The Properties of a new Orthogonal Function Associated with the Confluent Hypergeometric Function.

BY GEORGE E. CHAPPELL, Univ. College, Swansea.

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1. *Introduction.*

The function $M_k\left(\frac{z}{k}\right)$ whose properties are discussed in this note, is a special form of Whittaker's Confluent Hypergeometric Function, $W_{km}(z)$. It is the general solution of the Differential Equation

$$\frac{d^2u}{dz^2}+\left(\frac{1}{4z^2}+\frac{1}{z}-\frac{1}{4k^2}\right)u=0$$

and can be obtained in the form of a series, terminating only when k is half a positive odd integer, viz.,

$$M_k\left(\frac{z}{k}\right)=e^{-\frac{z}{2k}}\left(\frac{z}{k}\right)^{\frac{1}{2}}\sum_{n=0}^{\infty}\left\{\frac{(\frac{1}{2}-k)(\frac{3}{2}-k)\dots(n-\frac{1}{2}-k)}{n!\;n!}\left(\frac{z}{k}\right)^n\right\}.$$

There is a second solution of this equation, obtained in the usual way, viz.,

$$N_k\left(\frac{z}{k}\right)=M_k\left(\frac{z}{k}\right)\log\frac{z}{k}+e^{-\frac{z}{2k}}\left(\frac{z}{k}\right)^{\frac{1}{2}}Q_k\left(\frac{z}{k}\right)$$

where $Q_k\left(\frac{z}{k}\right)=\sum_{n=1}^{\infty}\left[\frac{(\frac{1}{2}-k)(\frac{3}{2}-k)\dots(n-\frac{1}{2}-k)}{n!\;n!}\right.$

$$\left.\left\{\frac{1}{\frac{1}{2}-k}+\dots\frac{1}{n-\frac{1}{2}-k}-2\left(1+\frac{1}{2}+\dots\frac{1}{n}\right)\right\}\left(\frac{z}{k}\right)^{n}\right].$$

This series is absolutely convergent for *all finite* values of z and k.

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2. Relation to $W_{km}(z)$.

Whittaker obtained two solutions of the Equation

$$\frac{d^2u}{dz^2}+\left(-\frac{1}{4}+\frac{k}{z}+\frac{\frac{1}{4}-m^2}{z^2}\right)u=0$$

viz., $$M_{km}(z)=z^{\frac{1}{2}+m}e^{-\frac{z}{2}}\left\{1+\sum_{n=1}^{\infty}\left(\frac{(\frac{1}{2}+m-k)\ldots(n-\frac{1}{2}+m-k)}{n!\,(2m+1)\ldots(2m+n)}z^n\right)\right\}$$

$$M_{k,-m}(z)=z^{\frac{1}{2}-m}e^{-\frac{z}{2}}\left\{1+\sum_{n=1}^{\infty}\left(\frac{(\frac{1}{2}-m-k)\ldots(n-\frac{1}{2}-m-k)}{n!\,(1-2m)\ldots(n-2m)}z^n\right\}\right)$$

and $$W_{km}(z)=\frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-k)}M_{km}(z)+\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}M_{k,-m}(z).$$

It will be noted that $M_k\left(\frac{z}{k}\right)=M_{k.m}\left(\frac{z}{k}\right)=M_{k,-m}\left(\frac{z}{k}\right)$ when $m=0$.

$$\therefore\quad W_{k.0}\left(\frac{z}{k}\right)=\frac{2}{\Gamma(\frac{1}{2}-k)}\,M_k\left(\frac{z}{k}\right)$$

$$\text{or}\quad M_k\left(\frac{z}{k}\right)=\frac{\Gamma(\frac{1}{2}-k)}{2}\,W_{k.0}\left(\frac{z}{k}\right).$$

3. Orthogonal Properties.

Consider the two functions $M_k\left(\frac{z}{k}\right)$ $M_\lambda\left(\frac{z}{\lambda}\right)$ where $\lambda \neq k$, satisfying the equations

$$M_k''\left(\frac{z}{k}\right)+\left(\frac{1}{4z^2}+\frac{1}{z}-\frac{1}{4k^2}\right)M_k\left(\frac{z}{k}\right)=0$$

$$M_\lambda''\left(\frac{z}{\lambda}\right)+\left(\frac{1}{4z^2}+\frac{1}{z}-\frac{1}{4\lambda^2}\right)M_\lambda\left(\frac{z}{\lambda}\right)=0.$$

