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ISOTROPIC IMMERSIONS INTO A REAL SPACE FORM

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ABSTRACT. The main purpose of this paper is to investigate isotropic immersions with low codimensions into a real space form.

1. **Introduction.** As an interesting class of isometric immersions, the notion of isotropic immersions was introduced by O'Neill [9]. Its definition is given as follows: Let M and $\tilde{M}$ be Riemannian manifolds and $f: M \longrightarrow \tilde{M}$ be an isometric immersion. We denote by σ the second fundamental form of f and call $\sigma(x,x)$ the *normal curvature vector* for a unit tangent vector x. An isometric immersion f is said to be *isotropic* provided that every normal curvature has the same length at each point, that is, the length of the normal curvature vector depends only on the point. In particular, if the length of the normal curvature vector is equal to λ (a function on M), then the immersion f is said to be λ-*isotropic*.

A totally umbilic immersion is clearly isotropic but some examples of isotropic immersions which are not totally umbilic are known. Now we give examples of isotropic immersions into a unit sphere S_1^N:

(1) M is a compact symmetric space of rank one and $f: M \longrightarrow S_1^N$ is a standard minimal immersion in the sense of Do Carmo and Wallach [3].

(2) M^n is an n-dimensional isotropic totally real submanifold with parallel second fundamental form of an n-dimensional complex projective space $P_{\mathbb{C}}^n(4)$ of constant holomorphic sectional curvature 4. We denote by $i: M^n \longrightarrow P_{\mathbb{C}}^n(4)$ its immersion and by $\pi: S_1^{2n+1} \longrightarrow P_{\mathbb{C}}^n(4)$ the Hopf fibration. Then we obtain the lift $f: M^n \longrightarrow S_1^{2n+1}$ of i with respect to π, that is, the following diagram holds:

$$\begin{array}{ccc} & & S_1^{2n+1} \\ & \overset{f}{\nearrow} & \downarrow \pi \\ M^n & \xrightarrow{\ i\ } & P_{\mathbb{C}}^n(4). \end{array}$$

We see that f is also isotropic. Such submanifolds M^n of $P_{\mathbb{C}}^n(4)$ are completely classified by Naitoh [8]. They are locally congruent to $S^1 \times S^{n-1}$ $(n \geq 2)$, SU(3)/ SO(3), SU(3), SU(6)/ Sp(3), E_6/F_4.

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(3) M is a Riemann surface and $f: M \longrightarrow S_1^4$ is a superminimal immersion in the sense of Bryant [1].

Here we remark that in the case of (1) the property of isotropic immersions does not characterize the standard minimal immersions among minimal immersions of spheres into spheres (for details, see Tsukada [12]). Also we note that λ is constant in examples (1) and (2) but in general λ is not constant in (3).

A complete and simply connected Riemannian manifold of constant curvature c is called a *real space form*, which is denoted by $\tilde{M}^N(c)$. In this paper we study isotropic immersions into a real space form with flat normal connection (Theorem 3.1) and with low codimension (Theorem 4.3). In Section 2, we are concerned with the second fundamental form of an isotropic immersion at one point and obtain the estimate of dimensions of the normal space (Theorem 2.7).

2. **Isotropy at one point.** In this section we investigate the second fundamental form at one point. Let V and W be the Euclidean vector spaces with inner products $\langle\ ,\ \rangle$, whose dimensions are n and k, respectively. We abstract the second fundamental form at one point to a symmetric bilinear form $\sigma: V \times V \longrightarrow W$. We adopt for σ the usual notation and terminology of isometric immersions. Let $S^2(V)$ be the space of all symmetric endomorphisms of V. Then we define the linear map $A: W \longrightarrow S^2(V)$ by $\langle A_\xi x, y\rangle = \langle \sigma(x,y), \xi\rangle$ for $x, y \in V$ and $\xi \in W$. The *mean curvature vector* $\mathfrak{h}$ is defined by $\mathfrak{h} = (1/n)\sum_{i=1}^n \sigma(e_i, e_i)$, where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of V. σ is said to be *umbilic* if it satisfies $\sigma(x,y) = \langle x,y\rangle \mathfrak{h}$ for any $x, y \in V$. σ is *minimal* if $\mathfrak{h}$ vanishes. We say that σ is λ-*isotropic* if there exists a real constant λ such that $\|\sigma(x,x)\| = \lambda$ for every unit vector $x \in V$.

