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Weak Semiprojectivity in Purely Infinite Simple C^*-Algebras

Huaxin Lin

Abstract. Let A be a separable amenable purely infinite simple C^*-algebra which satisfies the Universal Coefficient Theorem. We prove that A is weakly semiprojective if and only if $K_i(A)$ is a countable direct sum of finitely generated groups ($i = 0, 1$). Therefore, if A is such a C^*-algebra, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$ there exist $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: for any contractive positive linear map $L : A \to B$ (for any C^*-algebra B) with $\|L(ab) - L(a)L(b)\| < \delta$ for $a, b \in \mathcal{G}$ there exists a homomorphism $h\colon A \to B$ such that $\|h(a) - L(a)\| < \varepsilon$ for $a \in \mathcal{F}$.

1 Introduction

Purely infinite simple C^*-algebras were first defined and studied by J. Cuntz [Cu1, Cu2]. The Cuntz algebras are the classical separable purely infinite simple C^*-algebras which are generated by isometries with certain relations. Nowadays purely infinite simple C^*-algebras may arise as the graph C^*-algebras and as inductive limits, as well as dynamical systems and crossed products. The class of purely infinite simple C^*-algebras is a very important class of C^*-algebras. It has played an important role in the development of C^*-algebra theory. The work of classification of amenable purely infinite simple C^*-algebras started with M. Rørdam's work [Ro1]; see also [BSKR]. After a series of works by many (for example, [BEEK, ER, Ln2, LP1, LP2, Ro2, Ro3]), we now know from the work of Kirchberg and Phillips that amenable separable purely infinite simple C^*-algebras which satisfy the universal coefficient theorem (UCT) (or a weak version of it) are classified by their K-theoretical data [P1, K]. Since the Cuntz algebras $\mathcal{O}_n$ ($2 \le n < \infty$) and the Cuntz–Krieger algebras are generated by finitely many isometries with stable relations, they are semiprojective. Blackadar proved that $\mathcal{O}_\infty$ is also semiprojective. More recently, J. Spielberg [Sp] and W. Szymanski [Sz] showed that all separable nuclear purely infinite simple C^*-algebras which satisfy the UCT with finitely generated K-groups and with free K_1-groups are semiprojective by realizing these C^*-algebras as graph C^*-algebras.

In C^*-algebra theory, one often encounters the following stability question. For a given C^*-algebra A, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, are there $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: for any contractive completely positive

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linear map $L\colon A \to B$ (for any C^*-algebra B) with

$$\|L(ab) - L(a)L(b)\| < \delta \quad \text{for } a, b \in \mathcal{G}$$

there exists a homomorphism $h\colon A \to B$ such that $\|h(a) - L(a)\| < \varepsilon$ for $a \in \mathcal{F}$? This is a weak version of semiprojectivity. It is clear that if A is semiprojective, then the answer to the above question is affirmative. In Loring's terminology, C^*-algebras which have an affirmative answer to the above question are called weakly semiprojective (or weakly stable). It turns out that this weak semiprojectivity is a very useful notion. It is often related to certain perturbation problems which are becoming increasingly important.

In this paper, we will show that all separable nuclear purely infinite simple C^*-algebras satisfying the UCT are weakly semiprojective provided that their K-groups are countable direct sums of finitely generated abelian groups. As shown earlier [Ln7], the condition that the K-groups are countable direct sums of finitely generated abelian groups is also necessary.

The method used in this paper is closedly related to those used in the classification of amenable C^*-algebras. In particular, it is related to N. C. Phillips's paper [P1]. The main technical results are: (i) a uniqueness theorem (see Theorem 6.5) which roughly says that two full (almost multiplicative) contractive linear maps from a nuclear purely infinite simple C^*-algebra with the same partial KK-data are approximately unitarily equivalent; (ii) all possible KK-data can be realized by homomorphisms.

The paper is organized as follows: Section 2 serves as a preliminary. In Section 3 we study the class of **D** of C^*-algebras that admit a full embedding from $\mathcal{O}_2$. In Section 4 we study asymptotically multiplicative sequential morphisms to C^*-algebras in **D**. We introduce a functor from **D** to abelian groups. It turns out that the class **D** does not behave as well as we would like it to. However, in Section 5 we prove a version of a theorem of Higson which can be applied to the functor E_A. Applying a uniqueness theorem of the author, we prove in Section 6 that E_A is the same as the restriction of $KL(A, -)$ for certain amenable C^*-algebras A. In Section 7 we present an application of the results of Section 6: we show that a separable amenable purely infinite simple C^*-algebra satisfying the UCT is weakly semiprojective if and only if its K-groups are countable direct sums of finite abelian groups.

After a preliminary version of this paper started to circulate, we learned that J. Spielberg also obtained the results in Section 7 of this paper independently by studying graph C^*-algebras.

2 Preliminaries

We will use the following conventions. Let A and B be C^*-algebras and $\phi, \psi\colon A \to B$ be two maps. Let $\varepsilon > 0$ and $\mathcal{F} \subset A$.

(i) We write $\phi \approx_\varepsilon \psi$ on $\mathcal{F}$ if $\|\phi(a) - \psi(a)\| < \varepsilon$ for all $a \in \mathcal{F}$.

(ii) We write $\phi \overset{u}{\sim}_\varepsilon \psi$ on $\mathcal{F}$ if there is a unitary U in B (or in $\tilde{B}$ if B is not unital) such that $\|\operatorname{ad} U \circ \phi(a) - \psi(a)\| < \varepsilon$ for all $a \in \mathcal{F}$.

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(iii) We write $\phi \overset{u}{\sim} \psi$ if there exists a sequence of unitaries $\{u_n\}$ in B (or in $\tilde{B}$ if B is not unital) such that $\lim_{n\to\infty} \|\operatorname{ad} u_n \circ \phi(a) - \psi(a)\| = 0$ for all $a \in A$.

(iv) Let $\{B_n\}$ be a sequence of C^*-algebras. We will use the following notation:

$$c_0(\{B_n\}) = \bigoplus_{n=1}^{\infty} B_n, \quad l^\infty(\{B_n\}) = \prod_{n=1}^{\infty} B_n, \quad q_\infty(\{B_n\}) = l^\infty(\{B_n\})/c_0(\{B_n\}).$$

(v) A linear map $L\colon A \to B$ is said to be *full* if the (closed two-sided) ideal generated by $L(a)$ is B for any nonzero $a \in A$.

(vi) Let $L\colon A \to B$ be a linear map, let $\mathcal{G}$ be a subset of A and let $\varepsilon > 0$. We say L is $\mathcal{G}$-ε-*multiplicative* if $\|L(ab) - L(a)L(b)\| < \varepsilon$ for all $a, b \in \mathcal{G}$.

Definition 2.1 Let A be a separable amenable C^*-algebra. We shall say that A is *weakly stable* if, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ satisfying the following condition:

> for any C^*-algebra B and any positive linear contraction $L\colon A \to B$ which is $\mathcal{G}$-δ-multiplicative, there exists a homomorphism $h\colon A \to B$ such that $h \approx_\varepsilon L$ on $\mathcal{F}$.

It should be noted that here δ and $\mathcal{G}$ depend only on ε and $\mathcal{F}$. They do not depend on B.

Definition 2.2 Let A be a separable amenable C^*-algebra. We say that A is *weakly semiprojective* with respect to $\mathbf{D}$, if for any sequence $B_n \in \mathbf{D}$ and a homomorphism $\phi\colon A \to q_\infty(\{B_n\})$, there exists a homomorphism $h\colon A \to l^\infty(\{B_n\})$ such that $\pi \circ h = \phi$, where $\pi\colon l^\infty(\{B_n\}) \to q_\infty(\{B_n\})$ is the quotient map. This definition can be found, with some slight modification, in T. Loring's book [Lo].

The following is proved in [Ln7, Theorem 2.4], which is basically the same as [Lo, 19.13].

Theorem 2.3 *Let A be a separable amenable C^*-algebra. Then A is weakly stable if and only if it is weakly semiprojective.*

In what follows, we will not distinguish weakly stable from weakly semiprojective.

Definition 2.4 Let C_n be a commutative C^*-algebra with $K_0(C_n) = \mathbb{Z}/n\mathbb{Z}$ and $K_1(C_n) = 0$. Suppose that A is a C^*-algebra. Then $K_i(A, \mathbb{Z}/k\mathbb{Z}) = K_i(A \otimes C_k)$ (see [S3]). One has the following six-term exact sequence [S3]:

$$\begin{array}{ccccc} K_0(A) & \longrightarrow & K_0(A,\mathbb{Z}/k\mathbb{Z}) & \longrightarrow & K_1(A) \\ \mathbf{k}\big\uparrow & & & & \big\downarrow\mathbf{k} \\ K_0(A) & \longleftarrow & K_1(A,\mathbb{Z}/k\mathbb{Z}) & \longleftarrow & K_1(A) \end{array}.$$

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As in [DL], we use the notation

$$\underline{K}(A) = \bigoplus_{i=0,1,n\in\mathbb{Z}_+} K_i(A;\mathbb{Z}/n\mathbb{Z}).$$

There is a second six-term exact sequence (see [S3]):

$$\begin{tikzcd}
K_0(A,\mathbb{Z}/mk\mathbb{Z}) \arrow[r] & K_0(A,\mathbb{Z}/k\mathbb{Z}) \arrow[r] & K_1(A,\mathbb{Z}/m\mathbb{Z}) \arrow[d] \\
K_0(A,\mathbb{Z}/m\mathbb{Z}) \arrow[u] & K_1(A,\mathbb{Z}/k\mathbb{Z}) \arrow[l] & K_1(A,\mathbb{Z}/mk\mathbb{Z}) \arrow[l]
\end{tikzcd}$$

Definition 2.5 We denote by $\mathcal{N}$ the class of separable amenable C^*-algebras which satisfy the UCT.

Definition 2.6 By $\mathrm{Hom}_\Lambda(\underline{K}(A),\underline{K}(B))$ we mean all homomorphisms from $\underline{K}(A)$ to $\underline{K}(B)$ which respect the direct sum decomposition and the above two six-term exact sequences (see [DL]). It follows from the definition in [DL] that if $x \in KK(A,B)$, then the Kasparov product (associate with x) gives a homomorphism

$$\Gamma(x)\colon\ \mathrm{Hom}_\Lambda(\underline{K}(A),\underline{K}(D)) \to \mathrm{Hom}_\Lambda(\underline{K}(B),\underline{K}(D))$$

for any C^*-algebra D. Dadarlat and Loring [DL] showed that if A is in $\mathcal{N}$, then for any σ-unital C^*-algebra B, Γ is surjective and $\ker\Gamma = \mathrm{Pext}(K_*(A),K_*(B))$.

Note 2.7 We note that $KK(A,-)$ is a (covariant) functor from C^*-algebras to abelian groups. $\mathrm{Hom}_\Lambda(\underline{K}(A),\underline{K}(-))$ is also a (covariant) functor from C^*-algebras to abelian groups. It is easy to see that Γ is a natural transformation from $KK(A,-)$ to $\mathrm{Hom}_\Lambda(\underline{K}(A),\underline{K}(-))$. If we consider only separable C^*-algebras, then $KK(A,-)$ and $\mathrm{Hom}_\Lambda(\underline{K}(A),\underline{K}(-))$ both are homotopy invariant, stable and split exact. In particular, if $0 \to I\otimes\mathcal{K}\xrightarrow{\imath}B\xrightarrow{\pi}B/I \to 0$ is a split short exact sequence of separable C^*-algebras, then the map $[\imath]\colon\ \mathrm{Hom}_\Lambda(\underline{K}(A),\underline{K}(I)) \to \mathrm{Hom}_\Lambda(\underline{K}(A),\underline{K}(B))$ has a left inverse $[\imath]^{-1}$. It follows a theorem of Higson [H] that Γ is the unique natural transformation from $KK(A,-)$ to $\mathrm{Hom}_\Lambda(\underline{K}(A),\underline{K}(-))$ which sends $[\mathrm{id}_A]$ to $[\mathrm{id}_A]$.

Definition 2.8 Let $0 \to I \to B \to A \to 0$ be a short exact sequence of C^*-algebras. B is an essential extension of A by I, if I is an essential ideal of B. Suppose that $I \cong I\otimes\mathcal{K}$. Let $\tau\colon A \to M(I\otimes\mathcal{K})/I\otimes\mathcal{K}$ denote the Busby invariant. The extension τ is said to be *absorbing* if for any trivial extension $\tau_0\colon A \to M(I\otimes\mathcal{K})/I\otimes\mathcal{K}$, there is a unitary $u \in M(I\otimes\mathcal{K})$ such that $\mathrm{ad}\,\pi(u)\circ(\tau\oplus\tau_0) = \tau$, where $\pi\colon M(I\otimes\mathcal{K}) \to M(I\otimes\mathcal{K})/I\otimes\mathcal{K}$ is the quotient map. If $\tau_1,\tau_2\colon A \to M(I\otimes\mathcal{K})/I\otimes\mathcal{K}$ are essential absorbing trivial extensions, then there is a unitary $u \in M(I\otimes\mathcal{K})$ such that $\mathrm{ad}\,\pi(u)\circ\tau_1 = \tau_2$. In other words, all absorbing essential trivial extensions are unitarily equivalent.