Multiplying these equations by $M_\lambda\left(\frac{z}{\lambda}\right)$, $M_k\left(\frac{z}{k}\right)$ respectively, subtracting and integrating the result we have

$$\frac{\lambda^2-k^2}{4\lambda^2k^2}\int_0^1 M_k\left(\frac{z}{k}\right)M_\lambda\left(\frac{z}{\lambda}\right)dz=\Big[M_\lambda\left(\frac{z}{\lambda}\right)M_k'\left(\frac{z}{k}\right)$$
$$-M_k\left(\frac{z}{k}\right)M_\lambda'\left(\frac{z}{\lambda}\right)\Big]_0^1.$$

When $z=0$, it is clear that

$$M_\lambda\left(\frac{z}{\lambda}\right) M_k'\left(\frac{z}{k}\right) - M_k\left(\frac{z}{k}\right) M_\lambda'\left(\frac{z}{\lambda}\right)$$

$$= \frac{1}{2\sqrt{k\lambda}}.$$

$$\therefore \int_0^1 M_k\left(\frac{z}{k}\right) M_\lambda\left(\frac{z}{\lambda}\right) dz = \frac{4\lambda^2 k^2}{\lambda^2 - k^2}\left\{M_\lambda\left(\frac{1}{\lambda}\right) M_k'\left(\frac{1}{k}\right) - M_k\left(\frac{1}{k}\right) M_\lambda'\left(\frac{1}{\lambda}\right)\right\}$$

and as $M_k'\left(\frac{z}{k}\right)$ and $M_\lambda'\left(\frac{z}{\lambda}\right)$ are finite for $z=1$, it follows that

$$\int_0^1 M_k\left(\frac{z}{k}\right) M_\lambda\left(\frac{z}{\lambda}\right) dz = 0$$

provided k, λ are any two zeros of the Function $M_k\left(\frac{1}{k}\right)$.

Hence $M_k\left(\frac{z}{k}\right)$, $M_\lambda\left(\frac{z}{\lambda}\right)$ are orthogonal over the range 0 to 1 provided k, λ are two zeros of $M_k\left(\frac{1}{k}\right)$; *i.e.* of

$$1 + \frac{\frac{1}{2k} - 1}{1^2} + \frac{\left(\frac{1}{2k} - 1\right)\left(\frac{3}{2k} - 1\right)}{1^2 . 2^2} + \ldots$$

It may be noted that the orthogonal property can be extended to the range 0 to α, provided k, λ are two zeroes of $M_k\left(\frac{\alpha}{k}\right)$.

4. *The Zeroes of the Function.*

$M_k\left(\frac{z}{k}\right)$ has a zero at $z=0$ and at all the zeros of

$$J_k\left(\frac{z}{k}\right) = 1 + \frac{\frac{1}{2} - k}{1^2}\frac{z}{k} + \frac{\left(\frac{1}{2} - k\right)\left(\frac{3}{2} - k\right)}{1^2 . 2^2}\frac{z^2}{k^2} + \ldots$$

The position of the zeroes of this function can be ascertained by adopting the *Methods of Sturm* in connection with the equation

$$\frac{d^2 M_k\left(\frac{z}{k}\right)}{dz^2} - \left(\frac{1}{4k^2} - \frac{1}{z} - \frac{1}{4z^2}\right) M_k\left(\frac{z}{k}\right) = 0.$$

The effect of reducing the coefficient of $M_k\left(\frac{z}{k}\right)$ in this equation (*i.e.* of increasing the value of $|k|$), is to increase the frequency of oscillation of the solutions. It follows that there must be at least one zero of $M_\lambda\left(\frac{z}{\lambda}\right)$ between each pair of consecutive zeroes of $M_k\left(\frac{z}{k}\right)$ provided $|\lambda| > |k|$. Further, if $\lambda \to \infty$, all zeroes of $M_k\left(\frac{z}{k}\right)$ are separated by at least one zero of

$$1 - \frac{z}{1^2} + \frac{z^2}{1^2\,2^2} - \dots (-)^n \frac{z^n}{n!\,n!} \dots$$

which is the *Clifford Bessel Function* $J_0(2\sqrt{z})$. As $J_0(2\sqrt{z})$ obviously has no negative zeroes, it follows that *all the zeroes of* $M_k\left(\frac{z}{k}\right)$ *are positive and none of them can be less than* ζ_1 $(=1{\cdot}446\dots)$ *the first positive zero of* $J_0(2\sqrt{z})$.