The following lemma is due to O'Neill [9].

LEMMA 2.1. *A λ-isotropic symmetric bilinear form satisfies*

$$\langle \sigma(x,y), \sigma(z,w)\rangle + \langle \sigma(x,z), \sigma(w,y)\rangle + \langle \sigma(x,w), \sigma(y,z)\rangle = \lambda^2\{\langle x,y\rangle\langle z,w\rangle + \langle x,z\rangle\langle w,y\rangle + \langle x,w\rangle\langle y,z\rangle\} \quad \textit{for any } x, y, z, w \in V.$$

From now on, we assume that σ is λ (>0)-isotropic. Then σ induces a map $\hat{\sigma}: S_1^{n-1} \longrightarrow S_\lambda^{k-1}$ defined by $\hat{\sigma}(x) = \sigma(x,x)$ for $x \in S_1^{n-1}$, where S_1^{n-1} and S_λ^{k-1} denote an $(n-1)$-dimensional sphere of radius 1 in V and a $(k-1)$-dimensional sphere of radius λ in W, respectively. We shall investigate the map $\hat{\sigma}$ from a differential geometric point of view. For instance, σ is umbilic if and only if $\hat{\sigma}$ is a constant map, that is, the image of $\hat{\sigma}$ is exactly one point of S_λ^{k-1}. σ is minimal if and only if $\hat{\sigma}$ is a harmonic eigenmap corresponding to the second eigenvalue of S_1^{n-1} (*cf.* Toth and D'ambra [11]). It is known that these harmonic eigenmaps can be parametrized by a compact convex body lying in a finite dimensional vector space E, where $\dim E = (n-3)n(n+1)(n+2)/12$ for $n \geq 4$ ([11]).

We shall now describe the inverse image $\hat{\sigma}^{-1}(\xi)$ for $\xi \in S_\lambda^{k-1}$: Since a symmetric endomorphism A_ξ of V satisfies $\langle A_\xi x, x\rangle = \langle \sigma(x,x), \xi\rangle \leq \|\sigma(x,x)\|\,\|\xi\| = \lambda^2$ for any

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unit vector $x \in V$, the eigenvalues of A_ξ lie in the closed interval $[-\lambda^2, \lambda^2]$. This implies that for $x \in S_1^{n-1}$, $\hat{\sigma}(x) = \xi$ if and only if x is an eigenvector of A_ξ with eigenvalue λ^2. We denote by V_ξ the eigenspace of A_ξ with eigenvalue λ^2. Then if $\hat{\sigma}^{-1}(\xi)$ is not empty, we have $\hat{\sigma}^{-1}(\xi) = V_\xi \cap S_1^{n-1}$. It is easily seen that $\sigma(x,y) = \langle x,y\rangle\xi$ for $x,y \in V_\xi$ and that $V_\xi \cap V_\eta = \{0\}$ if $\xi \neq \eta$ for $\xi, \eta \in S_\lambda^{k-1}$. From the argument above, we obtain the following:

LEMMA 2.2. *If $\hat{\sigma}^{-1}(\xi)$ is not empty for $\xi \in S_\lambda^{k-1}$, then $\hat{\sigma}^{-1}(\xi)$ is a totally geodesic sphere of S_1^{n-1} (it may occur that the dimension of $\hat{\sigma}^{-1}(\xi) = 0$, that is, $\hat{\sigma}^{-1}(\xi)$ consists of one point and its antipodal point).*

The tangent space $T_xS_1^{n-1}$ at $x \in S_1^{n-1}$ is naturally identified with the subspace $\{v \in V \mid \langle x, v\rangle = 0\}$. Under this identification, the differential $d\hat{\sigma}_x$ of $\hat{\sigma}$ at $x \in S_1^{n-1}$ is given by $d\hat{\sigma}_x(v) = 2\sigma(x, v)$ for $v \in T_xS_1^{n-1}$. Moreover we have

LEMMA 2.3. $\ker d\hat{\sigma}_x = V_{\hat{\sigma}(x)} \cap T_xS_1^{n-1}$.