3 $\mathcal{O}_2$ Embeddings

Definition 3.1 By $\mathcal{O}_2$ we mean the Cuntz algebra generated by two isometries s_1 and s_2 with $s_1s_1^*+s_2s_2^*=1$. We will use the fact that $\mathcal{O}_2$ is amenable and $K_i(\mathcal{O}_2)=\{0\}$.

We denote by $\mathbf{D}$ the class of separable C^*-algebras D such that there is a full embedding $j_o\colon \mathcal{O}_2 \to D$. Note that every separable purely infinite simple C^*-algebra is in $\mathbf{D}$. Furthermore, if B is a separable C^*-algebra which contains a proper infinite full projection, then B is in $\mathbf{D}$.

Proposition 3.2 *Class $\mathbf{D}$ has the following properties:*

(i) *If $D_1,\dots,D_n$ are finitely many C^*-algebras in $\mathbf{D}$, then $\bigoplus_{i=1}^n D_i \in \mathbf{D}$.*
(ii) *If $D\in\mathbf{D}$, then $D\otimes M_n$ and $D\otimes\mathcal{K}\in\mathbf{D}$.*
(iii) *If D is in $\mathbf{D}$ then $C([0,1],D)\in\mathbf{D}$.*
(iv) *If D is in $\mathbf{D}$, then for any ideal I, D/I in $\mathbf{D}$.*
(v) *If D_1 and D_2 are in $\mathbf{D}$, and D_3 is given by the following split short exact sequence $0\to D_1\xrightarrow{\imath}D_3\xrightarrow{s}D_2\to 0$, then $D_3\in\mathbf{D}$.*

Proof It is easy to see that (i) and (ii) hold. To see (iii), we let $j\colon \mathcal{O}_2\to D$ be a full embedding. Define $\imath\colon D\to C([0,1],D)$ by $\imath(a)=a$, the constant map. One then easily sees that $\imath\circ j$ is a full embedding.

To see (iv), let $j\colon\mathcal{O}_2\to D$ be a full embedding. Let $\pi\colon D\to D/I$ be the quotient map. Then $\pi\circ j\colon \mathcal{O}_2\to D/I$ is also full.

For (v), let $h\colon D_2\to D_3$ be the homomorphism such that $s\circ h=\mathrm{id}_{D_2}$. We identify $\mathcal{O}_2$ with a full C^*-subalgebra of D_2. There are two non-zero mutually orthogonal projections $e_1,e_2\in h(\mathcal{O}_2)$ such that e_1+e_2 is a proper projection. Then there is a unitary $v\in h(\mathcal{O}_2)$ such that $v^*e_1v=e_1+e_2$. Denote by 1 the identity of $\tilde{D}_3$. We then obtain an isometry $w\in\tilde{D}_3$ such that $(1-e_1)D_3(1-e_1)=wD_3w^*$. Let $j_o'\colon\mathcal{O}_2\to D_1$ be a full embedding. Then $j_o''\colon \mathcal{O}_2\to(1-e_1)D_3(1-e_1)$ defined by $j_o''(x)=w(\imath\circ j_o'(x))w^*$ is an embedding. There is also an embedding $i_o\colon\mathcal{O}_2\to e_1D_3e_1$ such that $i_o(\mathcal{O}_2)=e_1h(\mathcal{O}_2)e_1$. Define $j_o\colon\mathcal{O}_2\to D_3$ by $j_o(x)=j_o''(x)\oplus i_o(x)$ for $x\in\mathcal{O}_2$. We claim that j_o is full. Let $x\in(\mathcal{O}_2)_+$. It suffices to show that the ideal generated by $j_o(x)$ is D_3. Since $j_o(x)\geq j_o''(x)$, the ideal generated by $j_o(x)$ contains the ideal generated by $j_o''(x)$. This ideal in turn contains the ideal generated by $w^*j_o''(x)w=\imath\circ j_o'(x)$. Thus it contains $\imath(D_1)$, since j_o' is full. Since $s\circ j_0=s\circ i_o$, $s\circ j_o(x)\in s(e_1)\mathcal{O}_2s(e_1)$. It follows that j_o is full. ■

The following is a version of the so-called uniqueness theorem originally proved in [Ln3]. The following version is proved in [Ln7, 6.1]. This theorem plays an important role in this paper.

Theorem 3.3 *Let A be a separable unital amenable C^*-algebra and B a σ-unital C^*-algebra. Suppose that $h_1,h_2\colon A\to B$ are two homomorphisms such that $[h_1]=[h_2]$ in $KK(A,B)$. Suppose that $h_0\colon A\to B$ is a full monomorphism such that $[h_0(1_A)]=[h_1(1_A)]=[h_2(1_A)]$. Then, for any $\varepsilon>0$ and finite subset $\mathcal{F}\subset A$, there are an integer n and a unitary $w\in U(M_{n+1}(B))$ such that*

$$\|w^*\,\mathrm{diag}(h_1(a),h_0(a),\dots,h_0(a))w-\mathrm{diag}(h_2(a),h_0(a),\dots,h_0(a))\|<\varepsilon$$

for all $a \in \mathcal{F}$.

From the above we obtain the following. Note, it follows from [KP, Theorem 2.8] that for any separable amenable C^*-algebra A, there is an embedding $\imath\colon A \to \mathcal{O}_2$.

Corollary 3.4 *Let A be a separable amenable C^*-algebra and $B \in \mathbf{D}$. Let $\imath\colon A \to \mathcal{O}_2$ be an embedding and let $j_0, j_1\colon k\mathcal{O}_2 \to B$ be two full embeddings. Then for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist an integer n and a unitary $U \in M_{n+1}(\tilde{B})$ such that*

$$\| \operatorname{ad} U \circ (\operatorname{diag}(j_0 \circ \imath, \dots, j_0 \circ \imath)(a) - \operatorname{diag}(j_1 \circ \imath, j_0 \circ \imath, \dots, j_0 \circ \imath)(a)\| < \varepsilon$$

for all $a \in \mathcal{F}$.

It was shown by Rørdam [Ro1] that unital homomorphisms from $\mathcal{O}_2$ to a purely infinite simple C^*-algebra are approximately unitarily equivalent. It actually holds for some C^*-algebras which are not purely infinite and simple. The following theorem will play a similar important role in this paper.

Corollary 3.5 *Let A be a separable amenable C^*-algebra and $B \in \mathbf{D}$. Let $\imath\colon A \to \mathcal{O}_2$ be an embedding and let $j_0, j_1\colon \mathcal{O}_2 \to B$ be two full embeddings. Then for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exists an isometry $U \in M_2(\tilde{B})$ such that*

$$\| j_0 \circ \imath(a) - \operatorname{ad} U^* \operatorname{diag}(j_1 \circ \imath, j_0 \circ \imath)(a)\| < \varepsilon$$

for all $a \in \mathcal{F}$.

Proof For each n, any $\varepsilon > 0$ and any finite subset $\mathcal{G} \subset \mathcal{O}_2$, there is an isometry $V \in M_{n+1}(\mathcal{O}_2)$ such that $VV^* = 1_{\mathcal{O}_2}$, $V^*V = 1_{M_{n+1}(\mathcal{O}_2)}$ and

$$\| \operatorname{ad} V^* \circ (j_0(a), j_0(a), \dots, j_0(a)) - j_0(a)\| < \varepsilon/2$$

for all $a \in \mathcal{G}$. Then the corollary follows from this and Corollary 3.4. ∎

Lemma 3.6 *Let A be a separable amenable C^*-algebra and $B \in \mathbf{D}$. Let $\imath\colon A \to \mathcal{O}_2$ be an embedding. Suppose that $j_0, j_n\colon \mathcal{O}_2 \to B$ are full embeddings.*

Suppose that $\phi_n, \psi_n, \colon A \to B \otimes \mathcal{K}$ are two contractive linear maps for which there exists a sequence of unitaries $\{u_n\}$ in $\widetilde{B \otimes \mathcal{K}}$ such that

$$\lim_{n\to\infty} \| \operatorname{ad} u_n \circ (\phi_n(a) \oplus j_n \circ \imath(a)) - \psi_n(a) \oplus j_n \circ \imath(a)\| = 0$$

for all $a \in A$. Then there exists another sequence of unitaries v_n in $\widetilde{B \otimes \mathcal{K}}$ such that

$$\lim_{n\to\infty} \| \operatorname{ad} v_n \circ (\phi_n(a) \oplus j_0 \circ \imath(a)) - \psi_n(a) \oplus j_0 \circ \imath(a)\| = 0$$

for all $a \in A$.

Proof Let $\delta > 0$, $N > 0$ be an integer and $\mathcal{G} \subset \mathcal{O}_2$ be a finite subset of $\mathcal{O}_2$. Denote by C the image of j_0. Note that $C \cong \mathcal{O}_2$. It follows from [Ro1, 3.6] that there exists $v \in M_N(C)$ such that $v^*v = 1_C$, $vv^* = 1_{M_N(C)}$ and for all $a \in \mathcal{G}$,

$$\| \operatorname{ad} v \circ \operatorname{diag}(j_0(a), \dots, j_0(a)) - j_0(a)\| < \delta.$$

Let $\varepsilon > 0$ and $\mathcal{F} \subset A$ be a finite subset. We may assume that there is an integer $m > 0$ such that

$$\| \operatorname{ad} u_n \circ (\phi_n(a) \oplus j_n \circ \imath(a)) - \psi_n(a) \oplus j_n \circ \imath(a)\| < \varepsilon/6$$

for all $a \in \mathcal{F}$ for all $n \geq m$. By Corollary 3.4, there is a unitary $w_n \in M_{n+1}(\tilde{B})$ such that

$$\begin{aligned}\Big\| \operatorname{ad} w_n \circ \operatorname{diag}(j_0 \circ \imath(a), j_0 \circ \imath(a), \dots, j_0 \circ \imath(a)) \\ - \operatorname{diag}(j_n \circ \imath(a), j_0 \circ \imath(a), \dots, j_0 \circ \imath(a))\Big\| < \varepsilon/6\end{aligned}$$

for all $a \in \mathcal{F}$. Therefore we obtain a unitary $z_n \in \widetilde{B \otimes \mathcal{K}}$ such that

$$\| \operatorname{ad} z_n \circ \operatorname{diag}(\phi_n(a), j_0\circ\imath(a), \dots, j_0\circ\imath(a)) - \operatorname{diag}(\psi_n(a), j_0\circ\imath(a), \dots, j_0\circ\imath(a))\| < \varepsilon/2$$

for all $a \in \mathcal{F}$, where $j_0 \circ \imath(a)$ repeats $n+1$ many times. Thus, with sufficiently small δ and large $\mathcal{G}$, from the first paragraph of the proof, we finally obtain a unitary $v_n \in \widetilde{B \otimes \mathcal{K}}$ such that

$$\| \operatorname{ad} v_n \circ \operatorname{diag}(\phi_n(a), j_0 \circ \imath(a)) - \operatorname{diag}(\psi_n(a), j_0 \circ \imath(a))\| < \varepsilon$$

for all $a \in \mathcal{F}$. ∎

Lemma 3.7 *Let J be a separable C^*-algebra in* **D**. *Then $J \otimes \mathcal{K} \cong C \otimes \mathcal{K}$, where C is unital and there is a unital embedding from $\mathcal{O}_2$ to C.*

Proof Let $j_o \colon \mathcal{O}_2 \to J$ be a full embedding. Put $e = j_o(1_{\mathcal{O}_2})$. Since j_o is full, eJe is a full hereditary C^*-subalgebra of J. It follows from L. G. Brown's stable isomorphism theorem [Br] that $eJe \otimes \mathcal{K} \cong J \otimes \mathcal{K}$. ∎

Lemma 3.8 *Let*

$$0 \to J \otimes \mathcal{K} \to B \to C \to 0$$

be an essential absorbing extension of C^*-algebras in **D**. *Suppose that C is exact. Then there exists an approximate identity $\{e_n\}$ of $J \otimes \mathcal{K}$ consisting of projections such that for each n, there is a full embedding $j_n \colon \mathcal{O}_2 \to (e_{2n} - e_{2n-1})J(e_{2n} - e_{2n-1})$ ($e_0 = 0$) such that $j(a) = \sum_{n=1}^{\infty} j_n(a) \in B$ for all $a \in A$ (converging in the strict topology).*

Proof By Lemma 3.7 we may assume that $J \otimes \mathcal{K} = D \otimes \mathcal{K}$, where D is unital and there is a unital full embedding $i_o \colon \mathcal{O}_2 \to D$. To simplify notation, we may well identify $\mathcal{O}_2$ with $i_o(\mathcal{O}_2)$. Since C is exact, there exists a unital embedding $h \colon C \to \mathcal{O}_2 \subset D$ by [KP, 2.8]. Put $h_n(a) = h(a) \otimes e_{nn}$ for all $a \in C$, where $\{e_{ij}\}$ is a system of matrix units for $\mathcal{K}$. Define $H \colon C \to M(D \otimes \mathcal{K})$ by $H(a) = \sum_{n=1}^{\infty} h_n(a)$ for all $a \in A$ (it converges in the strict topology). Let $\pi \colon M(D \otimes \mathcal{K}) \to M(D \otimes \mathcal{K})/D \otimes \mathcal{K}$ be the quotient map. Denote by τ_1 the extension determined by the given short exact sequence in the statement of the theorem. The assumption that τ_1 is absorbing implies that there is an isometry $Z \in M_2(D \otimes \mathcal{K})$ such that $\tau_1 = \pi(Z)(\tau_1 \oplus \pi \circ H)\pi(Z)^*$. Then the conclusion immediately follows. ■

