle$ we have $g([y, d], a, d) = 1$ by *(b)*. Thus (8) yields

(10) $\quad 1 = [y, d, d, a, d][y, d, a, d, a]$ for all $y \in \langle a^H \rangle$.

Combining (8), (9) and (10) we have $f(z, a, d) = 1$ by Lemma 1 and *(a)*; hence

(11) $1 = [z, a, d, d][z, d, a, d][z, d, d, a][z, a, d, a]$ for $z \in \langle a^H \rangle$.

Commuting (11) by a gives

(12) $\quad 1 = [z, d, a, d, a]$ for all $z \in \langle a^H \rangle$

by (9) and (1). To show (3) let $t^d = z$. Then by (1) and (12)

$$[z, d, a^d, a] = [[t, d, a]^d, a] = [t, d, a, a][t, d, a, d, a] = 1 .$$

Because *(b)* holds also for $g = ad$ we may replace d by ad in (11). For the individual terms of $f(z, a, ad)$ we obtain the following

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simplifications. Replacing z by $[z, a]$ in (7) we have by *(b)*,

$$[z, a, ad, ad] = [z, a, d, d][z, a, d, a] .$$

From (5) by *(b)* and (1),

$$[z, ad, a, ad] = [z, d, a, ad] = [z, d, a, d] .$$

From (7) we obtain by *(b)*, (9), (12) and (1),

$$\begin{aligned}[z, ad, ad, a] &= \big[[z, d, d], a\big]\big[[z, a, d]^d, a\big]\big[[z, d, a]^d, a\big] \\ &= [z, d, d, a][z, a, d, a] .\end{aligned}$$

Finally replacing z by $[z, a]$ in (5) we find by *(b)*,

$$[z, a, ad, a] = [z, a, d, a] .$$

Together this gives

$$1 = f(z, a, ad) = f(z, a, d)[z, a, d, a]^2 = [z, a, d, a]^2 ,$$

proving (4).

THEOREM. *The following conditions for a group* G *are equivalent:*

(ii) $c(\langle x^G \rangle) \leq 2$ *for all* $x \in G$ *;*

(iii) $\langle x^G \rangle$ *is a* 2-*Engel group for all* $x \in G$ *;*

(iv) G *is a* 3-*Engel group.*

Proof. As pointed out in the introduction it suffices to show that *(iv)* implies *(ii)*. Assume *(iv)* and that $c(\langle x^G \rangle) > 2$ for some $x \in G$. Then [5 , III.1.9] there are $r, s, t \in G$ such that $[x^r, x^s, x^t] \neq 1$. Since 3-Engel groups are locally nilpotent [1, Hauptsatz 2], the subgroup $U = \langle x, r, s, t \rangle$ is nilpotent, and satisfies the maximum condition [9, VI.6.a]. Let M be maximal among the normal subgroups of U such that $H = U/M$ does not satisfy condition *(ii)*. Since H is nilpotent, $Z(H) \neq 1$, and $H/Z(H)$ satisfies condition *(ii)*. This proves

$$\langle h^H \rangle_3 \subseteq Z(H) \quad \text{for all} \quad h \in H .$$

We may therefore apply Lemma 2.

Next we will evaluate the Jacobi identity

(13) $$1 = [u, v, w^u][w, u, v^w][v, w, u^v]$$

for various values of $u, v, w \in H$. First for $u = a$, $v = d$, $w = [z, a]$ and $z \in \langle a^H \rangle$ we have by *(b)* and (1),

$$1 = [a, d, [z, a]][d, [z, a], a^d] ;$$

hence by *(b)* and (4)

(14) $$[[a, z], [a, d]] = [z, a, d, a] .$$

Next let $u = a$, $v = d$, $w = z \in \langle a^H \rangle$ and commute (13) by a. We have by *(b)* and (3),

(15) $$1 = [a, d, z, a][z, a, d, a] .$$

Finally let $u = [a, d]$, $v = a$ and $w = z \in \langle a^H \rangle$. This gives by *(b)*,

(16) $$1 = [a, d, a, z][a, d, z, a]^{-1}[[a, z], [a, d]] .$$

Combining (14), (15), (16) and (2) we obtain

$$1 = [a, d, a, z] \text{ for all } a, d \in H \text{ and } z \in \langle a^H \rangle .$$

But

$$1 = [a, d, a, z] = [a^{-1}a^d, a, z] = [a^d, a, z] \text{ for all } z \in \langle a^H \rangle$$

and $a \in H$ proves $c(\langle a^H \rangle) \le 2$ for all $a \in H$, contrary to the choice of H.