PROOF. Since $\sigma(x, v) = \langle x, v\rangle\hat{\sigma}(x)$ for $v \in V_{\hat{\sigma}(x)}$, we find that $V_{\hat{\sigma}(x)}\cap T_xS_1^{n-1} \subset \ker d\hat{\sigma}_x$. Conversely let $v \in T_xS_1^{n-1}$ which satisfies $\sigma(x, v) = 0$. From Lemma 2.1, it follows that $\langle\sigma(x,x), \sigma(v, v)\rangle = \lambda^2\|v\|^2$ and hence $\langle A_{\hat{\sigma}(x)}, v, v\rangle = \lambda^2\|v\|^2$. This means that $v \in V_{\hat{\sigma}(x)}$. ∎

By virtue of Lemma 2.3, we see that the rank of $d\hat{\sigma}$ is constant on $\hat{\sigma}^{-1}(\xi)$ for $\xi \in S_\lambda^{k-1}$ and that rank of $d\hat{\sigma}_x + \dim\hat{\sigma}^{-1}(\xi) = n-1$, where $\xi = \hat{\sigma}(x)$.

Here we fix some notation:

$m = \mathrm{Max}\{\text{rank of } d\hat{\sigma}_x \mid x \in S_1^{n-1}\}$ $(\leq \min\{n-1, k-1\})$,
$M = \{x \in S_1^{n-1} \mid \text{rank of } d\hat{\sigma}_x = m\}$,
$B =$ the image of M by $\hat{\sigma}$ in S_λ^{k-1}.

LEMMA 2.4. *$S_1^{n-1} - M$ is a closed real analytic subset in S_1^{n-1} and hence M is open and dense in S_1^{n-1}.*

PROOF. For any $x \in V$, we consider the linear map $\sigma_x \colon V \longrightarrow W$ defined by $\sigma_x(v) = \sigma(x, v)$ for $v \in V$. Then we see that rank of $d\hat{\sigma}_x + 1 =$ rank of σ_x for $x \in S_1^{n-1}$. Let $\{e_1, \ldots, e_n\}$ and $\{\tilde{e}_1, \ldots, \tilde{e}_k\}$ be orthonormal bases of V and W, respectively. Setting $\sigma(e_i, e_j) = \sum_{a=1}^k \sigma_{ij}\tilde{e}_a$, we have

$$\sigma_x(e_i) = \sigma(x, e_i) = \sum_{a=1}^k \Big\{\sum_{j=1}^n x^j\sigma_{ij}\Big\}\tilde{e}_a \quad \text{for } x = \sum_{j=1}^n x^j e_j.$$

Let $\big((\sigma_x)_i^a\big)$ be the matrix representing the linear map σ_x with respect to the above bases. By the above, $(\sigma_x)_i^a = \sum_{j=1}^n x^j\sigma_{ij}^a$. Our assertion follows from $S_1^{n-1} - M = \big\{x \in S_1^{n-1} \mid$ all minor-determinants of order $m+1$ of the matrix $\big((\sigma_x)_i^a\big) = 0\big\}$. ∎

LEMMA 2.5. *B is an m-dimensional regular submanifold of S_λ^{k-1}.*

PROOF. We put $p = n - m - 1$, which denotes the dimension of $\hat{\sigma}^{-1}(\xi)$, $\xi \in B$. We consider the case that $p \geq 1$ (when $p = 0$, by the similar argument we can prove our assertion). By Lemma 2.4, M is an open submanifold of S_1^{n-1} and the foliation $\mathcal{F}$

consisting of p-dimensional totally geodesic spheres is defined in M. Fix an arbitrary point x in M. Then there exist a distinguished open set O with distinguished coordinates $u^1,\ldots,u^p,v^1,\ldots,v^m$ of $\mathcal{F}$ centered at x and a coordinate neighborhood U in S_λ^{k-1} with local coordinates $w^1,\ldots,w^{k-1}$ centered at $\hat\sigma(x)$ such that $\hat\sigma$ on O is represented as follows:

$$w^i\big(\hat\sigma(u^1,\ldots,u^p,v^1,\ldots,v^m)\big)=v^i \quad \text{for } 1\le i\le m$$
$$w^i\big(\hat\sigma(u^1,\ldots,u^p,v^1,\ldots,v^m)\big)=0 \quad \text{for } m+1\le i\le k-1$$

and that $\hat\sigma(O)\cap U=\{(w^i,\ldots,w^{k-1})\in U\mid w^{m+1}=\cdots=w^{k-1}=0\}$. Denote by $S(O)$ the saturation of O, which is defined by $S(O)=\hat\sigma^{-1}\big(\hat\sigma(O)\big)$. $S(O)$ is open in M and hence open in S_1^{n-1}. $S_1^{n-1}-S(O)$ is a compact subset of S_1^{n-1} and so $\hat\sigma\big(S_1^{n-1}-S(O)\big)$ is a compact subset of S_λ^{k-1} which does not contain $\hat\sigma(x)$. We choose a neighborhood $\tilde U$ of $\hat\sigma(x)$ such that $\tilde U\subset U$ and $\tilde U\cap\hat\sigma\big(S_1^{n-1}-S(O)\big)=\emptyset$ (*i.e.*, empty). Then we have $B\cap\tilde U=\{(w^1,\ldots,w^{k-1})\in\tilde U\mid w^{m+1}=\cdots=w^{k-1}=0\}$. Therefore our assertion is proved. ∎