Lemma 3.9 *Let $\sigma \colon B \to M(J \otimes \mathcal{K})$ be a monomorphism which gives an absorbing trivial extension, $0 \to J \otimes \mathcal{K} \to B \to C \to 0$. Suppose that J and B are in* **D** *and C is exact. Then there exists an approximate identity $\{e_n\}$ of $J \otimes \mathcal{K}$ consisting of projections such that $\|e_n x - x e_n\| \to 0$ as $n \to \infty$ for all $x \in B$. Moreover, there is for each n a full embedding $j_n \colon \mathcal{O}_2 \to (e_n - e_{n-1})J(e_n - e_{n-1})(e_0 = 0)$ such that $j(a) = \sum_{n=1}^{\infty} j_n(a) \in B$ (converging in the strict topology) for all $a \in A$.*

Proof Let τ_1 be the essential trivial absorbing extension. It follows from [Ln5, 5.5.6] that H gives an essential trivial absorbing extension. Since all essential trivial absorbing extensions are equivalent, we may assume that τ_1 is $\pi \circ H$. Then clearly the lemma follows. ■

4 The Functor E_A

Definition 4.1 Let A and B be two C^*-algebras and $\psi_n \colon A \to B$ be a sequence of maps from A to B. We say that $\{\psi_n\}$ is *asymptotically linear*, if

$$\lim_{n\to\infty} \|[\alpha\psi_n(a) + \beta\psi_n(b)] - \psi_n(\alpha a + \beta b)\| = 0 \quad \text{for all } a, b \in A \text{ and } \alpha, \beta \in \mathbb{C},$$

is *asymptotically selfadjoint* if $\lim_{n\to\infty} \|\psi_n(a^*) - \psi_n(a)^*\| = 0$ for all $a, b \in A$ and is *asymptotically multiplicative* if $\lim_{n\to\infty} \|\psi_n(a)\psi_n(b) - \psi_n(ab)\| = 0$ for all $a, b \in A$, respectively.

Definition 4.2 Let $\phi_n, \psi_n \colon A \to B$ be two sequences of maps. We say that $\{\phi_n\}$ and $\{\psi_n\}$ are *asymptotically the same* if $\lim_{n\to\infty} \|\phi_n(a) - \psi_n(a)\| = 0$ for all $a \in A$. We say $\{\phi_n\}$ and $\{\psi_n\}$ are *asymptotically approximately unitarily equivalent*, if there exists a sequence of unitaries $\{u_n\}$ in $\tilde{B}$ such that $\lim_{n\to\infty} \| \operatorname{ad} u_n \circ \phi_n(a) - \psi_n(a)\| = 0$ for all $a \in A$.

Definition 4.3 Let A and B be two C^*-algebras and $L \colon A \to B$ be a contractive completely positive linear map. Let C_n be as in Definition 2.4. We will still use L for $L \otimes \mathrm{id}_{M_n} \colon M_n(A) \to M_n(B)$, $L \otimes \mathrm{id}_{M_n} \otimes \mathrm{id}_{C_n} \colon M_n(A) \otimes C(C_n) \to M_n(B) \otimes C_n$, its extension from $\tilde{M}_n(A) \otimes C(C_n)$ to $\tilde{M}_n(B) \otimes C(C_n)$ and the $L \otimes \mathrm{id}_{M_n} \otimes \mathrm{id}_{C(S^1) \otimes C(C_n)}$ and its unitization. Let $\mathbf{P}$ be the set of projections in $M_n(A)$, $M_n(\tilde{A} \otimes C_n)$ and $M_n(\tilde{A} \otimes C_n \otimes C_(S^1))$. As discussed in [Ln4] and other places such as [DE], given a finite subset

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$\mathcal{P} \subset \mathbf{P}(A)$, there exists $\delta > 0$ and a finite subset $\mathcal{F}$, such that any $\mathcal{F}$-δ-multiplicative contractive completely positive linear map $L\colon A \to B$ uniquely defined a map from $[\mathcal{P}]$ to $\underline{K}(B)$. Let G be the group generated by $\mathcal{P}$, with even larger $\mathcal{F}$ and smaller δ, L gives a group homomorphism $[L]$ from G to $\underline{K}(A)$. In what follows, for a contractive completely positive linear map $L\colon A \to B$, whenever we write $[L]|_{\mathcal{P}}$ we mean L is $\mathcal{F}$-δ-multiplicative with sufficiently large $\mathcal{F}$ and sufficiently small δ so that $[L]|_{\mathcal{P}}$ is well defined.

Definition 4.4 Let A be a separable unital amenable C^*-algebra. Let B be a C^*-algebra in $\mathbf{D}$. An *asymptotic sequential morphism* $\phi = \{\phi_n\}$ from A to $B \otimes \mathcal{K}$ is a sequence of asymptotically multiplicative contractive completely positive linear maps $\{\phi_n\}$ from A to $B\otimes\mathcal{K}$ such that there is an element $\alpha \in \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B\otimes\mathcal{K}))$ with the property that $[\phi_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$ for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$ and any sufficiently large n.

We say two asymptotic sequential morphisms $\phi = \{\phi_n\}$ and $\psi = \{\psi_n\}$ are *equivalent* if there exists a sequence of unitaries $u_n \in \widetilde{B \otimes \mathcal{K}}$ such that for all $a \in A$,

$$\| \operatorname{ad} u_n \circ (\phi_n \oplus j)(a) - (\psi_n \oplus j)(a)\| \to 0 \text{ as } n \to \infty,$$

where $j\colon A \to \mathcal{O}_2 \to B \otimes \mathcal{K}$ is a full embedding. It follows from Lemma 3.6 that the above definition does not depend on the choices of j.

We will write $\phi \sim \psi$ if ϕ and ψ are equivalent. Denote by $E_A(B)$ the equivalent classes of asymptotic sequential morphisms from A to B. If $\phi = \{\phi_n\}$ is an asymptotic sequential morphism from A to $B \otimes \mathcal{K}$ we denote by $\langle\phi\rangle$ the equivalence class containing ϕ.

Given $\langle\phi\rangle$ and $\langle\psi\rangle$, by $\langle\phi\rangle + \langle\psi\rangle$ we mean $\langle\phi \oplus \psi\rangle$, where the direct sum is the orthogonal sum as usual.

Proposition 4.5 *Let $\phi = \{\phi_n\}$ and $\psi = \{\psi_n\}$ be two asymptotic sequential morphisms. If $\phi \sim \psi$, then there is unique $\alpha \in \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ such that for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$, $[\phi_n]|_{\mathcal{P}} = [\psi_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$ for all sufficiently large n.*

Proof Since $j\colon A \to \mathcal{O}_2 \to B \otimes \mathcal{K}$, $[j] = 0$ in $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$. Therefore $[\phi_n]|_{\mathcal{P}} = ([\phi_n] \oplus [j])|_{\mathcal{P}}$ and $[\psi_n]|_{\mathcal{P}} = ([\psi_n] \oplus [j])|_{\mathcal{P}}$. Suppose that there is $\alpha \in \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B\otimes\mathcal{K}))$, for any finite subset $\mathcal{P} \subset \widetilde{\mathbf{P}(A)}$, $[\phi_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$ for all large n. Since there exists a sequence of unitaries $\{u_n\}$ in $\widetilde{B \otimes \mathcal{K}}$ such that

$$\| \operatorname{ad} u_n \circ \phi_n(a) - \psi_n(a)\| \to 0 \quad \text{as } n \to \infty$$

for all $a \in A$. This implies that $[\phi_n]|_{\mathcal{P}} = [\psi_n]|_{\mathcal{P}}$ for all large n. ∎

Proposition 4.6 *$E_A(B)$ is a group.*

Proof From the definition, it is immediate that any homomorphism $j'\colon A \to \mathcal{O}_2 \to B \otimes \mathcal{K}$ represents the zero element. Let $\phi = \{\phi_n\}$ be an asymptotic sequential morphism from A to $B \otimes \mathcal{K}$. It follows [Ln7, 4.5] that there are a sequence of asymptotically contractive completely positive linear maps $\{\bar{\phi}_n\}$ from A to $B \otimes \mathcal{K}$ and a sequence of unitaries $u_n \in \widetilde{B \otimes \mathcal{K}}$ such that

$$\| \operatorname{ad} u_n \circ j(a) - (\phi_n \oplus \bar{\phi}_n)(a)\| \to 0 \quad \text{as } n \to \infty$$

for all $a \in A$. Suppose that $\alpha \in \operatorname{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ such that, for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$, $[\phi_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$ for all large n. Note that $[j] = 0$. Hence $([\phi_n]+[\bar{\phi}_n])|_{\mathcal{P}} = 0$, for all large n. Therefore $([\bar{\phi}_n])|_{\mathcal{P}} = -\alpha|_{\mathcal{P}}$ for all n. Thus $\bar{\phi} = \{\bar{\phi}_n\}$ is an asymptotic sequential morphism from A to $B \otimes \mathcal{K}$. This shows that $E_A(B)$ is a group. ∎

Proposition 4.7 *Let A be a unital separable amenable C^*-algebra. Then E_A is a functor from* $\mathbf{D}$ *and $*$-homomorphisms to abelian groups.*

Proof It follows from the definition and Proposition 4.6 that $E_A(B)$ is an abelian group for every C^*-algebra $B \in \mathbf{D}$. For functoriality, let C be another separable C^*-algebra and let $h\colon B \to C$ be a homomorphism. We extend it to obtain a homomorphism $\tilde{h}\colon B \otimes \mathcal{K} \to C \otimes \mathcal{K}$. Therefore $\{\phi_n\} \mapsto \{\tilde{h} \circ \phi_n\}$ sends asymptotic sequential morphisms to asymptotic sequential morphisms. Moreover, one checks $\langle\{\tilde{h} \circ \phi_n\}\rangle$ is in $E_A(C)$ if $\langle\{\phi_n\}\rangle$ is in $E_A(B)$. Therefore h induces a homomorphism $h_*\colon E_A(B) \to E_A(C)$.

If $g\colon C \to D$ is also a homomorphism, then it is easy to check that $(g \circ h)_* = g_* \circ h_*$. Moreover, it is obvious that $(\mathrm{id}_B)_* = \mathrm{id}_{E_A(B)}$. ∎

Definition 4.8 Let B be a C^*-algebra and fix a rank one projection $e \in \mathcal{K}$. Define a homomorphism $\tilde{e}\colon B \to B \otimes \mathcal{K}$ by $\tilde{e}(b) = b \otimes e$ for $b \in B$.

Lemma 4.9 *Let A be a separable amenable C^*-algebra. Then E_A is stable,* i.e., *the map $\tilde{e}_*\colon E_A(B) \to E_A(B \otimes \mathcal{K})$ is an isomorphism.*

Proof Let $s\colon \mathcal{K} \otimes \mathcal{K} \to \mathcal{K}$ be an isomorphism. We will show the following (known facts): for any finite subset $\mathcal{G} \subset \mathcal{K}$ and any $\delta > 0$, there exists a unitary $U \in \tilde{\mathcal{K}}$ such that

$$\| \operatorname{ad} U \circ (s \circ \tilde{e})(a) - a\| < \delta$$

for all $a \in \mathcal{G}$. But this follows from the fact that $(s \circ \tilde{e})_{*0} = \mathrm{id}_{K_0(\mathcal{K})}$.

Now let $\{\phi_n\}$ be an asymptotic sequential morphism from $A \to B \otimes \mathcal{K}$. It follows from what we have just proved that for any finite subset $\mathcal{F} \subset B \otimes \mathcal{K}$ and any $\varepsilon > 0$,

$$\| \operatorname{ad} 1 \otimes U(s \circ \tilde{e}) \circ \phi_n(a) - \phi_n(a)\| < \varepsilon$$

for all $a \in \mathcal{F}$. This implies that $(s \circ \tilde{e})_* = \mathrm{id}_{E_A(B)}$. ∎

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Lemma 4.10 ([P2, 3.16]) *Let A be a unital separable amenable C^*-algebra. Let*

$$0 \to I \xrightarrow{\imath} B \xrightarrow{\pi} B/I \to 0$$

be a split short exact sequence of separable C^*-algebras in **D** *such that the extension is essential and absorbing. Suppose also that B/I is amenable. Then the sequence*

$$E_A(I) \xrightarrow{\imath_*} E_A(B) \xrightarrow{\pi_*} E_A(B/I)$$

is exact in the middle.