It can further be shown that if $|k|$ lies between $\frac{2s-1}{2}$ and $\frac{2s+1}{2}$, where s is a positive integer, there will be exactly s positive zeroes of the Function. For values of the variable exceeding $\left|\frac{2k\,s^2}{2s-1-2k}\right|$, the series for $J_k\left(\frac{z}{k}\right)$ is positive or negative according as s is an even or an odd integer. It thus appears that the effect of increasing k by unity is to change the sign of $J_k\left(\frac{z}{k}\right)$ for sufficiently large values of z. An examination of this series shows that there cannot be more than s positive zeroes, and as the frequency of the zeroes increases as k increases, it follows that there will be exactly s zeroes when $|k|$ lies between the limits stated.

If we regard $M_k\left(\frac{z}{k}\right)$ as a function of k for any assigned value of z, we find that if z lies between the n^{th} and $(n+1)^{\text{th}}$ zeros of the Clifford Bessel Function, $J_0(2\sqrt{z})$, the function will vanish for n positive values of k, and for n negative values equal and opposite to these.

5. *Definite Integrals involving* $M_k\left(\frac{z}{k}\right)$

(a) $\int_0^1 M_k\left(\frac{z}{k}\right)dz.$

Let $C_{nk} = \dfrac{(\frac{1}{2}-k)(\frac{3}{2}-k)\ldots(n-\frac{1}{2}-k)}{n!\;n!}$.

Then $M_k\left(\frac{z}{k}\right) = \sum_{n=0}^{\infty}\left(C_{nk}\,e^{-\frac{z}{2k}}\left(\frac{z}{k}\right)^{n+\frac{1}{2}}\right)$.

Writing $\quad s=\dfrac{z}{2k}$, we have

$$\int_0^1 M_k\left(\frac{z}{k}\right)dz = \sum_{n=0}^{\infty} 2k\,.\,C_{nk}\int_0^{\frac{1}{2k}}(2s)^{n+\frac{1}{2}}e^{-s}\,ds$$
$$= \sum_0^{\infty}\left\{2^{n+\frac{3}{2}}\,k\,.\,C_{nk}\;\gamma\left(n+\tfrac{3}{2},\,\frac{1}{2k}\right)\right\}$$

where $\quad \gamma(n,x)=\int_0^x t^{n-1}e^{-t}\,dt$ is the Incomplete Gamma Function which has been tabulated by Prof. Karl Pearson and others. The integral is therefore available as an infinite series of these functions. It remains to examine the convergency of this series.

We have $\left|\,\gamma\left(n+\tfrac{3}{2},\,\frac{1}{2k}\right)\right| = \left|\int_0^{\frac{1}{2k}} t^{n+\frac{1}{2}}\,e^{-t}\,dt\,\right|$

$$\leq \int_0^{\frac{1}{2|k|}}\frac{e^{-t}}{|\,2k\,|^{n+\frac{1}{2}}}\,dt.$$
$$\leq \frac{M}{|\,2k\,|^{n+\frac{1}{2}}},\ \text{where } M=1-e^{\frac{-1}{2|k|}}.$$

In the series we have

$$|\,u_{n+1}\,| = \left|\,2k\,.\,C_{nk}\,.\,2^{n+\frac{1}{2}}\,\gamma\left(n+\tfrac{3}{2},\,\frac{1}{2k}\right)\right|$$
$$\leq \left|\,2k\,\frac{(\frac{1}{2}-k)\ldots(n-\frac{1}{2}-k)}{n!\;n!}\,\frac{M}{k^{n+\frac{1}{2}}}\right|$$
$$\leq |\,2k\,|\;\frac{(\frac{1}{2}+|\,k\,|)\ldots(n-\frac{1}{2}+|\,k\,|)}{n!\;n!}\,\frac{M}{(\,|\,k\,|\,)^{n+\frac{1}{2}}}$$
$$\leq |\,2k^{\frac{1}{2}}\,|\left(\frac{1}{2\,|\,k\,|}+1\right)^n.\;\frac{1\,.\,3\,.\,5\ldots 2n-1}{n!\;n!}\,M$$
$$\leq |\,2k^{\frac{1}{2}}\,|\left(\frac{1}{|\,k\,|}+2\right)^n.\;\frac{M}{n!}$$
$$\leq |\,2k^{\frac{1}{2}}\,\frac{G^n\,.\,M}{n!}\ \text{where } G=\frac{1}{|\,k\,|}+2$$
$$\rightarrow 0 \text{ as } n\rightarrow\infty \text{ if } G \text{ is finite.}$$

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$\therefore$ Thus the series is absolutely convergent for all values of $|k|$ except $k=0$.