By virtue of Lemma 2.5, $\hat\sigma|_M$ is a differentiable map of M onto B. Applying a theory of Hermann [6] to our discussion, we obtain the following:

LEMMA 2.6. *$\hat\sigma\colon M\longrightarrow B$ is a fibre bundle whose fibre is S^{n-m-1}.*

To state Theorem 2.7, we recall the invariant ν_n defined by Ferus [4]. Let $V'_{t,r}$ be a Stiefel manifold of ordered r-tuples of linearly independent vectors in R^t. We denote by $\rho(t)$ the largest integer such that the natural fibration $V'_{t,\rho(t)}\longrightarrow V'_{t,1}$ has a global cross section. For every positive integer n we define ν_n as the largest integer such that $\rho(n-\nu_n)\ge\nu_n+1$. By definition of ν_n the following inequality is clear: $\nu_n\le(n-1)/2$. Some numerical values and estimates for ν_n are found in [4].

We are now in a position to prove the following:

THEOREM 2.7. *Let $\sigma\colon V\times V\longrightarrow W$ be a λ (>0)-isotropic symmetric bilinear form, where* $\dim V=n\,(\ge 2)$ *and* $\dim W=k$. *Suppose that σ is not umbilic. Then $k\ge n-\nu_{n-1}$.*

PROOF. We use the notation of the preceding lemmas. Since σ is not umbilic, we have $m=\dim B\ge 1$. Fix $\xi\in B$. Let V_ξ denote the eigenspace of A_ξ with eigenvalue λ^2 and $V_\xi^\perp$ denote the orthogonal complement of V_ξ in V. Put $\dim V_\xi=\nu+1$. We here note that $\nu+m=n-1$. We take $x\in V_\xi\cap S_1^{n-1}$. Then $\hat\sigma(x)=\xi$ and hence $x\in M$. By Lemma 2.6, we have $d\hat\sigma_x(T_xM)=T_\xi B$. Since M is an open submanifold of S_1^{n-1}, $T_xM=T_xS_1^{n-1}$ and hence $d\hat\sigma_x(T_xS_1^{n-1})=T_\xi B$. This, together with Lemma 2.3, implies that $d\hat\sigma_x$ restricted on $V_\xi^\perp$ is a linear isomorphism of $V_\xi^\perp$ onto $T_\xi B$. Noticing that $d\hat\sigma_x(v)=2\sigma(x,v)$ for $v\in T_xS_1^{n-1}$, we can define the bilinear form $F\colon V_\xi\times V_\xi^\perp\longrightarrow T_\xi B$ as $F(x,v)=\sigma(x,v)$ for $x\in V_\xi$, $v\in V_\xi^\perp$. Moreover F satisfies that for x $(\ne 0)$ the map defined by $v\longmapsto F(x,v)$ is a linear isomorphism of $V_\xi^\perp$ onto $T_\xi B$ and for v $(\ne 0)$ the map defined by $x\longmapsto F(x,v)$ is an injective linear map of V_ξ into $T_\xi B$.

Let $\{e_1,\ldots,e_{\nu+1}\}$ be an orthonormal basis of V_ξ. We denote by f the inverse map of the linear isomorphism of $V_\xi^\perp$ onto $T_\xi B$ defined by $v\longmapsto F(e_1,v)$. For any nonzero vector

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$\eta \in T_\xi B$, $\{F(e_1, f(\eta)) = \eta, F(e_2, f(\eta)), \ldots, F(e_{\nu+1}, f(\eta))\}$ is an ordered $(\nu + 1)$-tuples of linearly independent vectors in $T_\xi B$. Therefore the fibration $V'_{m,\nu+1} \longrightarrow V'_{m,1}$ has a cross section. In particular, we have $\rho(m) = \rho(n - 1 - \nu) \geq \nu + 1$ and hence $\nu \leq \nu_{n-1}$ by definition of ν_{n-1}. This implies that $k = \dim W \geq \dim B + 1 = n - \nu \geq n - \nu_{n-1}$. ∎

By the theorem above the following clearly holds, which is an improvement of a result of Kleinjohann and Walter (Proposition 5.A in [7]).