Proof It follows from Proposition 4.7 that $\pi_* \circ \imath_* = 0$. To show that $\ker(\pi_*) \subset \operatorname{im}(\imath_*)$, we let $\langle \phi \rangle = \{\phi_n\}$ be an asymptotic sequential morphism from $A \to B \otimes \mathcal{K}$ such that $\imath_*(\langle \phi \rangle) = 0$. Let j be as in Lemma 3.8 and $h\colon A \to \mathcal{O}_2$ be an embedding. Then there exits a sequence of unitaries $v_n \in \widetilde{(B/I) \otimes} \mathcal{K}$ such that

$$\| \operatorname{ad} v_n \circ \pi \circ (\phi_n(a) \oplus j \circ h(a)) - j \circ h(a)\| \to 0 \quad \text{as } n \to \infty$$

for all $a \in A$. By replacing v_n by $v_n \oplus v_n^*$, we may assume that $v_n \in U_0(((B/I) \otimes \tilde{\mathcal{K}}))$. Let $u_n \in U_0((B \otimes \tilde{\mathcal{K}}))$ such that $\pi(u_n) = v_n$. We have

$$\|\pi(u_n^*(\phi_n \oplus j \circ h(a))u_n - j \circ h(a))\| \to \quad \text{as } n \to \infty$$

for all $a \in A$. Let $\sigma\colon ((B/I) \otimes \tilde{\mathcal{K}}) \to \widetilde{B \otimes \mathcal{K}}$ be a continuous (not necessarily linear) cross section of π satisfying $\sigma(0) = 0$ (given by [BG]). Define $\psi_n'\colon A \to B$ by

$$\psi_n'(a) = u_n^*(\phi_n(a) \oplus j \circ h(a))u_n - (\sigma \circ \pi)(u_n^*\phi_n(a)u_n - j \circ h(a))$$

for $a \in A$. Since σ is continuous, we have

$$\lim_{n\to\infty} \|(\sigma \circ \pi)(u_n^*(\phi_n(a) \oplus j \circ h(a))u_n - j \circ h(a))\| = 0$$

for all $a \in A$. Since

$$\pi(\psi_n'(a) - j(a)) = 0,$$

$\psi_n'(a) \in I \otimes \mathcal{K} + j \circ h(A)$ for all A.

Therefore $\{\psi_n'\}$ is an asymptotically linear, self adjoint and multiplicative map (not necessarily linear, or positive) from A to $I \otimes \mathcal{K} + j \circ h(A)$. By the construction of j as in Lemma 3.9, there is an approximate identity $\{e_n\}$ of $I \otimes \mathcal{K}$ consisting of projections such that

$$e_n j \circ h(a) = j \circ h(a)e_n \quad \text{and} \quad \|(1 - e_n)(\psi_n'(a) - j \circ h(a))\| \to 0$$

as $n \to \infty$ for all $a \in A$. Let $\psi_n''(a) = e_n\phi_n'(a)e_n$ for all $a \in A$ and $j_n'(a) = (1 - e_n)j \circ h(a)(1 - e_n)$ for $a \in A$. Since A is amenable, it follows from [P2, 1.1.5] that there is a sequence of asymptotically multiplicative contractive completely positive linear maps $\psi_n\colon A \to e_n(I \otimes \mathcal{K})e_n$ such that

$$\|\psi_n(a) - \psi_n''(a)\| \to \quad \text{as } n \to \infty$$

for all $a \in A$. Then

$$\lim_{n\to\infty} \| \operatorname{diag}(\psi_n(a), j_n'(a)) - \operatorname{ad} u_n \circ \operatorname{diag}(\phi_n(a), j \circ h(a))\| \to 0$$

for all $a \in A$. It follows from Lemma 3.6 that there are unitaries $z_n \in \widetilde{B \otimes \mathcal{K}}$ such that

$$\lim_{n\to\infty} \| \operatorname{diag}(\imath \circ \psi_n(a), j \circ h(a)) - \operatorname{ad} z_n \circ (\phi_n(a), j \circ h(a))\| = 0$$

for all $a \in A$. So there is $\alpha \in \operatorname{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ such that, for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$, $[\imath\circ\psi_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$ for all sufficiently large n. Since the extension splits, by Lemma 3.6, there is a left inverse $[\imath]^{-1}\colon \operatorname{Hom}_\Lambda((\underline{K}(A), \underline{K}(B)) \to \operatorname{Hom}_\Lambda((\underline{K}(A), \underline{K}(I))$ such that $[\imath]^{-1} \circ [\imath] = [\mathrm{id}_I]$. Let $\beta = [\imath]^{-1}(\alpha)$. Then it follows that

$$[\psi_n]|_{\mathcal{P}} = \beta|_{\mathcal{P}}$$

for all large n. In other words, $\psi = \langle\{\psi_n\}\rangle$ is in $E_A(I)$. Furthermore, from last limit formula above, we conclude that $\langle\phi\rangle = \imath_*(\langle\psi\rangle)$. ■

Theorem 4.11 *Let A be a separable amenable C^*-algebra. Let*

$$0 \to I \otimes \mathcal{K} \xrightarrow{\imath} B \xrightarrow{\pi} B/I \to 0$$

be a split short exact sequence of separable C^*-algebras in **D** *which gives an absorbing essential trivial extension.*

Then one has the following split short exact sequence.

$$0 \to E_A(I) \xrightarrow{\imath_*} E_A(B) \xrightarrow{\pi_*} E_A(B/I) \to 0.$$

Proof Denote by $g\colon B/I \to B$ the splitting map so that $\pi \circ g = \mathrm{id}_{B/I}$. We first show that $\imath_*$ is injective. To simplify the notation, we may assume that $B = B \otimes \mathcal{K}$.

Let $\langle\phi\rangle = \{\phi_n\}$ be an asymptotic sequential morphism from $A \to I\otimes\mathcal{K}$. Suppose that $\imath_* \circ \langle\phi\rangle = 0$. In other words, there is a sequence of unitaries $u_n \in \tilde{B}$ such that

$$\lim_{n\to\infty} \| \operatorname{ad} u_n \circ (\phi_n(a) + j(a)) - j(a)\| = 0$$

for all $a \in A$, where $j\colon A \to B$ is a monomorphism which factors through $\mathcal{O}_2$. It follows from Lemma 3.9 that there exists an approximate identity $\{e_n\}$ of $I \otimes \mathcal{K}$ consisting of projections such that

$$\|e_n x - x e_n\| \to 0 \quad \text{as } n \to \infty$$

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for all $x \in B$. Moreover, there is, for each n, an embedding

$$j_n\colon \mathcal{O}_2 \to (e_n - e_{n-1})I(e_n - e_{n_1})$$

$(e_0 = 0)$ such that

$$j'(a) = \sum_{n=1}^{\infty} j_n(a) \in B$$

for all $a \in \mathcal{O}_2$. Fix an embedding $h\colon A \to \mathcal{O}_2$. It follows from Lemma 3.6 that we may assume that $j = j' \circ h$. Let $\{x_n\}$ be a dense sequence of A. We choose $e_{m(n)}$ such that

(i) $\|e_{m(n)}u_n - u_n e_{m(n)}\| < 1/2^{n+1}$,
(ii) $\|e_{m(n)}\phi_n(a)e_{m(n)} - \phi_n(a)\| < 1/2^{n+1}$ for all $a \subset \{x_1, \dots, x_n\}$.

Note that (i) is possible because $u_n \in \tilde{B}$. There is a unitary w_n in $e_{m(n)}Ie_{m(n)}$ such that

$$\|w_n - e_{m(n)}u_n e_{m(n)}\| < 1/2^n$$

for all n. Set $V_n = (1 - e_{m(n)}) + w_n$. We may view V_n as a unitary in $\widetilde{I \otimes \mathcal{K}}$. Put $\bar{j}_n(a) = e_{m(n)}j(a)e_{m(n)} = \sum_{k=1}^{m(n)} j_k \circ h(a)$ for $a \in A$. Then we have

$$\lim_{n\to\infty} \|\operatorname{ad} V_n \circ (\phi_n(a) \oplus \bar{j}_n(a) - \bar{j}_n(a)\| = 0$$

for all $a \in A$. Note that $\bar{j}_n(a) = (\sum_{k=1}^{m(n)} j_k) \circ h$. Let $j_o\colon \mathcal{O}_2 \to I$ be an embedding. It follows from Lemma 3.6 that there exists another sequence of unitaries z_n in $\widetilde{B}$ such that

$$\lim_{n\to\infty} \|\operatorname{ad} z_n \circ (\phi_n(a) \oplus j_0 \circ h(a)) - j_0 \circ h(a)\| = 0$$

for all $a \in A$. This implies that $(\imath)_*$ is injective.

By Lemma 4.10, it remains to show that

$$\pi_* \circ g_* = \mathrm{id}_{E_A(B/I)} .$$

Note the above also show that π_* is surjective. But $\pi \circ g = \mathrm{id}_{B/I}$. Therefore $\pi_* \circ g_* = (\pi \circ g)_* = \mathrm{id}_{E_A(B/I)}$. ∎

The following is obvious.

Lemma 4.12 $E_A(B \oplus C) = E_A(B) \oplus E_A(C)$.

Proposition 4.13 *Let A be a separable amenable C^*-algebra. Then $E_A(-)$ is homotopy invariant in the following sense. Suppose that $f_{i*}\colon E_A(B) \to E_A(C)$ $(i = 1, 2)$ are homomorphisms and there is a homomorphism $g_*\colon E_A(B) \to E_A(C([0,1], C))$ such that $\delta_0 \circ g_* = f_{1*}$ and $\delta_1 \circ g_* = f_{2*}$, where $\delta_t\colon C([0,1], B) \to B$ is the point evaluation at t, then $f_{1*} = f_{2*}$;*

Proof Let $\langle\phi\rangle \in E_A(B)$. Suppose that $g_*(\langle\phi\rangle)$ is represented by $\{\Phi_n\}$, where $\Phi_n\colon A \to C([0,1],B)$ is a sequence of contractive completely positive linear maps. Then $\delta_t \circ g_*(\langle\phi\rangle)$ is represented by $\{\delta_t \circ \Phi_n\}$. It follows from [Ln7, 4.6] that $\langle\{\delta_0 \circ \Phi_n\}\rangle = \langle\{\delta_1 \circ \Phi_n\}\rangle$. However, $f_{0*}(\langle\phi\rangle) = \langle\{\delta_0 \circ \Phi_n\}\rangle$ and $f_{1*}(\langle\phi\rangle) = \langle\{\delta_1 \circ \Phi_n\}\rangle$. Therefore $f_{1*} = f_{2*}$. ∎

5 A Theorem of Higson

Definition 5.1 Let A be a unital separable amenable C^*-algebra and B be a separable C^*-algebra. Given $\langle\phi\rangle \in E_A(B)$ which is represented by an asymptotic sequential morphism $\{\phi_n\}$ from $A \to B \otimes \mathcal{K}$. It follows from Proposition 4.5 that there is a unique $\alpha \in \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B \otimes \mathcal{K}))$ such that $[\phi_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$ for all finite subset $\mathcal{P} \subset \mathbf{P}$ and all sufficiently large n. Let $\beta_B(\langle\phi\rangle) = \alpha$. Then β_B gives a (well-defined) homomorphism from $E_A(B)$ to $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B \otimes \mathcal{K})$. This defines a natural transformation β from the functor E_A to the functor $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(-))$. To see this, let $h\colon B \to D$ be a homomorphism and $\langle\phi\rangle = \langle\{\phi_n\}\rangle \in E_A(B)$. Suppose that $\xi = \beta_B(\langle\phi\rangle)$. Then it is clear that $h_*(\xi) = \beta_D(\{h \circ \phi_n\})$. Therefore

$$\beta_D(E_A(h))(\langle\phi\rangle) = \beta_D(\langle\{h \circ \phi_n\}\rangle) = h_* \circ \beta_B(\langle\phi\rangle).$$

Definition 5.2 Fix a class of separable C^*-algebras $\mathbf{A}$ which satisfies the conditions (i)–(iv) in Proposition 3.2.

Let F be a covariant functor from $\mathbf{A}$ to abelian groups. F is said to be

(a) *homotopy invariant*, if for any pair of homomorphisms $f_{1*}, f_{2*}\colon F(A) \to F(B)$ with the property that there exists a homomorphism $g\colon F(A) \to F(C([0,1],B)$ such that $\delta_{0*} \circ g = f_{1*}$ and $\delta_{1*} \circ f_{2*}$, where $\delta_t\colon C([0,1],B) \to B$ is the evaluation at $t \in [0,1]$, then $f_{1*} = f_{2*}$;
(b) *stable,* if $\tilde{e}_*$ is an isomorphism;
(c) *additive,* if B_1 and B_2 in $\mathbf{A}$ then $F(B_1 \oplus B_2) = F(B_1) \oplus F(B_2)$, and
(d) A-*absorbing split exact,* if for fixed $A \in \mathbf{D}$, and, for any $B \in \mathbf{A}$ and whenever

$$0 \to B \otimes \mathcal{K} \xrightarrow{\imath} E \xrightarrow{\pi} A \to 0$$

is a split short exact sequence which gives an absorbing essential trivial extension, then one has the following split short exact sequence:

$$0 \to F(B) \xrightarrow{\imath_*} F(E)) \xrightarrow{\pi_*} F(A) \to 0.$$

Fix a C^*-algebra A in $\mathbf{A}$. In the rest of this section, F is always a covariant functor from $\mathbf{A}$ to abelian groups which satisfies the conditions (i)–(iv) of Proposition 3.2 with a fixed C^*-algebra $A \in \mathbf{A}$.

Let C be a separable amenable C^*-algebra. It follows from Proposition 4.13, Lemma 4.9, Lemma 4.12 and Theorem 4.11 that $E_C(-)$ satisfies (i)–(iv) for any amenable C^*-algebra $A \in \mathbf{D}$,

Remark 5.3 To apply Higson's original theorem, one would need to assume that the functor in question is split exact, as well as stable and homotopy invariant. However, we only know that E_A is stable and homotopy invariant. But split exact holds only for some special cases. In what follows, we will show that Higson's theorem actually holds for our functor E_A. The proof is almost a line by line modification of Higson's original proof.