COROLLARY. *If a 3-Engel group G is generated by n elements then* $c(G) \le 2n$.

Proof. If $G = \langle x_1, \ldots, x_n \rangle$, then G is the product of the n normal subgroups $\langle x_i^G \rangle$, which have class at most 2 by the theorem. Hence $c(G) \le 2n$ [5, III.4.1].

3. Examples

In this section a torsion free group $G(\infty)$ of class precisely 4 is constructed satisfying condition (i). Example 3 is a homomorphic image of $G(\infty)$ which fails to satisfy (i). Since conditions (ii), (iii) and (iv)

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obviously define varieties, this shows that (i) does not follow from the other conditions. Example 2, another quotient group of $G(\infty)$, shows that finite p-groups satisfying (i) may have class greater than 3 , and that unlike the situation in a class of groups recently investigated by Heineken [4], there are also no obvious restrictions on the structure of G_4.

The following lemmas will be needed in establishing that Examples 1 and 2 satisfy condition (i).

LEMMA 3. *Maximal class* 2 *subgroups are normal in* G *if and only if*

(17) *for all* $x, a, b, c \in G$:

$$[x, a, b, c] = 1 \text{ whenever } c(\langle a, b, c\rangle) \le 2 .$$

Proof. Let $c(\langle a, b, c\rangle) \le 2$ and U a maximal class 2 subgroup of G containing $\langle a, b, c\rangle$. Then $U^x = U$ implies $\langle [x, a], b, c\rangle \subseteq U$, hence $[x, a, b, c] = 1$.

Conversely let V be a maximal class 2 subgroup and $u, v, w \in V$. Applying (17) with $x = y^u$, $a = u^{-1}$, $b = v^u$ and $c = w$ we obtain

$$[u^y, v, w] = \left[[y, u^{-1}]u, v, w\right] =$$

$$= \left[[y, u^{-1}, v]^u, w\right]^{[u,v]}[u, v, w] = [y^u, u^{-1}, v^u, w]^{[u,v]} = 1 .$$

This implies

(18) $$[v, u^y, w] = 1 ,$$

and since y is arbitrary,

(19) $$[u^z, v, w^z] = \left[u, v^{z^{-1}}, w\right]^z = 1 .$$

Similarly follows $[v, u^z, w^z] = 1$.

To prove $c(\langle u^y, V\rangle) \le 2$ it remains to be shown that $[v, w, u^y] = 1$. The Jacobi identity gives

$$1 = [v, w, t^v][t, v, w^t][w, t, v^w] .$$

With $t = u^y$ we have $[w, t, v^w] = 1$ from (18), and from (19),

$$[t, v, w^t] = [t, v, t^{-1}wt] = [t, v, t][t, v, w]^t[t, v, t^{-1}]^{wt} = 1 .$$

Thus $1 = [v, w, u^{yv}]$ for all $u, v, w \in V$, in particular

$$1 = [v, w^v, u^{yv}] = [v, w, u^y]^v .$$

LEMMA 4. *Let R be the ring of integers Z or the ring Z/Zp^n of integers modulo a prime power p^n. Let $\{\beta_{\lambda i} \mid 1 \leq \lambda \leq s, 1 \leq i \leq t\}$ be a set of not necessarily distinct elements $\beta_{\lambda i} \in R$ such that*

$$(\beta_{\lambda i}\beta_{\mu j}-\beta_{\lambda j}\beta_{\mu i})\beta_{\lambda k} = 0 \tag{20}$$

for all λ, μ, i, j, k. Then

$$\beta_{\sigma i}\beta_{\tau j}\beta_{\rho k} = \beta_{\pi(\sigma)i}\beta_{\pi(\tau)j}\beta_{\pi(\rho)k} \tag{21}$$

for all i, j, k, any triple $\sigma, \tau, \rho \in \{1, 2, \ldots, s\}$ of 3 distinct indices and any permutation π of the set $\{\sigma, \tau, \rho\}$.

Proof. For convenience let $\{\sigma, \tau, \rho\} = \{1, 2, 3\}$. The permutations $\pi \in S_3$ such that (21) holds form a subgroup Δ. Note that (21) holds trivially if $\beta_{1r} = 0$ for all r.