Other integrals between the same limits can be determined in similar form, *e.g*,

$$(b)\ \int_0^1 z^s M_k\left(\frac{z}{k}\right) dz$$

$$= \sum_{n=0}^{\infty} C_{nk}\, k \int_0^1 e^{-\frac{z}{2k}} \left(\frac{z}{k}\right)^{n+s+\frac{1}{2}} dz$$

$$= \sum_{n=0}^{\infty} 2^{s+n+\frac{3}{2}}\, k^{s+1}\, C_{nk} \int_0^{\frac{1}{2k}} t^{s+n+\frac{1}{2}} e^{-t}\, dt$$

$$\text{where } t = \frac{z}{2k}$$

$$= \sum_{n=0}^{\infty} \left\{ 2^{s+n+\frac{3}{2}}\, k^{s+1}\, C_{nk}\, \gamma\left(n + s + \tfrac{3}{2},\ \frac{1}{2k}\right)\right\}.$$

$$(c)\ \int_0^1 \left(\frac{z}{k}\right)^{s+\frac{1}{2}} e^{\mu z} M_k\left(\frac{z}{k}\right) dz$$

$$= \sum_{n=0}^{\infty} k\,.\,C_{nk} \int_0^1 e^{-z\left(\frac{1}{2k}-\mu\right)} \left(\frac{z}{k}\right)^{n+s+1} d\left(\frac{z}{k}\right)$$

$$= \sum_{n=0}^{\infty} \frac{k\, C_{nk}}{(k\lambda)^{n+s+2}} \int_0^{\lambda} e^{-t}\, t^{n+s+1}\, dt$$

$$\text{where } t = z\lambda,\ \lambda = \frac{1}{2k} - \mu.$$

$$= \sum_{n=0}^{\infty} \left\{ \frac{k\,.\,C_{nk}}{(k\lambda)^{n+s+2}}\,.\,\gamma(n+s+2,\ \lambda)\right\}.$$

In the particular case where $\mu = \dfrac{-1}{2k}$, $\lambda k = 1$.

$$\therefore\ \int_0^1 \left(\frac{z}{k}\right)^{s+\frac{1}{2}} e^{-\frac{z}{2k}} M_k\left(\frac{z}{k}\right) dz.$$

$$= k \sum_{n=0}^{\infty} \left\{ C_{nk}\, \gamma\left(n+s+2,\ \frac{1}{k}\right)\right\}.$$

From this we develope

$$(d)\ \int_0^1 \left\{ M_k\left(\frac{z}{k}\right)\right\}^2 dz$$

$$= \sum_{s=0}^{\infty} \int_0^1 \left(\frac{z}{k}\right)^{s+\frac{1}{2}} C_{sk}\, e^{-\frac{z}{2k}} M_k\left(\frac{z}{k}\right) dz.$$

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Denoting by $J_{r,k}$, $\int_0^1 \left(\frac{z}{k}\right)^{r+\frac{1}{2}} e^{-\frac{z}{2k}} M_k\left(\frac{z}{k}\right) dz$ we have

$$\frac{J_{0,k}}{k} = C_{0k}\,\gamma_2 + C_{1k}\,\gamma_3 + \ldots \qquad + C_{nk}\,\gamma_{n+2} + \ldots$$

$$\frac{J_{1k}}{k} = \qquad C_{0k}\,\gamma_3 + \ldots \qquad + C_{n-1,k}\,\gamma_{n+2} + \ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{J_{rk}}{k} = \qquad C_{0k}\,\gamma_{r+2} \ldots + C_{n-r,k}\,\gamma_{n+2} + \ldots$$

where γ_r denotes $\gamma\left(r, \frac{1}{k}\right)$.

$$\therefore \int_0^1 \left\{M_k\left(\frac{z}{k}\right)\right\}^2 dz = C_{0k}\,J_{0k} + C_{1k}\,J_{1k} + \ldots \text{ to } \infty$$

$$= k\sum_{n=0}^{\infty}\left\{A_n\,\gamma\left(n+2, \frac{1}{k}\right)\right\}$$

where $A_n = \sum_{r=0}^{n} (C_{rk}\,C_{n-r.k})$.