Definition 5.4 ([H, 3.3]) Let A be a separable C^*-algebra in $\mathbf{A}$. Let B be a σ-unital C^*-algebra. Let $\Phi = (\phi_+, \phi_-, 1)$ be a $KK(A, B)$-cycle (for which $U = 1$) in the Cuntz picture. Let $\sigma\colon A \to M(B \otimes \mathcal{K})$ be a monomorphism such that it gives an essential trivial absorbing extension of A by $B \otimes \mathcal{K}$. By replacing ϕ_+ and ϕ_- by $\phi_+ \oplus \sigma$ and $\phi_- \oplus \sigma$, we may assume that ϕ_+ (and ϕ_-) gives an absorbing extension. Set

$$A_\Phi = \{a \oplus x \in A \oplus M(B \otimes \mathcal{K}) : \phi_+(a) = x, \text{modulo } B \otimes \mathcal{K}\}.$$

Define $\widehat{\phi}_\pm\colon A \to A_\Phi$ by $\widehat{\phi}_\pm(a) = a \oplus \phi_\pm(a)$, define $j\colon B \otimes \mathcal{K} \to A_\Phi$ by $j(x) = 0 \oplus x$ and define $\pi\colon A_\Phi \to A$ by $\pi(a \otimes x) = a$. These maps combine to give the following essential trivial absorbing extension:

$$0 \to B \otimes \mathcal{K} \xrightarrow{j} A_\Phi \xrightarrow{\pi} A \to 0$$

which splits by either $\widehat{\phi}_\pm\colon A \to A_\Phi$.

Suppose that both A and B are in $\mathbf{A}$. Then by the assumption in Definition 5.2, A_Φ is in $\mathbf{A}$. Let F be a covariant functor from $\mathbf{A}$ to abelian groups as above. As in [H, 3.4], we get a homomorphism from $F(A)$ to $F(B)$ as follows.

Definition 5.5 ([H, 3.4]) Let $A \in \mathbf{A}$ and let F be a covariant functor from $\mathbf{A}$ to abelian groups which is stable, homotopy invariant, additive and A-absorbing split exact. Suppose that B is in $\mathbf{A}$. Let $\Phi_*\colon F(A) \to F(B)$ be the following composition of homomorphisms:

$$F(A) \xrightarrow{\widehat{\phi}_{+*} - \widehat{\phi}_{-*}} F(A_\Phi) \xrightarrow{l} F(B \otimes \mathcal{K}) \xrightarrow{\bar{e}_*^{-1}} F(B),$$

where $l\colon F(A_\Phi) \to F(B \otimes \mathcal{K})$ is a left inverse of $j_*\colon F(B \otimes \mathcal{K}) \to F(A_\Phi)$. This exists because the short exact sequence is A-absorbing split exact; also, since $\widehat{\phi}_{+*} - \widehat{\phi}_{-*}$ maps into the kernel of π_*, Φ_* does not depend on the particular choice of l (see [H, 3.3]). Note that $\operatorname{im} j_* = \ker \pi_*$. Denote by $p\colon F(A_\Phi) \to \operatorname{im} j_*$. We may choose $l = j_*^{-1} \circ p$.

Remark 5.6 Let $h_\pm\colon A \to B$ be two homomorphisms. We have a $KK(A, B)$-cycle $\Phi = (\phi_+, \phi_-, 1)$, where $\phi_+ = \tilde{e} \circ h_1$ and $\phi_- = \tilde{e} \circ h_-$. If we do not add any degenerate cycle, we have $A_\Phi = A \oplus B \otimes \mathcal{K}$. It is still an extension of A by $B \otimes \mathcal{K}$. We have $F(A_\Phi) = F(A) \oplus F(B \otimes \mathcal{K})$. Therefore we can also define Φ_* exactly the same way as

Definition 5.5. Let $\Psi = (\psi, \psi, 1)$ be a degenerate cycle such that $\psi\colon A \to M(B \otimes \mathcal{K})$ gives an essential absorbing trivial extension of A by $B \otimes \mathcal{K}$. Then

$$A_{\Phi\oplus\Psi} = \Big\{ a \oplus (x_{ij}) \in M_2(M(B \otimes \mathcal{K})) : \\ (x_{ij}) = \begin{pmatrix} \phi_+(a) & 0 \\ 0 & \psi(a) \end{pmatrix}, \text{modulo } M_2(B \otimes \mathcal{K}) \Big\}.$$

Define $f\colon A_\Phi \to A_{\Phi\oplus\Psi}$ by

$$f(a \oplus x) = a \oplus \begin{pmatrix} x & 0 \\ 0 & \psi(a) \end{pmatrix}.$$

With suitable choices of $\tilde{e}$, we have the following commutative diagram:

$$\begin{tikzcd}
& A_\Phi \arrow[dd, "f"] & & B \otimes \mathcal{K} \arrow[ll, "j"'] \arrow[dd, "\tilde{e}"] & \\
A \arrow[ur, "\widehat{\phi}_\pm"] \arrow[dr, "\widehat{\gamma}"'] & & & & B \arrow[ul, "\tilde{e}"'] \arrow[dl, "\tilde{e}"] \\
& A_{\Phi\oplus\Psi} & & M_2(B \otimes \mathcal{K}) \arrow[ll, "j"] &
\end{tikzcd}$$

where $\gamma = \left(\begin{smallmatrix} \phi_\pm & 0 \\ 0 & \psi \end{smallmatrix}\right)$. Note that A_Φ and $A_{\Phi\oplus\Psi}$ are in $\mathbf{A}$ if B is in $\mathbf{A}$. This implies that $\Phi_* = (\Phi \oplus \Psi)_*$. We note that the above holds for any $\Phi = (\phi_+, \phi_-, 1)$, not just the case that $\phi_\pm = h_\pm$.

For the cycle $\Phi = (\mathrm{id}_A, 0, 1)$, $A_\Phi = A \oplus A \otimes \mathcal{K}$. Moreover, $\widehat{\phi_+}(a) = a \oplus \tilde{e}(a)$, $\widehat{\phi_-}(a) = a \oplus 0$, and $l\colon F(A_\Phi) \to F(A \otimes \mathcal{K})$ may be chosen to be q_*, where $q(a \oplus x) = x$. It follows that $\Phi_* = (\mathrm{id}_A, 0, 1)_* = \mathrm{id}_{F(A)}$. From the above, if $\Psi = (\psi, \psi, 1)$ is a degenerate cycle such that $\psi\colon A \to M(A \otimes \mathcal{K})$ gives an essential absorbing trivial extension, then $(\Phi \oplus \Psi)_* = \mathrm{id}_{F(A)}$.

In the following $KK(A, -)$ denote the KK functor restricted on $\mathbf{A}$.

Theorem 5.7 ([H, Theorem 3.7]) *Let F be as in Definition* 5.5 *and A be a fixed C^*-algebra in* $\mathbf{A}$*. If $x \in F(A)$, then there exists a unique natural transformation $\alpha\colon KK(A, -) \to F$ such that $\alpha_A([\mathrm{id}_A]) = x$.*

Proof Let $\alpha\colon KK(A, -) \to F$ be a natural transformation and Φ be a cycle as in Definition 5.4. To verify $\alpha_B \circ \Phi_* = \Phi_* \circ \alpha_A$, we choose $l = j_*^{-1} \circ p$ (see Definition 5.5). Since α is a natural transformation, we have $\alpha_{A_\Phi} \circ KK(j) = F(j) \circ \alpha_{B\otimes\mathcal{K}}$.

In particular, α_{A_Φ} maps $\operatorname{im} KK(j)$ to $\operatorname{im} F(j)$. Therefore we obtain the following commutative diagram:

$$\begin{array}{ccc} \operatorname{im} KK(j) & \xrightarrow{\alpha_{A_\Phi}} & \operatorname{im} F(j) \\ \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle l} \\ KK(A, B\otimes\mathcal{K}) & \xrightarrow{\alpha_{B\otimes\mathcal{K}}} & F(B\otimes\mathcal{K}) \end{array}$$

We also have $\alpha_{A_\Phi}\circ(\widehat{\phi}_+ - \widehat{\phi}_-) = (\widehat{\phi}_+ - \widehat{\phi}_-)\circ\alpha_A$. Since $KK(\widehat{\phi}_+ - \widehat{\phi}_-)$ maps $KK(A,A)$ to $\ker KK(\pi) = \operatorname{im} KK(j)$, $F(\widehat{\phi}_+ - \widehat{\phi}_-)$ maps $F(A)$ to $\ker F(\pi) = \operatorname{im} F(j)$, from the above commutative diagram, we obtain

$$\alpha_{B\otimes\mathcal{K}}\circ(l_*\circ(\widehat{\phi}_+ - \widehat{\phi}_-)_* = (l_*\circ(\widehat{\phi}_+ - \widehat{\phi}_-)_*\circ\alpha_A.$$

It follows that $\alpha_B\circ\Phi_* = \Phi_*\circ\alpha_A$.

Hence, by [H, 3.5], as in the proof of [H, 3.3], $\alpha_B([\Phi]) = \alpha_B(\Phi_*([\mathrm{id}_A])) = \Phi_*(\alpha_A([\mathrm{id}_A])$. Here the first Φ_* is a homomorphism from $KK(A,A)$ to $KK(A,B)$ and second Φ_* is a homomorphism from $F(A)$ to $F(B)$. Moreover, $[\mathrm{id}_A]$ is an element in $KK(A,A)$. This implies that $\alpha_A([\mathrm{id}_A])$ determines $\alpha_B([\Phi])$. This proves the uniqueness.

If $x\in F(A)$, define $\bar\alpha_B([\Phi]) = \Phi_*(x)$. We must prove that it is well defined. Note that here $\Phi = (\phi_+, \phi_-, 1)$ and both ϕ_+ and ϕ_- are assumed to give essential absorbing trivial extensions. If $\Psi = (\psi, \psi, 1)$ is a degenerate cycle then by Remark 5.6, $\Phi_* = (\Phi\oplus\Psi)_*$. Also, if Φ_0 and Φ_1 are homotopic via a homotopy $\Phi = (\Phi_+, \Phi_-, 1)$, a $KK(A, C([0,1], B)$-cycle such that $\Phi_{i*} = \delta_{i*}\circ\Phi_*$, where $\delta_i\colon C([0,1],B)\to B$ is point evaluation at $i = 0, 1$. Hence $\Phi_{0*} = \Phi_{1*}$ by homotopy invariance. As in the proof of [H, 3.7], applying [H, 3.6], $\bar\alpha_B$ gives a homomorphisms $\alpha_B\colon KK(A,B)\to F(B)$. From Remark 5.6, $\alpha_A([\mathrm{id}_A]) = x$. Finally, if Φ is special (as of [H]) then it is easy to check that $(g_\#\Phi)_* = g_*\circ\Phi_*$ if $g\colon B\to C$ is a homomorphism. Therefore α is a natural transformation. ■

6 Isomorphism from $E_A(B)$ to $KL(A,B)$

Definition 6.1 Let A be a separable amenable C^*-algebra. We use the identification $KK(A,B) = \mathrm{Ext}(SA,B)$ and $KK^1(A,B) = \mathit{Ext}(A,B)$. We denote by $\mathcal{T}(A,B)$ the set of equivalence classes of stably approximately trivial extensions (see [Ln6]). It was shown that $\mathcal{T}$ is a subgroup of $\mathit{Ext}(A,B)$.

Let A be a separable amenable C^*-algebra and B be a σ-unital C^*-algebra. Recall that

$$KL(A,B) = KL^0(A,B) = \mathrm{Ext}(SA,B)/\mathcal{T}(SA,B), \quad KL^1(A,B) = \mathrm{Ext}(A,B)/\mathcal{T}(A,B)$$

(see [Ln6]). We will use $\Pi\colon KK(A,B)\to KL(A,B)$ for the quotient map. It should be noted we now defined $KL(A,B)$ without the UCT (see [Ln6]).

It follows from [DL] that there is homomorphism

$$\Gamma\colon KK(A,B) \to \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B)).$$

It is shown by Dadarlat and Loring [DL] that if A is in $\mathcal{N}$, then for any σ-unital C^*-algebra B, Γ is surjective and $\ker\Gamma = \mathrm{Pext}(K_*(A), K_*(B))$.

It follows from [Ln6] that (with A amenable) $\mathcal{T}(A,B)$ is in the kernel of Γ. Thus we obtain the induced map $\tilde{\Gamma}$ from $KL(A,B)$ to $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$.

Recall (see [Ln6]) that A is said to satisfy the approximate universal coefficient theorem (AUCT) if the map $\tilde{\Gamma}$ is an isomorphism.