For $R = Z$ and $\beta_{1r} \neq 0$ for suitable r condition (20) implies

$$\beta_{1i}\beta_{\mu j} = \beta_{1j}\beta_{\mu i} \quad \text{for all } i, j, \mu .$$

Hence the permutations (12) and (13) are in Δ and $\Delta = S_3$. For $R = Z/Zp^n$ we may assume without loss of generality that p^m divides all $\beta_{1i}, \beta_{2j}, \beta_{3k}$, but p^{m+1} does not divide β_{1r} for some r. Condition (20) implies

$$\beta_{1i}\beta_{\mu j} \equiv \beta_{1j}\beta_{\mu i} \bmod p^{n-m} \quad \text{for all } i, j, \mu .$$

Hence

$$\beta_{1i}\beta_{\mu j}\beta_{vk} \equiv \beta_{\mu i}\beta_{1j}\beta_{vk} \bmod p^n$$

for all i, j, k and $v, u \in \{2, 3\}$. Thus the permutations (12) and (13) are in Δ and $\Delta = S_3$, proving (21).

LEMMA 5. *If G is nilpotent of class $c(G) < p$ then the set $\mho_{p,k}(G)$ of p^k-th powers of elements of G is a (normal) subgroup of G.*

This is proved as in the case of regular p-groups [5, III.10.5.6] using the Hall-Petrescu formula and the fact that p^k divides $\binom{p^k}{i}$ for all $0 < i < p$. The induction here however is on the class rather than the order of the group.

EXAMPLE 1. The construction of $G(\infty)$ follows usual practice (see, for example, [8]), starting from a group A_0 isomorphic to $G'(\infty)$. With A_i given, the group A_{i+1} is the semidirect product of A_i with an infinite cyclic group $\langle a_{i+1} \rangle$, where a_{i+1} induces an automorphism α_{i+1} on A_i, and $A_4 = G(\infty)$. The action of α_{i+1} on the generators of A_i is not given explicitly, but in the form

$$(R_{i+1}) \qquad [g, a_{i+1}] \in A_i ,$$

where g runs through the generators of A_i, and $g^{\alpha_{i+1}} = g[g, a_{i+1}]$. The defining relations of A_{i+1} are then those of A_i together with (R_{i+1}).

The lengthy but straightforward verification that α_{i+1} is an automorphism of A_i is omitted here. It involves checking that the images of the generating set of A_i under α_{i+1} generate A_i, and that α_{i+1} preserves the defining relations of A_i.

The group A_0 is the direct product of free abelian groups $X = \langle x_1, \ldots, x_{12} \rangle$, $Y = \langle y_1, \ldots, y_8 \rangle$ of rank 12 and 8 respectively and a group $A = \langle u_r, z_i \rangle$ of class 2. Then

$$A_0 = \langle u_r, x_j, y_k, z_i \mid 1 \le r \le 6, 1 \le j \le 12, 1 \le k \le 8, 1 \le i \le 3 \rangle$$

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with defining relations

$$(R_0)\begin{cases}[u_1, u_4] = z_1^{-2}z_3^{-2},\ [u_2, u_5] = z_1^{-10}z_2^{8}z_3^{-6},\ [u_3, u_6] = z_1^{-6}z_2^{4}z_3^{-2}, \\ [u_i, u_j] = 1 \ \text{for} \ (i, j) \neq (1, 4), (2, 5), (3, 6), \\ [u_r, z_i] = [z_i, z_j] = [x_j, z_i] = [x_j, u_r] = [x_i, x_j] = 1, \\ [x_j, y_k] = [y_k, u_r] = [y_i, y_k] = [y_k, z_i] = 1 ;\end{cases}$$

$$(R_1)\begin{cases}[u_1, a_1] = x_1,\ [u_2, a_1] = x_3,\ [u_3, a_1] = x_5, \\ [u_4, a_1] = y_5^{-1}y_6,\ [u_5, a_1] = y_3^{-1}y_4,\ [u_6, a_1] = y_1^{-1}y_2, \\ [y_7, a_1] = z_1^{-5}z_2^{4}z_3^{-2},\ [y_8, a_1] = z_1^{-7}z_2^{5}z_3^{-4}, \\ [x_j, a_1] = [z_i, a_1] = 1 \ \text{for all} \ i, j;\ [y_k, a_1] = 1 \ \text{for} \ k \neq 7, 8 ;\end{cases}$$