COROLLARY 2.8. *Let f be an isotropic immersion of an $n\,(\geq 2)$-dimensional Riemannian manifold M^n into an $(n+k)$-dimensional Riemannian manifold $\tilde{M}^{n+k}$. If $k < n - \nu_{n-1}$, then f is totally umbilic.*

Now we consider the extremal case in Theorem 2.7.

LEMMA 2.9. *In addition to the assumption of Theorem 2.7, we suppose that $k = n - \nu_{n-1}$. Then $\hat{\sigma}$ is a surjective map of S_1^{n-1} onto S_λ^{k-1} and for $x \in M$ σ_x is also a surjective linear map of V onto W, where σ_x is defined by $\sigma_x(v) = \sigma(x, v)$ for $v \in V$.*

PROOF. Since $\dim W = \dim B + 1$, B is an open submanifold of S_λ^{k-1}. We define the function d on S_λ^{k-1} by $d(\xi) = \det(A_\xi - \lambda^2 \,\mathrm{Id})$ for $\xi \in S_\lambda^{k-1}$. From the argument in Lemma 2.2 it follows that $\hat{\sigma}(S_1^{n-1}) = d^{-1}(0)$. Since d is a real analytic function on S_λ^{k-1} and $B \subset d^{-1}(0)$, d vanishes identically on S_λ^{k-1} and hence $\hat{\sigma}(S_1^{n-1}) = S_\lambda^{k-1}$. The fact that rank of $d\hat{\sigma}_x + 1 =$ rank of σ_x gives us the second part of this lemma. ∎

We note that $\nu_{n-1} \leq (n-2)/2$ and hence $n - \nu_{n-1} \geq (n+2)/2$. So we shall consider the case of $\dim W = (n+2)/2$. We get the following:

PROPOSITION 2.10. *In addition to the assumption of Theorem 2.7, we suppose that $k = (n+2)/2$. Then it occurs only when $n = 2, 4, 8$ or 16 and $\hat{\sigma}$ gives fibrations of S^{2m-1} onto S^m with fibres S^{m-1}, where $n = 2m$. Moreover for an arbitrary unit vector $\xi \in W$, the vector space V has the orthogonal decomposition $V = V_\xi + V_{-\xi}$ such that $\dim V_\xi = \dim V_{-\xi}$ and that V_ξ and $V_{-\xi}$ are the eigenspaces of A_ξ with eigenvalues λ and $-\lambda$, respectively. In particular, σ is minimal.*

PROOF. Put $n = 2m$ and hence $k = m + 1$. By virtue of Lemma 2.9, $\hat{\sigma}^{-1}(\xi)$ is not empty for an arbitrary $\xi \in S_\lambda^m$. Since $\dim V_\xi = \dim\ker d\hat{\sigma}_x + 1$ for $x \in \hat{\sigma}^{-1}(\xi)$, we have $\dim V_\xi \geq m$. The eigenspace $V_{-\xi}$ of $A_{-\xi}$ with eigenvalue λ^2 coincides with the eigenspace of A_ξ with eigenvalue $-\lambda^2$ so that $V_{-\xi} \subset V_\xi^\perp$. Since $\dim V_\xi \geq m$ and $\dim V_{-\xi} \geq m$, we have $\dim V_\xi = \dim V_{-\xi} = m$ and obtain the orthogonal decomposition $V = V_\xi + V_{-\xi}$. From Lemma 2.1, it follows that $\|\sigma(x,y)\| = \lambda\|x\|\,\|y\|$ for $x \in V_\xi$, $y \in V_{-\xi}$. Therefore we obtain the bilinear form F: $V_\xi \times V_{-\xi} \longrightarrow T_\xi S_\lambda^m$ such that $\dim V_\xi = \dim V_{-\xi} = \dim T_\xi S_\lambda^m = m$ and $\|F(x,y)\| = \lambda\|x\|\,\|y\|$. This implies that $m = 1, 2, 4$ or 8. Since $\dim\ker d\hat{\sigma}_x = \dim V_\xi - 1 = m - 1$ at every point $x \in S_1^{n-1}$, the differential $d\hat{\sigma}_x$ has the same rank m. From this and Lemma 2.6, it follows that $\hat{\sigma}$ is a fibration of S^{2m-1} onto S^m with fibres S^{m-1}. ∎

We here provide a characterization of umbilic bilinear forms.