As an example of different type we note

$$\int_0^\infty \sqrt{\frac{k}{z}}\, e^{-\mu z} M_k\left(\frac{z}{k}\right) dz$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2k}-1\right)\ldots\left(\frac{2n-1}{2k}-1\right)}{n!\;n!} \int_0^\infty z^n e^{-z\left(\frac{1}{2k}+\mu\right)} dz.$$

Let $\mu + \frac{1}{2k} = \lambda$.

$$\therefore \int_0^\infty z^n e^{-\lambda z}\, dz = \frac{n!}{\lambda^{n+1}} \text{ if } \lambda \text{ is positive.}$$

$$\therefore \int_0^\infty \left(\frac{k}{z}\right)^{\frac{1}{2}} e^{-\mu z}\; M_k\left(\frac{z}{k}\right) dz$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}-k\right)\ldots\left(\frac{2n-1}{2}-k\right)}{\lambda\,.\,n!\;n!} \cdot \frac{n!}{(\lambda k)^n}$$

$$= \frac{1}{\lambda}\left(1-\frac{1}{k\lambda}\right)^{k-\frac{1}{2}} \text{ provided } |\,k\lambda\,| > 1.$$

$$= \frac{2k}{2k\mu+1}\left(\frac{2k\mu-1}{2k\mu+1}\right)^{k-\frac{1}{2}}.$$

provided $\mu > \frac{1}{2k}$ and $\mu + \frac{1}{2k} > 0$.

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6. The following relations are obtained without difficulty

(1) $z \,.\, M_k'\left(\frac{z}{k}\right) = \left(k - \frac{z}{2k}\right) M_k\left(\frac{z}{k}\right) + (\frac{1}{2} - k)\, M_{k-1}\left(\frac{z}{k}\right).$

(2) $z \,.\, M'_{k-1}\left(\frac{z}{k}\right) = (k-\frac{1}{2})\, M_k\left(\frac{z}{k}\right) - \left(k-1-\frac{z}{2k}\right) M_{k-1}\left(\frac{z}{k}\right).$

(3) $(\frac{3}{2} - k)\, M_{k-2}\left(\frac{z}{k}\right) = 2\left(\frac{z}{2k} - k + 1\right) M_{k-1}\left(\frac{z}{k}\right)$

$$- (\tfrac{1}{2} - k)\, M^k\left(\frac{z}{k}\right).$$

(4) $$J_k\left(\frac{z}{k}\right) = e^{\frac{z}{k}}\, J_{-k}\left(\frac{z}{-k}\right).$$

(5) $$M_k\left(\frac{z}{k}\right) = i\, M_{-k}\left(\frac{z}{-k}\right).$$

7. The orthogonal properties of this function can be used to obtain expansions of simple functions in series of $M_k\left(\frac{z}{k}\right)$ functions, *e.g.*,

Expansion of $\sqrt{z}\,.\, J_0(2\sqrt{z})$.

This satisfies the equation

$$\frac{d^2 u}{dz^2} + \left(\frac{1}{4z^2} + \frac{1}{z}\right) u = 0.$$

$$\text{As} \quad \frac{d^2 M_k}{dz^2} + \left(\frac{1}{4z^2} + \frac{1}{z} - \frac{1}{4k^2}\right) M_k = 0$$

we derive without difficulty

$$\int_0^1 \sqrt{z}\, J_0(2\sqrt{z}) M_k\left(\frac{z}{k}\right) dz = 4k^2 \left[u \,.\, M_k'\left(\frac{z}{k}\right) - u' M_k\left(\frac{z}{k}\right)\right]_0^1$$

and $u\, M_k'\left(\frac{z}{k}\right) = u' M_k\left(\frac{z}{k}\right)$ at lower limit.

$$\therefore \int_0^1 \sqrt{z}\, J_0(2\sqrt{z})\, M_k\left(\frac{z}{k}\right) dz = 4k^2 \,.\, J_0(2)\, M_k'\left(\frac{1}{k}\right)$$

if k is a zero of $M_k\left(\frac{1}{k}\right)$.