$$(R_2)\begin{cases}[a_1, a_2] = u_1,\ [u_1, a_2] = x_2,\ [u_2, a_2] = y_2,\ [u_3, a_2] = y_4, \\ [u_4, a_2] = y_7^{-1}y_8,\ [u_5, a_2] = x_9,\ [u_6, a_2] = x_7, \\ [y_5, a_2] = z_1^{6}z_2^{-5}z_3^{3},\ [y_6, a_2] = z_1^{6}z_2^{-4}z_3^{3}, \\ [x_j, a_2] = [z_i, a_2] = 1 \ \text{for all} \ i, j;\ [y_k, a_2] = 1 \ \text{for} \ k \neq 5, 6 ;\end{cases}$$

$$(R_3)\begin{cases}[a_1, a_3] = u_2,\ [a_2, a_3] = u_6,\ [u_1, a_3] = y_1,\ [u_2, a_3] = x_4, \\ [u_3, a_3] = y_6,\ [u_4, a_3] = x_{11},\ [u_5, a_3] = y_8,\ [u_6, a_3] = x_8, \\ [y_3, a_3] = z_1^{-3}z_2^{3}z_3^{-1},\ [y_4, a_3] = z_3,\ [y_k, a_3] = 1 \ \text{for} \ k \neq 3, 4 ; \\ [x_j, a_3] = [z_i, a_3] = 1 \ \text{for all} \ i, j ;\end{cases}$$

$$(R_4)\begin{cases}[a_1, a_4] = u_3,\ [a_2, a_4] = u_5,\ [a_3, a_4] = u_4, \\ [u_1, a_4] = y_3,\ [u_2, a_4] = y_5,\ [u_3, a_4] = x_6, \\ [u_4, a_4] = x_{12},\ [u_5, a_4] = x_{10},\ [u_6, a_4] = y_7, \\ [y_1, a_4] = z_1^{-5}z_2^{3}z_3^{-3},\ [y_2, a_4] = z_1^{-4}z_2^{3}z_3^{-3}, \\ [y_k, a_4] = 1 \ \text{for} \ k \neq 1, 2;\ [x_j, a_4] = [z_i, a_4] = 1 \ \text{for all} \ i, j .\end{cases}$$

For convenience let $G = G(\infty)$. We have $G' = A_0$, $G_3 = [A_0, G] = \langle X, Y, G_4 \rangle$, $G_4 = \langle z_1, z_2, z_3 \rangle$, $G_5 = 1$, $G'' = A_0' \neq 1$

and $Z(G) = X \times G_4$. From the defining relations of G it follows that G/G', G'/G_3, $G_3/G_4 \cong X \times Y$ and G_4 are free abelian of rank 4, 6, 20 and 3 respectively. In particular G is torsion free.

To show that G satisfies (i) we apply Lemma 3. Each $d_i \in G$ $(i = 1, 2, 3)$ has a unique presentation

$$d_i \equiv a_1^{\beta_{1i}} a_2^{\beta_{2i}} a_3^{\beta_{3i}} a_4^{\beta_{4i}} \bmod G'$$

with integers $\beta_{\lambda i}$. Then

$$[d_i, d_j, d_k] \equiv \prod_s x_s^{\rho(s)} \bmod \langle Y, G_4 \rangle$$

where $\rho(s)$ is of the form

$$\rho(s) = (\beta_{\lambda i}\beta_{\mu j} - \beta_{\lambda j}\beta_{\mu i})\beta_{\rho k}, \quad 1 \le s \le 12,$$

with the following combinations $(s; \lambda, \mu, \rho)$:

(1; 1, 2, 1), (2; 1, 2, 2), (3; 1, 3, 1), (4; 1, 3, 3),
(5; 1, 4, 1), (6; 1, 4, 4), (7; 2, 3, 2), (8; 2, 3, 3),
(9; 2, 4, 2), (10; 2, 4, 4), (11; 3, 4, 3), (12; 3, 4, 4).

Assume now that $c(\langle d_1, d_2, d_3 \rangle) \le 2$. Then $[d_i, d_j, d_k] = 1$ for all $1 \le i, j, k \le 3$. Since $X \cong G_3/\langle Y, G_4 \rangle$ is freely generated by the x_s, we have $\rho(s) = 0$; hence (20) holds for $\lambda \ne \mu$ and trivially for $\lambda = \mu$. Because $c(G) = 4$ it suffices to prove $[x, d_1, d_2, d_3] = 1$ for $x = a_1, a_2, a_3, a_4$. We have

$$[x, d_1, d_2, d_3] = \prod_{\sigma,\tau,\rho} [x, a_\sigma, a_\tau, a_\rho]^{\beta_{\sigma 1}\beta_{\tau 2}\beta_{\rho 3}}$$