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PROPOSITION 2.11. *Let $\sigma: V \times V \longrightarrow W$ be a symmetric bilinear form. Then the following are equivalent:*

(i) σ is umbilic.

(ii) σ is isotropic and for any $\xi, \eta \in W$ $A_\xi A_\eta = A_\eta A_\xi$.

PROOF. (i) $\Rightarrow$ (ii): Let $\mathfrak{f}$ be the mean curvature vector of σ. Since $\sigma(x,y) = \langle x, y\rangle \mathfrak{f}$, we have $A_\xi = \langle \xi, \mathfrak{f}\rangle$ Id for any $\xi \in W$ so that A_ξ and A_η commute for any $\xi, \eta \in W$.

(ii) $\Rightarrow$ (i): We assume that σ is λ (> 0)-isotropic. We take a unit vector $x \in V$ and put $\xi = \sigma(x,x) \in W$. Note that x is an eigenvector of A_ξ with eigenvalue λ^2. We choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of V such that $A_\xi e_i = \lambda_i e_i$ $(i = 1, \ldots, n)$, where $e_1 = x$ and $\lambda_1 = \lambda^2$. We fix i (≥ 2). From Lemma 2.1 it follows that

$$\langle \sigma(x, e_i), \sigma(x, e_j)\rangle = (\lambda^2 - \lambda_i)/2 \cdot \delta_{ij} \quad \text{for any } j,$$

and hence $A_{\sigma(x,e_i)} x = (\lambda^2 - \lambda_i)/2 \cdot e_i$. Since $A_\xi A_{\sigma(x,e_i)} x = A_{\sigma(x,e_i)} A_\xi x$, we obtain

$$(\lambda^2 - \lambda_i)\lambda_i/2 = (\lambda^2 - \lambda_i)\lambda^2/2$$

so that $\lambda_i = \lambda^2$ for $i \geq 2$, that is, V is the eigenspace of A_ξ with eigenvalue λ^2. And hence by the argument in Lemma 2.2, σ is umbilic. ∎

3. **Isotropic immersions with flat normal connection.**

Proposition 2.11 gives us the following statement on submanifolds in a real space form.

THEOREM 3.1. *Let M be a submanifold immersed in a real space form $\tilde{M}(c)$. Then the following are equivalent:*

(i) M is a totally umbilic submanifold,

(ii) M is an isotropic submanifold with flat normal connection.

PROOF. By the equation of Ricci, we see that a submanifold in a real space form has a flat normal connection if and only if $A_\xi A_\eta = A_\eta A_\xi$ for any normal vector fields ξ, η (*c.f.* Chen [2]). From this and Proposition 2.11, it follows that two conditions in Theorem 3.1 are equivalent. ∎

As an immediate consequence of Theorem 3.1 we obtain the following:

COROLLARY 3.2. *Let M be a submanifold immersed in a real space form $\tilde{M}(c)$. Then the following are equivalent:*

(i) M is a totally geodesic submanifold.

(ii) M is an isotropic minimal submanifold with flat normal connection.

Note that the statement above does not hold if we remove one of three notions "isotropic", "minimal" and "flat normal connection" of condition (ii) in Corollary 3.2.

REMARK. In case that the ambient space $\tilde{M}$ is a complex projective space, Theorem 3.1 does not hold. Our discussion is as follows:

Let $P^n_{\mathbb{C}}(4)$ be an n-dimensional complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 4. Now we shall construct a flat manifold T^n

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in $P^n_{\mathbb{C}}(4)$. We consider $M^{n+1} = S^1(1/\sqrt{n+1}) \times \cdots \times S^1(1/\sqrt{n+1})$ in $S^{2n+1}(1)$, where $S^1(1/\sqrt{n+1})$ is a circle with radius $1/\sqrt{n+1}$. Making use of this manifold M^{n+1}, we get a fibration $S^1 \longrightarrow M^{n+1} \longrightarrow T^n$ which is compatible with the Hopf fibration $S^1 \longrightarrow S^{2n+1} \longrightarrow P^n_{\mathbb{C}}(4)$ (*cf.* [13]).