Theorem 6.2 *Let A be a unital separable amenable C^*-algebra in* $\mathbf{D}$. *Then, for each separable C^*-algebra $B \in \mathbf{D}$, the image of the map β_B contains $\tilde{\Gamma}(KL(A,B))$.*

Proof It is easy to see that $\tilde{\Gamma}$ is a natural transformation from the functor $KK(A,-)$ to the functor $\mathrm{Hom}_\Lambda(\underline{K}(A),-))$. It follows from Theorem 5.7 that there is a unique natural transformation α from $KK(A,-)$ to E_A with $\alpha_A([\mathrm{id}_A]) = \langle \mathrm{id}_A\rangle$. Let $\beta\colon E_A \to \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(-))$ be the natural transformation defined in Definition 5.1. Then $\beta_A(\langle \mathrm{id}_A\rangle) = [\mathrm{id}_A]$. Therefore $\beta_A \circ \alpha_A([\mathrm{id}_A]) = [\mathrm{id}_A]$. Since $\tilde{\Gamma}([\mathrm{id}_A]) = [\mathrm{id}_A]$, It follows from Theorem 5.7 that

$$\tilde{\Gamma} \circ \Pi = \beta \circ \alpha.$$

Thus $\beta_B(E_A(B)) \supset \tilde{\Gamma}(KL(A,B)) = \tilde{\Gamma}((KL(A,B))$. ■

Corollary 6.3 *Let A be a unital separable amenable C^*-algebra in* $\mathbf{D}$ *which satisfies the AUCT. Then, for each separable C^*-algebra $B \in \mathbf{D}$, the map $\beta_B\colon E_A(B) \to$ $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ is surjective.*

Theorem 6.4 *Let A be a unital separable amenable C^*-algebra in* $\mathbf{D}$ *and B be a separable unital C^*-algebra. Suppose that $\tilde{\Gamma}\colon KL(A, q_\infty(B)) \to \tilde{\Gamma}(KL(A, q_\infty(B)))$ is injective. Then $\beta_B\colon E_A(B) \to \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ is also injective.*

Proof By Theorem 6.2, it suffices to show that β_B is injective. Let

$$\alpha \in \tilde{\Gamma}(KL(A,B)) \subset \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$$

and $\langle\phi\rangle$, $\langle\psi\rangle \in E_A(B)$ such that $\beta_B(\langle\phi\rangle) = \beta_B(\langle\psi\rangle)$. Suppose that $\langle\phi\rangle$ and $\langle\psi\rangle$ are represented by $\phi = \{\phi_n\}$ and $\psi = \{\psi_n\}$, respectively. For any finite subset $\mathcal{P} \subset \mathbf{P}(A)$, there exists $N > 0$ such that $[\phi_n]|_{\mathcal{P}} = [\psi_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$ for all large $n \geq N$. It follows

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from [GL1, 2.9] (see also [GL2, Remark 2.1]) that

$$K_i(l^\infty(B\otimes\mathcal{K}))=\prod_n K_i(B),$$

$$K_i(l^\infty(B\otimes\mathcal{K}))=\prod_n K_i(B,\mathbb{Z}/k\mathbb{Z}),$$

$$K_i(q_\infty(B\otimes\mathcal{K}))=\prod_n K_i(B)/\oplus_n K_i(B),$$

$$K_i(q_\infty(B\otimes\mathcal{K},\mathbb{Z}/k\mathbb{Z}))=\prod_n K_i(B,\mathbb{Z}/k\mathbb{Z})/\oplus_n K_i(B,\mathbb{Z}/k\mathbb{Z}).$$

Define $\Phi,\Psi\colon A\to\prod_{n\geq N}B\otimes\mathcal{K}$ by $\Phi_N(a)=\{\phi_n(a)\}_n\geq N$ and $\Psi_N(a)=\{\psi_n(a)\}$ for $a\in A$, respectively. We then have $[\Psi]|_{\mathcal{P}}=[\Phi]|_{\mathcal{P}}$. Let $h_1=\pi'\circ\Phi$ and $h_2=\pi'\circ\Psi$, where $\pi'\colon l^\infty(B\otimes\mathcal{K})\to q_\infty(B\otimes\mathcal{K})$. Regarding $[h_1]$ and $[h_2]$ as elements in $KL(A,q_\infty(B\otimes\mathcal{K}))$, we have $\tilde{\Gamma}([h_1])=\tilde{\Gamma}([h_2])$. By the assumption that $\tilde{\Gamma}$ is injective, we obtain $[h_1]=[h_2]$ in $KL(A,q_\infty(B\otimes\mathcal{K}))$. Fix a full embedding $j\colon A\to\mathcal{O}_2\to B\otimes\mathcal{K}$. Let $J\colon A\to\mathcal{O}_2\to l^\infty(B\otimes\mathcal{K})$ be defined by $J(a)=\{j(a),j(a),\dots\}$. Put $h_3=h_1\oplus\pi'\circ J$. Note h_3 is full. It follows from Theorem 3.3 that there is a sequence of integers $\{m(n)\}$ and a sequence of unitaries $w_n\in q_\infty\widetilde{(B\otimes\mathcal{K})}$ such that

$$\|\operatorname{ad}w_n\circ(h_1(a)\oplus d_{m(n)}\circ h_3(a))-h_2(a)\oplus d_{m(n)}\circ h_3(a)\|\to 0$$

as $n\to\infty$ for all $a\in A$. Let $\{\bar{\phi}'_n\}$ be the sequence of the maps which represents the inverse of $\langle\{\phi_n\}\rangle$ given by [Ln7, 4.5] corresponding to $\{\phi_n\}$. Define $\bar{\Phi}\colon A\to l^\infty(B\otimes\mathcal{K})$ by $\bar{\Phi}(a)=\{\bar{\phi}_n(a)\}$ for $a\in A$. Set $h_4=\pi'\circ\bar{\Phi}$. Then

$$\lim_{n\to\infty}\|\operatorname{ad}w'_n\circ(h_1(a)\oplus d_{m(n)}\circ h_3\oplus h_4(a))-h_2(a)\oplus d_{m(n)}\circ h_3\oplus h_4(a)\|=0$$

for all $a\in A$, where w'_n is a unitary in $U(q_\infty\widetilde{(B\otimes\mathcal{K})})$. However, by the choice of $\bar{\phi}'_n$, we have

$$\lim_{n\to\infty}\|\operatorname{ad}z_n\circ(h_1(a)\oplus d_{m(n)}\circ\pi'\circ J(a))-h_2(a)\oplus d_{m(n)}\circ\pi'\circ J(a)\|=0$$

for all $a\in A$, where z_n is a unitary in $U(q_\infty\widetilde{(B\otimes\mathcal{K})})$.

By applying Lemma 3.6, we have

$$\lim_{n\to\infty}\|\operatorname{ad}z'_n\circ(h_1\oplus\pi'\circ J)(a)-h_2(a)\oplus\pi'\circ J(a)\|=0$$

for all $a\in A$, where z'_n is another sequence of unitaries. There is a unitary $U_{n,k}=\{v_k^{(n)}\}_{k\geq 1}\in l^\infty\widetilde{(B\otimes\mathcal{K})}$ such that $\pi(U_{n,k})=z'_n$, $n=1,2,\dots$. Let $u_n=v_n^{(n)}$. Then we have $\lim_{n\to\infty}\|\operatorname{ad}u_n\circ(\phi'_n(a)\oplus j(a))-\psi'_n(a)\oplus j(a)\|=0$ for all $a\in A$. From here we conclude that $\langle\phi\rangle=\langle\psi\rangle$. ∎

Theorem 6.5 *Let A be a unital separable amenable C^*-algebra. Then for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist $\delta > 0$, a finite subset $\mathcal{G}$ and a finite subset $\mathcal{P} \subset \mathbf{P}(A)$ satisfying the following: if B is a unital C^*-algebra in $\mathbf{D}$ and $\phi, \psi \colon A \to B \otimes \mathcal{K}$ are two $\mathcal{G}$-δ- multiplicative contractive completely positive linear maps with $[\psi]|_{\mathcal{P}} = [\phi]|_{\mathcal{P}}$, then there exists a unitary $u \in \widetilde{B \otimes \mathcal{K}}$ such that $\operatorname{ad} \circ (\phi \oplus j) \approx_{\varepsilon} \psi \oplus j$ on $\mathcal{F}$, where $j \colon A \to \mathcal{O}_2 \to B$ is a full embedding.*

Proof Suppose the theorem is false. Let $\{\delta_n\}$ be a decreasing sequence of positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$, $\{\mathcal{F}_n\}$ be an increasing sequence of finite subsets of A such that $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is dense in A and $\{\mathcal{P}_n\}$ be an increasing sequence of finite subsets such that $\bigcup_{n=1}^{\infty} \mathcal{P}_n = \mathbf{P}(A)$. Then there exits $\varepsilon_0 > 0$, a finite subset $\mathcal{F}_0 \subset A$, a sequence of unital C^*-algebras $\{B_n\} \subset \mathbf{D}$ and two sequences of asymptotically multiplicative contractive completely positive linear maps $\psi_n, \phi_n \colon A \to B_n \otimes \mathcal{K}$ such that $[\psi_n]|_{\mathcal{P}_n} = [\phi_n]|_{\mathcal{P}_n}$, $n = 1, 2, \ldots$ and

$$\sup_n \{\max\{\| \operatorname{ad} u_n \circ (\psi_n(a) \oplus j(a)) - (\phi_n(a) \oplus j(a))\| : a \in \mathcal{F}_0\}\} \geq \varepsilon_0$$

for all unitaries $u_n \in U(\widetilde{B_n \otimes \mathcal{K}})$.

Define $\Psi \colon A \to l^{\infty}(\{B_n \otimes \mathcal{K}\})$ and $\Phi \colon A \to l^{\infty}(\{B_n\})$ by $\Psi(a) = \{\psi_n(a)\}$ and $\Phi(a) = \{\phi_n(a)\}$ for $a \in A$, respectively. Put $h_1 = \pi' \circ \Psi$ and $h_2 = \pi' \circ \Phi$, where $\pi' \colon l^{\infty}(\{B_n \otimes \mathcal{K}\}) \to q_{\infty}(\{B_n \otimes \mathcal{K}\})$ is the quotient map. It follows from the proof of Theorem 6.4 that $\tilde{\Gamma}_1([h_1]) = \tilde{\Gamma}_1([h_2])$. The same proof shows that there exists an integer m and a unitary in $w \in M_m(q_{\infty}(\{B_n \otimes \mathcal{K}\})$ such that

$$\| \operatorname{ad} w \circ (h_1(a) \oplus \pi' \circ J(a)) - (h_2(a) \oplus \pi' \circ J(a))\| < \varepsilon_0/2$$

for all $a \in \mathcal{F}_0$. There exists a unitary $U = \{v_n\} \in U(q_{\infty}(\{B_n \otimes \mathcal{K}\}))$ such that $\pi'(U) = w$. Therefore there exists an integer N such that for all $n \geq N$,

$$\| \operatorname{ad} v_n \circ (\psi_n(a) \oplus j(a)) - (\phi(a) \oplus j(a))\| < \varepsilon_0$$

for all $a \in \mathcal{F}_0$. This gives a contradiction. ∎

Lemma 6.6 *Let A be a separable amenable C^*-algebra in $\mathbf{D}$. Then elements in $E_A(B)$ can be represented by homomorphisms from A to $B \otimes \mathcal{K}$.*

Proof Let $\langle \phi \rangle \in E_A(B)$ be represented by $\{\phi_n\}$ and $\xi = \beta_B(\langle \phi \rangle)$. Fix $\varepsilon > 0$ and a finite subset $\mathcal{F}$. Let δ, $\mathcal{G}$ and $\mathcal{P}$ be as required in Theorem 6.5. There exists $N > 0$ such that ϕ_n are $\mathcal{G}$-δ-multiplicative and $[\phi_n]|_{\mathcal{P}} = \xi|_{\mathcal{P}}$ for all $n \geq N$. It follows from Theorem 6.5 that there exists, for each n, a unitary $u_{n,k} \in \widetilde{B \otimes \mathcal{K}}$ such that

$$\| \operatorname{ad} u_{n,k} \circ (\phi_n \oplus j)(a) - (\phi_{n+k} \oplus j)(a)\| < \varepsilon$$

for all $a \in \mathcal{F}$ and $n \geq N$. By applying [Ln7, 4.7] we obtain a subsequence $\{m(k)\}$ and a sequence of unitaries $w_k \in \widetilde{B \otimes \mathcal{K}}$ such that $h(a) = \lim_{n\to\infty} w_k^*(\phi_{m(k)} \oplus j)(a) w_k$ converges for each $a \in A$ and $h \colon A \to B \otimes \mathcal{K}$ is a homomorphism. Moreover, $[h] = \xi$. By applying Theorem 6.5 again, we obtain that $\langle \phi \rangle = \langle h \rangle$. ∎

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Theorem 6.7 *Let A be a unital separable amenable C^*-algebra in $\mathbf{D}$ which satisfies the AUCT and B be a separable C^*-algebra in $\mathbf{D}$. Then $\beta_B\colon E_A(B) \to \operatorname{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ is an isomorphism.*

Proof If A satisfies the AUCT, the map $\tilde{\Gamma}$ in Theorem 6.5 is injective. So the theorem follows immediately from Theorem 6.5. ■

7 Weak Stability

Lemma 7.1 *Let A be a unital purely infinite simple C^*-algebra. There are $\delta > 0$ and a finite subset $\mathcal{G}$ such that for any C^*-algebra B, if there exists a contractive completely positive linear map $L\colon A \to B$ which is $\mathcal{G}$-δ-multiplicative, then there exists a projection $e \in B$ such that there is a full embedding from $\mathcal{O}_2$ to eBe. If we assume that $L(1_A)$ is a projection, then we can choose $e = L(1_A)$. Moreover, if A is separable, there exists a separable C^*-algebra $C \in \mathbf{D}$ such that $C \subset B$ and $L(A) \subset C$.*

Proof There is an embedding $j\colon \mathcal{O}_2 \to A$. There is $x \in A$ such that $x^* j(1_{\mathcal{O}_2})x = 1_A$. Since $\mathcal{O}_2$ is semiprojective, for any finite subset $\mathcal{F}$ and $\varepsilon > 0$, there is a finite subset $\mathcal{G}_1 \subset j(\mathcal{O}_2)$ and $\delta > 0$ such that, if L is $\mathcal{G} \cup \{x, x^*\}$-$\delta$-multiplicative, then there is a monomorphism $h\colon j(\mathcal{O}_2) \to D$ such that $\|L(x) - h(x)\| < \varepsilon$ for all $x \in \mathcal{F}$. With a larger finite subset $\mathcal{G}_2 \subset A$, we may assume that there is a projection $e \in B$ such that $\|e - L(1_A)\| < \varepsilon/2$ and $\|L(x)^* h(1_{\mathcal{O}_2})L(x) - e\| < \varepsilon$. This implies that $h(1_{\mathcal{O}_2})$ is a full projection in eBe.