In particular

(22) $[a_1, d_1, d_2, d_3] =$

$$[y_1, a_4]^{\beta_{21}\beta_{32}\beta_{43}} [y_2, a_4]^{\beta_{22}\beta_{31}\beta_{43}} [y_3, a_3]^{\beta_{21}\beta_{42}\beta_{33}}$$
$$[y_4, a_3]^{\beta_{41}\beta_{22}\beta_{33}} [y_5, a_2]^{\beta_{31}\beta_{42}\beta_{23}} [y_6, a_2]^{\beta_{41}\beta_{32}\beta_{2-}}$$

All exponents in (22) are equal by Lemma 4 and

$$[y_1, a_4][y_2, a_4][y_3, a_3][y_4, a_3][y_5, a_2][y_6, a_2] = \left(z_1^{-5}z_2^3z_3^{-3}\right)\left(z_1^{-4}z_2^3z_3^{-3}\right)\left(z_1^{-3}z_2^3z_3^{-1}\right)z_3\left(z_1^6z_2^{-5}z_3^3\right)\left(z_1^6z_2^{-4}z_3^3\right) = 1 .$$

Hence $[a_1, d_1, d_2, d_3] = 1$. Similarly for $[a_2, d_1, d_2, d_3]$, $[a_3, d_1, d_2, d_3]$ and $[a_4, d_1, d_2, d_3]$ we obtain

$$\left([y_1, a_4]^{-2}[y_2, a_4][y_3, a_3]^{-2}[y_4, a_3][y_7, a_1][y_8, a_1]\right)^{\beta_{11}\beta_{32}\beta_{43}} = 1 ,$$

$$\left([y_1, a_4][y_2, a_4]^{-2}[y_5, a_2]^{-2}[y_6, a_2][y_7, a_1]^{-2}[y_8, a_1]\right)^{\beta_{11}\beta_{22}\beta_{43}} = 1 ,$$

$$\left([y_3, a_3][y_4, a_3]^{-2}[y_5, a_2][y_6, a_2]^{-2}[y_7, a_1][y_8, a_1]^{-2}\right)^{\beta_{11}\beta_{22}\beta_{33}} = 1 .$$

Thus, by Lemma 3, G satisfies (i).

EXAMPLE 2. This is the group $G(p^n) = G(\infty)/\mho_{p,n}(G(\infty))$ for $p > 3$ and $n \geq 1$. Then $G(p^n)$ is regular, hence of exponent p^n . To show that $G(p^n)$ also satisfies condition (i) we proceed as with $G(\infty)$. We have to show here that the x_i mod $\mho_{p,n}(G(\infty))$ are independent and of order precisely p^n . Assume

$$t = z_1^u z_2^v z_3^w \prod x_i^{m_i} \in \mho_{p,n}(G(\infty))$$

with integers $u, v, w, m_1, \ldots, m_{12}$. Then $t = g^{p^n}$ for some $g \in G(\infty)$ by Lemma 5. Note that $G(\infty)/\left(X \times G_4(\infty)\right)$ is torsion free, and hence $t \in X \times G_4(\infty)$ implies $g \in X \times G_4(\infty)$. Since $z_1, z_2, z_3, x_1, \ldots, x_{12}$ are free generators of the abelian group $X \times G_4(\infty)$ this proves p^n divides $u, v, w, m_1, \ldots, m_{12}$ and thus the assertion. We have also established that $G_4(p^n)$ has exponent p^n and order p^{3n} .

EXAMPLE 3. We finally construct a homomorphic image $G(\infty)/N$ which fails to satisfy (i). The subgroup

$$N = \langle X, y_1, y_2, z_1, z_2 z_3^{-1} \rangle$$

is free abelian, and normal since $Z(G(\infty)) = X \times G_4(\infty)$, $y_1, y_2 \in Z(A_3)$ and $[y_1, a_4], [y_2, a_4] \in \langle z_1, z_2 z_3^{-1} \rangle$. It is easily verified that $[a_i, a_j, a_k] \in N$ for $i, j, k \in \{1, 2, 3\}$. But

$$[a_4, a_1, a_2, a_3] = [u_3^{-1}, a_2, a_3] = [y_4^{-1}, a_3] = z_3^{-1} \notin N ,$$

and thus $G(\infty)/N$ fails (i) by Lemma 3.

We mention without proof that the corresponding homomorphic image $G(\infty)/N \cdot \mho_{p,n}(G(\infty))$ of Example 2 also does not satisfy condition (i).

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