The submanifold T^n (in $P^n_{\mathbb{C}}(4)$) thus obtained has various beautiful properties. In fact, for $n \geq 2$ T^n is a totally real minimal submanifold with flat normal connection. Moreover, the second fundamental form of T^n is parallel (*cf.* [13]). But T^n is not isotropic in $P^n_{\mathbb{C}}(4)$ in the case of $n \geq 3$. We emphasize the fact that T^2 is isotropic in $P^2_{\mathbb{C}}(4)$ (*cf.* [8]). Consequently $P^2_{\mathbb{C}}(4)$ admits a flat torus T^2 as an isotropic submanifold with flat normal connection. Of course, the submanifold T^2 is not totally umbilic in $P^2_{\mathbb{C}}(4)$.

4. **Isotropic immersions with low codimension.** Let M be an n (≥ 2)-dimensional λ-isotropic submanifold of a real space form $\tilde{M}(c)$. Noting that λ^2 is a differentiable function on M, we study the derivative of the second fundamental form σ.

LEMMA 4.1. *For any $x, y \in T_pM$ the following holds:*

$$(4.1) \qquad \langle (\bar{\nabla}_x\sigma)(x,x), \sigma(x,y)\rangle = d\lambda^2(x)\langle x,y\rangle\langle x,x\rangle - 1/2\cdot d\lambda^2(y)\langle x,x\rangle\langle x,x\rangle.$$

PROOF. We fix $p \in M$. We take arbitrary vectors $x, y \in T_pM$. Let $\gamma\colon (-\varepsilon,\varepsilon) \longrightarrow M$ be a differentiable curve satisfying $\gamma(0) = p$ and $\dot\gamma(0) = y$. We denote by $X(t)$ a parallel vector field along γ such that $X(0) = x$. Then for any $t \in (-\varepsilon,\varepsilon)$

$$(4.2) \qquad \Big\langle \sigma\big(X(t),X(t)\big), \sigma\big(X(t),X(t)\big)\Big\rangle = \lambda^2\langle X(t),X(t)\rangle^2.$$

Differentiating (4.2) at $t = 0$, we find

$$(4.3) \qquad \langle (\bar{\nabla}_y\sigma)(x,x), \sigma(x,x)\rangle = 1/2\cdot d\lambda^2(y)\langle x,x\rangle^2 \quad \text{for any } x,y \in T_pM.$$

In particular, putting $y = x$ in (4.3), we get

$$(4.4) \qquad \langle (\bar{\nabla}_x\sigma)(x,x), \sigma(x,x)\rangle = 1/2\cdot d\lambda^2(x)\langle x,x\rangle^2 \quad \text{for any } x \in T_pM.$$

So, using the symmetry of $\bar{\nabla}_\sigma$, we have

$$(4.5) \qquad \begin{aligned} &3\langle (\bar{\nabla}_y\sigma)(x,x), \sigma(x,x)\rangle + 2\langle (\bar{\nabla}_x\sigma)(x,x), \sigma(x,y)\rangle \\ &\qquad = 1/2\cdot d\lambda^2(y)\langle x,x\rangle^2 + 2d\lambda^2(x)\langle x,y\rangle\langle x,x\rangle \quad \text{for any } x,y \in T_pM. \end{aligned}$$

(4.3) and (4.5) yield (4.1). ∎

PROPOSITION 4.2. *Let M^n be an n (≥ 2)-dimensional λ-isotropic connected submanifold immersed in an $(n+k)$-dimensional real space form $\tilde{M}^{n+k}(c)$. If $k \leq n-1$, then λ is constant.*

PROOF. We shall prove that $d\lambda^2 = 0$ at every point $p \in M$. First we study at a point $p \in M$ such that $\lambda(p) = 0$. From (4.1), it follows that

$$1/2\cdot d\lambda^2(x)\langle x,x\rangle^2 = \langle (\bar{\nabla}_x\sigma)(x,x), \sigma(x,x)\rangle = 0 \quad \text{for any } x \in T_pM.$$

Therefore we have $d\lambda^2 = 0$ at such a point p.