Further, our recurrence relations show

$$M_k'\left(\frac{1}{k}\right) = (\tfrac{1}{2} - k)\, M_{k-1}\left(\frac{1}{k}\right)$$

Now let $z^{\frac{1}{2}} J_0(2\sqrt{z}) = \sum_1^{\infty} A_n\, M_{k_n}\left(\frac{z}{k_n}\right)$

where k_n $(n = 1,\ 2\ \ldots)$ are the zeros of $M_k\left(\dfrac{1}{k}\right)$

Then $\displaystyle\int_0^1 z^{\frac{1}{2}} J_0(2\sqrt{z})\, M_{k_n}\left(\frac{z}{k_n}\right) dz = A_n \int_0^1 \left\{M_{k_n}\left(\frac{z}{k_n}\right)\right\}^2 dz.$

$$\therefore\ A_n = \frac{4k_n^2(\tfrac{1}{2} - k_n)\, M_{k_n-1}\left(\dfrac{1}{k_n}\right)\ J_0(2)}{\displaystyle\int_0^1 \left\{M_{k_n}\left(\frac{z}{k_n}\right)\right\}^2 dz}$$

which is determinate.

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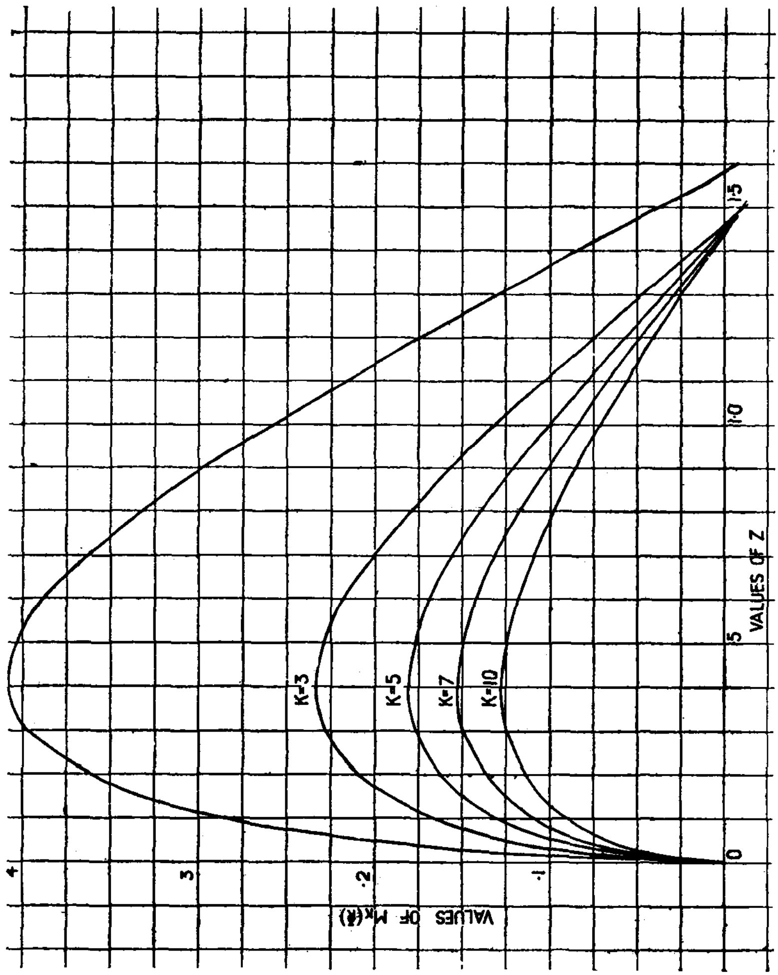
TABLE OF THE FUNCTION.