If A is separable, one can find a separable C^*-subalgebra $C \subset eBe$ with $1_C = e$ and $L(A) \cup h(A) \cup \{x\} \subset C$. It follows that $C \in \mathbf{D}$. ■

Lemma 7.2 *Let A be a unital separable amenable purely infinite simple C^*-algebra. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: for any separable C^*-algebra $B \in \mathbf{D}$, any $\mathcal{G}$-δ-multiplicative contractive completely positive linear map $L\colon A \to B$ such that $e = L(1_A)$ is a projection and any full embedding $j\colon A \to \mathcal{O}_2 \to eBe$, there exists an isometry $v \in M_2(\tilde{B})$ with $vv^* = 1_{\tilde{B}}$ such that* $\operatorname{ad} v^* \circ (L \oplus j) \approx_\varepsilon L$ *on $\mathcal{F}$.*

Proof Let $\delta > 0$ and $\mathcal{G}$ be a finite subset of A. It follows from [Ro4, 8.2.5] that there is an embedding $j_o\colon A \to \mathcal{O}_2 \to A$ such that there is an isometry $z \in M_2(\tilde{A})$ such that $\operatorname{ad} z^* \circ (\mathrm{id}_A \oplus j_o)) \approx_{\delta/2} \mathrm{id}_A$ on $\mathcal{F}$. Since $\mathcal{O}_2$ is semiprojective, without loss of generality we may assume that there is a homomorphism $h_o\colon A \to \mathcal{O}_2 \to A$ such that $L \circ j_o \approx_{\delta/2} h_0$ on $\mathcal{F}$. Moreover, as in Lemma 7.1, we may assume that h_0 is full in eBe. Note that for any $\eta > 0$ with sufficiently large $\mathcal{G}$ and small δ, we may assume that there is an isometry $u \in M_2(e\tilde{B}e)$ with $uu^* = 1_B$ and $u^*u = 1_{M_2(\tilde{e}Be)}$ such that $\|u - L(z)\| < \eta$. Thus, we may assume that $\operatorname{ad} u^* \circ \operatorname{diag}(L, h_o) \approx_{\varepsilon/4} L$ on $\mathcal{F}$.

It follows from Corollary 3.5 that there is an isometry $w \in M_2(B)$ such that

$$\operatorname{ad} w^* \circ (h_0 \oplus j) \approx_{\varepsilon/4} h_0 \quad \text{on } \mathcal{F}.$$

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Let $w_1 = 1_B \oplus w$. Then

$$L \approx_{\varepsilon/4} \operatorname{ad} z^* \circ (L \oplus h_o) \approx_{\varepsilon/4} \operatorname{ad} w_1^* \circ \operatorname{ad} z^* \circ (L \oplus h_o \oplus j)) \quad \text{on } \mathcal{F}.$$

However, there is an isometry $v \in M_2(\tilde{B})$ such that

$$\operatorname{ad} w_1^* \circ \operatorname{ad} z^*(L \oplus h_o \oplus j)) \approx_{\varepsilon/4} \operatorname{ad} v^* \circ (L \oplus j) \quad \text{on } \mathcal{F}.$$

Thus the lemma follows. ∎

Lemma 7.3 *Let A be a unital separable purely infinite simple C^*-algebras in $\mathcal{N}$. Suppose also that $K_i(A)$ is finitely generated for $i = 0, 1$. Then A is weakly semiprojective.*

Proof Let $\{B_n\}$ be a sequence of C^*-algebras and $h\colon A \to q_\infty(\{B_n\})$ be a monomorphism. Since A is amenable, there is a contractive completely positive linear map $L = \{L_n\}\colon A \to l^\infty(\{B_n\})$, where each $L_n\colon A \to B_n$ is a contractive completely positive linear map, such that $\pi \circ L = h$, where $\pi\colon l^\infty(\{B_n\}) \to q_\infty(\{B_n\})$ is the quotient map. Since $\|L_n(ab) - L_n(a)L_n(b)\| \to 0$ as $n \to \infty$ for all $a, b \in A$, without loss of generality we may assume that $L_n(1_A) = p_n$ is a projection. Moreover, by Lemma 7.1, we may assume that $p_n B_n p_n \in \mathbf{D}$ for all n. To simplify notation, we may assume that $B_n \in \mathbf{D}$. Since $B_n \subset B_n \otimes \mathcal{K}$, and B_n is full in $B_n \otimes \mathcal{K}$, $B_n \otimes \mathcal{K} \in \mathbf{D}$.

Since $K_i(A)$ $(i = 0, 1)$ is finitely generated, by [DL, 2.11], there is an integer $k_0 > 0$ such that $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(C)) = \mathrm{Hom}_\Lambda(F_{k_0}\underline{K}(A), F_{k_0}\underline{K}(C))$ for every σ-unital C^*-algebra C, where $F_{k_0}\underline{K}(D) = K_*(A) \oplus \bigoplus_{k \le k_0} K_*(D, \mathbb{Z}/k\mathbb{Z})$. Let $\tilde{\Gamma}([h]) \in \mathrm{Hom}_\Lambda(F_{k_0}\underline{K}(A), F_{k_0}\underline{K}(q_\infty(B_n))$.

Put

$$P_0^{(i)} = \prod_n K_i(B_n), P_k^{(i)} = \prod_n K_i(B_n, \mathbb{Z}/k\mathbb{Z}),$$
$$Q_0^{(i)} = \prod_n K_i(B_n)/ \oplus_n K_i(B_n),$$
$$Q_k^{(i)} = \prod_n K_i(B_n, \mathbb{Z}/k\mathbb{Z})/ \oplus_n K_i(B_n, \mathbb{Z}/k\mathbb{Z}).$$

It follows from [GL1, 2.9] that

$$K_i(l^\infty(B_n \otimes \mathcal{K})) = P_0^{(i)}, \qquad K_i(l^\infty(B_n \otimes \mathcal{K})) = P_k^{(i)},$$
$$K_i(q_\infty(B_n \otimes \mathcal{K})) = Q_0^{(i)}, \qquad K_i(q_\infty(B_n \otimes \mathcal{K}, \mathbb{Z}/k\mathbb{Z})) = Q_k^{(i)},$$

$k = 2, 3, \dots$. We have the following commutative diagrams:

$$
\begin{tikzcd}
K_0(A) \arrow[rr] \arrow[dr] & & K_0(A,\mathbb{Z}/k\mathbb{Z}) \arrow[rr] \arrow[d, "\tilde{\Gamma}([h])"] & & K_1(A) \arrow[dl] \arrow[ddd] \\
& Q_0^{(0)} \arrow[r] & Q_k^{(0)} \arrow[r] & Q_0^{(1)} \arrow[d] & \\
& Q_0^{(0)} \arrow[u] & Q_k^{(1)} \arrow[l] & Q_0^{(1)} \arrow[l] & \\
K_0(A) \arrow[uuu] \arrow[ur] & & K_1(A,\mathbb{Z}/k\mathbb{Z}) \arrow[ll] \arrow[u, "\tilde{\Gamma}([h])"'] & & K_1(A) \arrow[ll] \arrow[ul]
\end{tikzcd}
$$

$$
\begin{tikzcd}
K_0(A,\mathbb{Z}/mk\mathbb{Z}) \arrow[rr] \arrow[dr] & & K_0(A,\mathbb{Z}/k\mathbb{Z}) \arrow[rr] \arrow[d, "\tilde{\Gamma}([h])"] & & K_1(A,\mathbb{Z}/m\mathbb{Z}) \arrow[dl] \arrow[ddd] \\
& Q_{mk}^{(0)} \arrow[r] & Q_k^{(0)} \arrow[r] & Q_m^{(1)} \arrow[d] & \\
& Q_m^{(0)} \arrow[u] & Q_k^{(1)} \arrow[l] & Q_{mk}^{(1)} \arrow[l] & \\
K_0(A,\mathbb{Z}/m\mathbb{Z}) \arrow[uuu] \arrow[ur] & & K_1(A,\mathbb{Z}/k\mathbb{Z}) \arrow[ll] \arrow[u, "\tilde{\Gamma}([h])"'] & & K_1(A,\mathbb{Z}/mk\mathbb{Z}) \arrow[ll] \arrow[ul]
\end{tikzcd}
$$

It follows from [Ln7, 7.2] that for any $0 \leq k \leq k_0^2$, we obtain homomorphisms $\alpha\colon K_i(A,\mathbb{Z}/k\mathbb{Z}) \to P_k^{(i)}$ such that $\pi_* \circ \alpha = \tilde{\Gamma}([h])$ for all $0 \leq k \leq k_0^2$. Moreover, we have the following commutative diagrams for all $k \leq k_0^2$:

$$
\begin{tikzcd}
K_0(A) \arrow[rr] \arrow[dr] & & K_0(A,\mathbb{Z}/k\mathbb{Z}) \arrow[rr] \arrow[d, "\tilde{\Gamma}([h])"] & & K_1(A) \arrow[dl] \arrow[ddd] \\
& Q_0^{(0)} \arrow[r] & Q_k^{(0)} \arrow[r] & Q_0^{(1)} \arrow[d] & \\
& Q_0^{(0)} \arrow[u] & Q_k^{(1)} \arrow[l] & Q_0^{(1)} \arrow[l] & \\
K_0(A) \arrow[uuu] \arrow[ur] & & K_1(A,\mathbb{Z}/k\mathbb{Z}) \arrow[ll] \arrow[u, "\tilde{\Gamma}([h])"'] & & K_1(A), \arrow[ll] \arrow[ul]
\end{tikzcd}
$$

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$$\begin{tikzcd}
K_0(A,\mathbb{Z}/mk\mathbb{Z}) \arrow[rr] \arrow[dr, "\alpha"] & & K_0(A,\mathbb{Z}/k\mathbb{Z}) \arrow[rr] \arrow[d, "\alpha"] & & K_1(A,\mathbb{Z}/m\mathbb{Z}) \arrow[dl, "\alpha"'] \arrow[ddd] \\
& P^{(0)}_{mk} \arrow[r] & P^{(0)}_{k} \arrow[r] & P^{(1)}_{m} \arrow[d] & \\
& P^{(0)}_{m} \arrow[u] & P^{(1)}_{k} \arrow[l] & P^{(1)}_{mk} \arrow[l] & \\
K_0(A,\mathbb{Z}/m\mathbb{Z}) \arrow[uuu] \arrow[ur, "\alpha"] & & K_1(A,\mathbb{Z}/k\mathbb{Z}) \arrow[ll] \arrow[u, "\alpha"] & & K_1(A,\mathbb{Z}/mk\mathbb{Z}). \arrow[ll] \arrow[ul, "\alpha"']
\end{tikzcd}$$

So $\alpha = \{\alpha_n\} \in \mathrm{Hom}_\Lambda(F_{k_0}\underline{K}(A), F_{k_0}\underline{K}(l^\infty(B_n))$ such that $\pi_* \circ \alpha = \Gamma([h])$. Note that each $\alpha_n \in \mathrm{Hom}_\Lambda(F_{k_0}\underline{K}(A), F_{k_0}\underline{K}(B_n))$. It follows from Lemma 6.6 that there is a homomorphism $h_n \colon A \to B_n$ such that $[h_n] = \alpha_n$. Fix a sequence of finite subsets $\mathcal{F}_j$ such that $\mathcal{F}_j \subset \mathcal{F}_{j+1}$, $n = 1, 2, \dots$, and $\bigcup_{n=1}^\infty \mathcal{F}_n$ is dense in A, and a decreasing sequence of positive numbers ε_j such that $\lim_{j\to\infty} \varepsilon_j = 0$. For each j, let $\delta_j > 0$, $\mathcal{P}_j \subset \mathbf{P}(A)$ and $\mathcal{G}_j$ be finite subsets associated with ε_j, $\mathcal{F}_j$ and A as required by Theorem 6.5. There is $n(j) > 0$ such that $L_n \colon A \to B_n$ is $\mathcal{G}_n$-δ_n-multiplicative, $[L_n]|_{\mathcal{P}_j}$ is well defined for all $n \geq n(j)$. Furthermore, we may also assume (with perhaps even larger $n(j)$) that $[L_n]|_{\mathcal{P}_j} = (\alpha_n)|_{\mathcal{P}}$ for all $n \geq n(j)$. Let $j_n \colon A \to \mathcal{O}_2 \to B_n$ be full embedding. It follows from Theorem 6.5 that there is a unitary $u(j,n) \in \tilde{B}_n$ such that $L_n \oplus j_n \approx_{\varepsilon_j} \mathrm{ad}\, u(j,n) \circ h'_n$ on $\mathcal{F}_j$, where $h'_n = h_n \oplus j_n$. By applying Lemma 7.2, there exists a sequence of isometries $z_n \in \widetilde{B \otimes \mathcal{K}}$ with $z_n^* z_n = 1_{B\otimes\mathcal{K}}$ and $z_n z_n^* = 1_{M_2(B\otimes\mathcal{K})}$ such that

$$\lim_{n\to\infty} \|\,\mathrm{ad}\, z_n \circ (L_n \oplus j_n)(a) - L_n(a)\| = 0$$

for all $a \in A$. Thus, we obtain unitaries $w(n, j) \in \widetilde{B \otimes \mathcal{K}}$ (with $n > n(j)$) such that $L_n \approx_{2\varepsilon_j} \mathrm{ad}\, w(n, j) \circ h_n$ on $\mathcal{F}$.