Next we study at a point $p \in M$ such that $\lambda(p) > 0$. We denote by N_pM the normal space at p. Since $\dim N_pM \leq \dim T_pM - 1$, for any $y \in T_pM$ there exists a nonzero vector $x \in T_pM$ such that $\sigma(x, y) = 0$. From Lemma 2.1 it follows that

$$\langle \sigma(x,x), \sigma(x,y) \rangle = \lambda^2 \langle x, y \rangle \langle x, x \rangle \quad \text{so that } \langle x, y \rangle = 0.$$

So, from (4.1) we see that $d\lambda^2(y) = 0$ for any $y \in T_pM$, that is, $d\lambda^2 = 0$ at such a point p. ■

The purpose of this section is to prove the following:

THEOREM 4.3. *Let f be a λ-isotropic immersion of an n (≥ 2)-dimensional connected Riemannian manifold M^n into an $(n+k)$-dimensional real space form $\tilde{M}^{n+k}(c)$. Suppose that $k \leq \min\{n-1, n-\nu_{n-1}\}$. Then either f is totally umbilic or f is locally congruent to one of the following first standard minimal immersions of M^n into $\tilde{M}^{n+k}(c)$:*

(1) $M^n = P^2_{\mathbb{C}}(4c/3)$, $\tilde{M}^{n+k}(c) = S^7(c)$,

(2) $M^n = P^2_{\mathbb{H}}(4c/3)$, $\tilde{M}^{n+k}(c) = S^{13}(c)$,

(3) $M^n = P^2_{\mathrm{Cay}}(4c/3)$, $\tilde{M}^{n+k}(c) = S^{25}(c)$,

where $S^m(c)$ denotes an m-dimensional sphere of constant sectional curvature c and $P^2_{\mathbb{C}}(4c/3)$, $P^2_{\mathbb{H}}(4c/3)$ and $P^2_{\mathrm{Cay}}(4c/3)$ denote the projective planes of maximal sectional curvature $4c/3$ over the complex, quaternions and Cayley numbers, respectively.

PROOF. Since $k \leq n-1$, Proposition 4.2 tells us that λ is constant. If $k < n - \nu_{n-1}$, by Corollary 2.8 f is totally umbilic. So we shall study the case $k = n - \nu_{n-1}$. We denote by U the set of all umbilic points of f and put $U^c = M - U$. We suppose that U^c is not empty and we shall prove that $U^c = M$ and that the second fundamental form of f is parallel on M.

We denote by S_pM the unit sphere in T_pM at $p \in U^c$. We consider the linear map $\sigma_x \colon T_pM \longrightarrow N_pM$ defined by $v \in T_pM \longmapsto \sigma(x, v) \in N_pM$ for $x \in S_pM$. From Lemma 2.9 it follows that $\pi_p = \{x \in S_pM \mid \sigma_x \text{ is surjective}\}$ is an open and dense subset of S_pM. Since λ is constant, (4.1) yields that

$$\langle (\bar{\nabla}_x \sigma)(x,x), \sigma(x,y) \rangle = 0 \quad \text{for any } x, y \in T_pM.$$

Therefore we have

$$(\bar{\nabla}_x \sigma)(x,x) = 0 \quad \text{for } x \in \pi_p, \text{ and hence } (\bar{\nabla}_x \sigma)(x,x) = 0 \quad \text{for all } x \in S_pM.$$

By the symmetry of $\bar{\nabla}\sigma$, $\bar{\nabla}\sigma = 0$ at $p \in U^c$.

Now we define a differentiable function h on M by $h(p) = \|\sigma_p\|^2 - n\|\mathfrak{f}_p\|^2$, where $\|\sigma_p\|$ and $\|\mathfrak{f}_p\|$ denote the length of the second fundamental form σ and the mean curvature $\mathfrak{f}$ at $p \in M$, respectively. Note that h is nonnegative and $h^{-1}(0) = U$. In particular, U^c is open in M. Since $\bar{\nabla}\sigma = 0$ in U^c, h is locally constant in U^c. We fix $p_0 \in U^c$ and put $h_0 = h(p_0)$ (> 0). Then $h^{-1}(h_0)$ is a closed subset in M. On the other hand, since $h^{-1}(h_0) \subset U^c$, $h^{-1}(h_0)$ is open. By the connectedness of M, $h^{-1}(h_0) = M$ and hence $U^c = M$.

By the classification theorem of isotropic submanifolds with parallel second fundamental form in a real space form (Sakamoto [10] and also see Ferus [5]), we get our conclusion. ∎

REMARK. In case of (1), (2) and (3) in Theorem 4.3, they satisfy $k = (n+2)/2$. Therefore the second fundamental forms of the submanifolds satisfy the properties of Proposition 2.10.

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