Values of z.		$K=1$		$K=2$		$K=3$		$K=4$		$K=5$	
		J_K	M_K	J_K	M_K	J_K	M_K	J_K	M_K	J_K	M_K
1st Range	·1	·9494	·2856	·9255	·2017	·9177	·1648	·9139	·1430	·9116	·1276
	·2	·8974	·3615	·8519	·2562	·8375	·2091	·8304	·1811	·8263	·1620
	·3	·8441	·3979	·7793	·2800	·7593	·2284	·7497	·1977	·7439	·1768
	·4	·7893	·4078	·7076	·2864	·6832	·2334	·6716	·2020	·6646	·1806
	·5	·7330	·4036	·6369	·2810	·6091	·2288	·5961	·1980	·5883	·1770
	·6	·6750	·3874	·5672	·2674	·5371	·2173	·5230	·1870	·5147	·1679
	·7	·6154	·3628	·4984	·2475	·4670	·2008	·4526	·1735	·4442	·1550
	·8	·5539	·3321	·4307	·2230	·3990	·1803	·3846	·1556	·3764	·1390
	·9	·4905	·2967	·3640	·1945	·3330	·1570	·3255	·1380	·3113	·1207
	1·0	·4252	·2579	·2982	·1642	·2689	·1314	·2561	·1130	·2469	·0999
	1·1	·3577	·2165	·2336	·1316	·2068	·1042	·1924	·0879	·1892	·0795
	1·2	·2913	·1751	·1699	·0975	·1466	·0759	·1371	·0646	·1320	·0574
	1·3	·2159	·1285	·1073	·0625	·0884	·0469	·0812	·0412	·0774	·0347
	1·4	·1412	·0830	·0456	·0269	·0321	·0174	·0275	·0137	·0253	·0116
	1·5	·0639	·0369	—·0148	—·0086	—·0222	—·0122	—·0239	—·0121	—·0243	—·0115
2nd Range	2·0	—·3690	—·1920	—·3010	—·1826	—·2658	—·1555	—·2480	—·1366	—·2374	—·1229
	3·0	—1·562	—·6035	—·7872	—·4554	—·6163	—·3738	—·5447	—·3244	—·5056	—·2901
	4·0	—3·519	—·9525	—1·1479	—·5944	—·7957	—·4717	—·6654	—·4036	—·5987	—·3589
	5·0	—7·014	—1·289	—1·3647	—·6168	—·8176	—·4587	—·6406	—·3833	—·5558	—·3371
	6·0	—13·73	—1·674	—1·4190	—·5484	—·6963	—·3622	—·4991	—·2887	—·4126	—·2480
	7·0	—27·43	—2·192	—1·3091	—·4256	—·4473	—·2128	—·2690	—·1429	—·2011	—·1182
	8·0	—56·66	—2·935	—·9230	—·2487	—·0861	—·0371	+·0236	+·0122	+·0280	+·0159
	9·0	—121·2	—4·039	—·3088	—·0691	+·3695	+·1428				
	10·0	—267·5	—5·700	+·6691	—·1228						

TABLE OF THE FUNCTION—*Continued.*

Values of z.	$K=6$		$K=7$		$K=8$		$K=9$		$K=10$	
	J_K	M_K	J_K	M_K	J_K	M_K	J_K	M_K	J_K	M_K
·1	·9100	·1165	·9090	·1079	·9081	·1009	·9075	·0951	·9070	·0903
·2	·8234	·1478	·8215	·1369	·8200	·1280	·8189	·1207	·8180	·1145
·3	·7402	·1614	·7375	·1494	·7355	·1398	·7340	·1318	·7327	·1250
·4	·6601	·1649	·6569	·1526	·6545	·1431	·6527	·1346	·6513	·1277
·5	·5839	·1616	·5797	·1495	·5771	·1398	·5750	·1318	·5734	·1251
·6	·5095	·1533	·5058	·1419	·5030	·1327	·5009	·1251	·4991	·1186
·7	·4388	·1414	·4350	·1309	·4322	·1224	·4300	·1154	·4283	·1094
·8	·3711	·1268	·3674	·1173	·3647	·1097	·3624	·1034	·3608	·0980
·9	·3063	·1101	·3028	·1018	·3002	·0952	·2983	·0897	·2967	·0851
1·0	·2444	·0918	·2412	·0849	·2389	·0753	·2371	·0748	·2356	·0709
1·1	·1853	·0723	·1825	·0669	·1806	·0625	·1790	·0589	·1779	·0558
1·2	·1288	·0522	·1267	·0482	·1251	·0449	·1239	·0423	·1230	·0401
1·3	·0751	·0314	·0736	·0289	·0726	·0269	·0717	·0254	·0711	·0240
1·4	·0241	·0104	·0233	·0094	·0229	·0088	·0223	·0082	·0221	·0077
1·5	—·0244	—·0108	—·0244	—·0105	—·0244	—·0096	—·0243	—·0091	·0242	—·0087
2·0	—·2304	—·1126	—·2255	—·1045	—·2218	—·0978	—·2189	—·0924	—·2166	—·0877
3·0	—·4811	—·2649	—·4643	—·2453	—·4521	—·2295	—·4428	—·2164	—·4355	—·2052
4·0	—·5583	—·3267	—·5313	—·3018	—·5120	—·2820	—·4976	—·2656	—·4864	—·2519
5·0	—·5066	—·3049	—·4746	—·2807	—·4521	—·2615	—·4357	—·2460	—·4230	—·2329
6·0	—·3650	—·2214	—·3352	—·2022	—·3149	—·1874	—·3002	—·1756	—·2890	—·1662
7·0	—·1653	—·0996	—·1474	—·0894	—·1346	—·0813	—·1257	—·0751	—·1191	—·0702
8·0	+·0580	+·0344	+·0604	+·0365	+·0608	+·0369	+·0605	+·0366	+·0599	+·0359