Define $w_1 = 1, \dots, w_{n(1)-1} = 1, w_{n(j)+i} = u(1, n(j)+i), 0 \leq i \leq n(j+1)-n(j)-1$ and $\phi_n = \mathrm{ad}\, w_n \circ h_n$. Then, since $\bigcup_{j=1}^\infty \mathcal{F}_j$ is dense in A, we conclude that

$$\lim_{n\to\infty} \|L_n(a) - \phi_n(a)\| = 0 \ \text{ for all } \ a \in A.$$

Finally define $H(a) = \{\phi_n(a)\}$ for $a \in A$. Then $H \colon A \to l^\infty(\{B_n \otimes \mathcal{K}\})$ is a homomorphism. Moreover, $\pi \circ H = h$. If B_n are not stable, put $p_n = H_n(1_A)$. Since $\|p_n - L_n(1_A)\| \to 0$ (as $n \to \infty$) and $L_n(1_A) \in B_n$, there is a sequence of projections $e_n \in B_n$ such that $\|p_n - e_n\| \to 0$ as $n \to \infty$. We obtain unitaries $v_n \in \widetilde{B_n \otimes \mathcal{K}}$ such that

$$\|v_n - 1\| \to 0 \ \text{ as } \ n \to \infty, v_n^* p_n v_n = e_n \quad \text{and} \quad v_n e_n v_n^* = p_n \ \ n = 1, 2, \dots.$$

Put $H_n(a) = v_n^* \phi_n n(a) v_n$ for $a \in A$, $n = 1, 2, \dots$. Then $H_n \colon A \to B_n$ is a homomorphism and $\|H_n(a) - \phi_n(a)\| \to 0$ as $n \to \infty$. for all $a \in A$. Therefore

$$\lim_{n\to\infty} \|L_n(a) - H_n'(a)\| = 0 \quad \text{for all } a \in A.$$

Put $H'(a) = \{H_n(a)\}$. Then $\pi \circ H' = h$. ■

Lemma 7.4 *Let A be a unital separable purely infinite simple C^*-algebra in $\mathcal{N}$. Suppose that $K_i(A)$ is a (countable) direct sum of finitely generated abelian groups ($i = 0, 1$). Then $A = \lim_{n\to\infty}(A_n, j_n)$, where each A_n is a unital purely infinite simple C^*-algebra in $\mathcal{N}$ with finitely generated $K_i(A_n)$, $n = 1, 2, \dots$. Moreover, there exist for each n, a homomorphism $r_n \colon A \to A_n$ and a sequence of unitaries $u_{n(k)} \in A_n$ such that $\lim_{k\to\infty} \| \operatorname{ad} u_{n(k)} \circ r_n \circ j_n(a) - a\| = 0$ for all $a \in A_n$.*

Proof Write $K_i(A) = \bigoplus_{i=1}^{\infty} G(n, i)$, where each $G(n, i)$ is a finitely generated group ($i = 0, 1$ and $n = 1, 2, \dots$). It follows from [ER] that there is a unital separable amenable purely infinite simple C^*-algebra A_n in $\mathcal{N}$ such that

$$K_i(A_n) = \bigoplus_{m=1}^{n} G(m, i),$$

$i = 0, 1$. Let $\sigma_{n,i} \colon \bigoplus_{m=1}^{n} G(m, i) \to \bigoplus_{m=1}^{n+1} G(m, i)$ by defining $\sigma_n(g) = (g, 0)$ for $g \in \bigoplus_{m=1}^{n} G(m, i)$ (for $i = 0, 1$). It follows from [P2, 4.1.1] that there is a homomorphism $\phi_n \colon B_n \to B_{n+1}$ such that $(\phi_n)_{i*} = \sigma_{n,i}$. Let $B = \lim_{n\to\infty}(B_n, j_n)$. Since each A_n is purely infinite and simple, one sees that so is B. Since each B_n is in $\mathcal{N}$ and each ϕ_n is injective, by [S1], $B \in \mathcal{N}$. Moreover $K_i(A) = K_i(B)$. It follows from [P2] that $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$. Therefore there is a projection $e \in B \otimes \mathcal{K}$ such that $A \cong e(B \otimes \mathcal{K})e$. From this one checks that there are A_n which is a unital hereditary C^*-subalgebra of $B_n \otimes \mathcal{K}$ such that $A = \lim_{n\to\infty}(A_n, \phi_n)$. Note $K_i(A_n) = K_i(B_n)$ for $i = 0, 1$ and $n = 1, 2, \dots$.

To see the last part of the lemma, let $\gamma_{(n,i)} \colon \bigoplus_{m=1}^{\infty} G(m, i) \to \bigoplus_{m=1}^{n} G(m, i)$ be the projection $\gamma_{(n,i)}(g_1, g_2, \dots, g_n, g_{n+1}, \dots) = (g_1, \dots, g_n)$, where $g_m \in G(m, i)$ for $i = 0, 1$. It follows from [P2, 4.1.1] that there is a homomorphism $\psi_n \colon A \to A_n$ such that $(\psi_n)_{i*} = \gamma_{(n,i)}$, $i = 0, 1$ and $n = 1, 2, \dots$. It follows that $(\psi_n \circ \phi_n)_{i*} = \mathrm{id}_{K_i(A_n)}$. Let $z = [\phi_n] \times [\psi_n]$ in $KK(A_n, A_n)$. It follows from [RS] that there is $z_1 \in KK(A_n, A_n)$ such that $[z] \times [z_1] = [z_1] \times [z] = [\mathrm{id}_{A_n}]$. It follows again from [P2, 4.1.1] that there is a homomorphism $h_n \colon A_n \to A_n$ such that $[h_n] = [z_1]$. Put $r_n = h_n \circ \psi_n$. Then $[r_n] = [h_n \circ \psi_n \circ \phi_n] = [\mathrm{id}_{A_n}]$. It follows again from [P2, 4.1.1] that there exists a sequence of unitaries $u_{n(k)} \in A_n$ such that $\lim_{k\to\infty} \| \operatorname{ad} u_{n(k)} \circ r_n \circ \phi_n(a) - a\| = 0$ for all $a \in A_n$. ■

Theorem 7.5 *Let A be a unital separable purely infinite simple C^*-algebra in $\mathcal{N}$. Then A is weakly semiprojective if and only if $K_i(A)$ is a countable direct sum of finitely generated abelian groups for $i = 0, 1$.*

Proof For the "if" part, we apply Lemmas 7.3 and 7.4. By Lemma 7.4, we write $A = \lim_{n\to\infty}(A_n, \phi_n)$, where each A_n is a unital separable amenable purely infinite simple C^*-algebra in $\mathcal{N}$ and $K_i(A_n)$ is finitely generated ($i = 0, 1$). We may identify A_n with $j_{n,\infty}(A_n)$. So we assume that $A_n \subset A_{n+1}$.

Suppose that $\{B_n\}$ is a sequence of C^*-algebras such that there is a homomorphism $h\colon A \to q_\infty(\{B_n\})$. Let $L = \{L_n\}\colon A \to l^\infty(\{B_n\})$ be a contractive completely positive linear map such that $\pi \circ L = h$, where $\pi\colon l^\infty(\{B_n\}) \to q_\infty(\{B_n\})$ is the quotient map. Denote by h_n the restriction of h on A_n. For each m, by applying Lemma 7.3, there is a homomorphism $\Phi_m\colon A_m \to l^\infty(\{B_n\})$ such that $\pi \circ \Phi_m = h_m$. Write $\Phi_m = \{g_{(n,m)}\}$. Note that $g_{(n,m)}$ are homomorphisms from A_m to B_n.

Suppose that $\mathcal{F}_m \subset A_m$ is a finite subset such that $\mathcal{F}_m \subset \mathcal{F}_{m+1}$ and $\bigcup_{m=1}^\infty \mathcal{F}_m$ is dense in A. By Lemma 7.4, there is a homomorphism $s_n\colon A \to A_n$ such that

$$s_n \approx_{1/2^{n+1}} \mathrm{id}_{A_n} \quad \text{on } \mathcal{F}_n.$$

Since $\lim_{n\to\infty} \|g_{(n,m)}(a) - L_n(a)\| = 0$ for all $a \in A_m$, we choose $n(m) \geq m$ such that

$$g_{(k,m)} \approx_{1/2^{m+1}} L_k(a) \quad \text{on } \mathcal{F}_m$$

for all $k \geq n(m)$. Moreover, we require that $n(m+1) > n(m)$. Now define $\psi_k = g_{(k,m)} \circ s_m$ if $n(m) \leq k < n(m+1)$, $m = 1, 2, \dots$. Define $\Psi = \{\psi_k\}$. Then $\Psi\colon A \to l^\infty(\{B_m\})$ is a homomorphism. Furthermore for each m and $n(m+1) > k \geq n(m)$,

$$\psi_k = g_{(k,m)} \circ s_m \approx_{1/2^{k+1}} g_{(k,m)} \approx_{1/2^{m+1}} L_k(a) \quad \text{on } \mathcal{F}_m.$$

Since $\mathcal{F}_m \subset \mathcal{F}_{m+1}$, we have $\psi_k \approx_{1/2^l} L_k$ on $\mathcal{F}_m$ for all $k \geq n(l)$ and $l \geq m$. Therefore,

$$\lim_{n\to\infty} \|\psi_n(a) - L_n(a)\| = 0$$

for all $a \in \mathcal{F}_m$. Since $\bigcup_{n=1}^\infty \mathcal{F}_m$ is dense in A, we conclude that

$$\lim_{n\to\infty} \|\psi_n(a) - L_n(a)\| = 0$$

for all $a \in A$. This implies that $\pi \circ \Psi = h$. This shows that A is weakly semiprojective.

The "only if" part follows from part (1) of [Ln7, Theorem 8.4]. ∎

Corollary 7.6 *Let A be a unital separable purely infinite simple C^*-algebra in $\mathcal{N}$. Then, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exists $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ such that for any C^*-algebra B and any contractive positive linear map $L\colon A \to B$ satisfying $\|L(ab) - L(a)L(b)\| < \delta$ for all $a \in \mathcal{G}$, there exists a homomorphism $h\colon A \to B$ such that $\|L(a) - h(a)\| < \varepsilon$ for all $a \in \mathcal{F}$.*

Corollary 7.7 *A separable amenable purely infinite simple C^*-algebra A in $\mathcal{N}$ is weakly semiprojective if and only if $K_i(A)$ $(i = 0, 1)$ is a countable direct sum of finitely generated abelian groups.*

Proof Let $e \in A$ be a projection and C be a non-unital hereditary C^*-subalgebra of eAe. Then both C and A are stable. Moreover there exists an isometry $v \in A^{**}$ such that $v^*v = 1$, $vv^* = p$, where p is the open projection associated with C, and $v^*cv \in A$ for all $c \in C$ and $vav^* \in C$ for all $a \in A$. Moreover $\phi(a) = vav^*$ gives an isomorphism from A to C. The reference of this can be found in [Zh1] and Theorem 10, Corallary 11 and the last paragraph of [Ln1]. Fix a finite subset $\mathcal{F} \subset A$ and $\varepsilon > 0$. We may assume without loss of generality that $\mathcal{F} \subset eAe$. Set $\mathcal{F}_1 = \{vav^* : a \in \mathcal{F}\}$. It follows from Corollary 7.6 that there exists a finite subset $\mathcal{G}$ and $\delta > 0$ such that for any $\mathcal{G}$-δ-multiplicative contractive positive linear map $L\colon A \to B$ there is a homomorphism $h_0\colon eAe \to B$ such that the map $a \mapsto L(v^*av)$ from eAe to B is approximated by h_0 within $\varepsilon/2$ on $\mathcal{F}_1$: $\|L(v^*av) - h_0(a)\| < \varepsilon/2$ for all $a \in \mathcal{F}_1$. Define $h\colon A \to B$ by defining $h(a) = h_0(vav^*)$ for $a \in A$. Then,

$$\|L(a) - h(a)\| = \|L(v^*(vav^*)v) - h_0(vav^*)\| < \varepsilon$$

for all $a \in \mathcal{F}$. ∎

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Department of Mathematics
East China Normal University
Shangai
P.R. China
and
Department of Mathematics
University of Oregon
Eugene, OR 97403-1222
U.S.A.
hlin@uoregon.edu