Graphs of $M_k\left(\dfrac{z}{k}\right)$ from $z=0$ to $z=1\cdot5$



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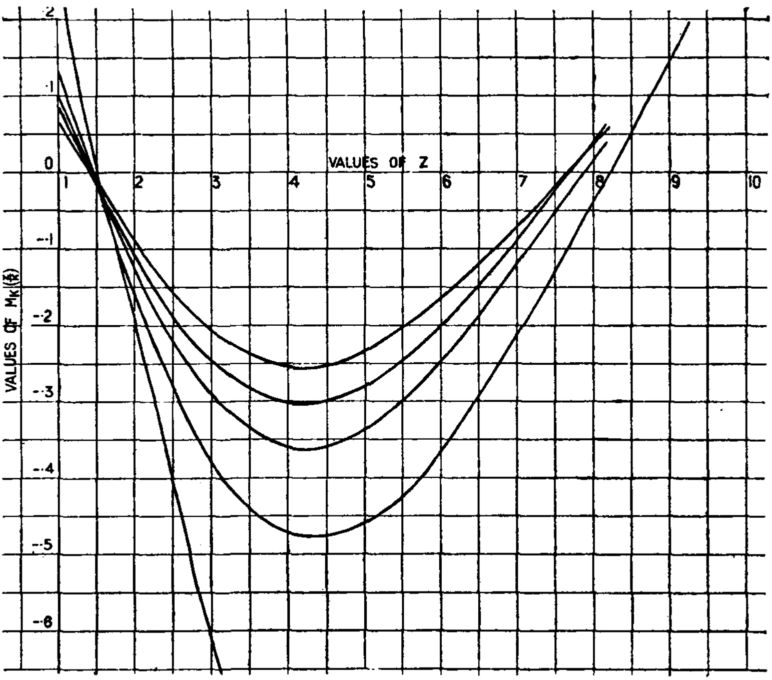
GRAPHS OF $M_k\left(\dfrac{z}{k}\right)$ FROM $z=1$ TO $z=8$



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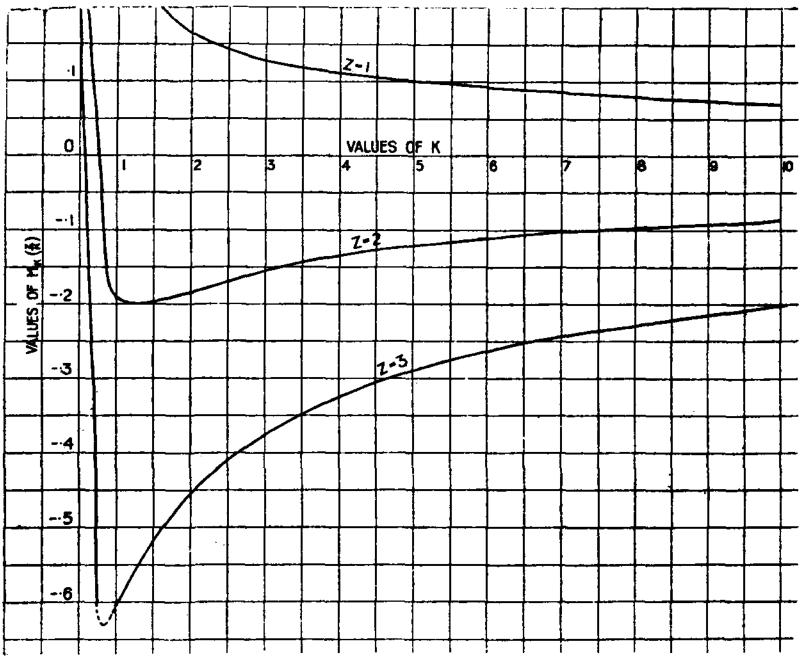
GRAPHS OF $M_k\left(\frac{z}{k}\right)$ AS A FUNCTION OF k FROM $k=\frac{1}{2}$ TO $k=10$



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