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INTERPOLATION AND SPECTRA OF REGULAR L^P-SPACE OPERATORS

BY
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ABSTRACT. We consider the Banach algebra consisting of linear operators T which are defined on the simple functions and have bounded extensions T_p on L^p for all values of $p \in [1, \infty]$. We show that the 'integral' operators in this algebra form a right ideal, and that each T_p associated to an integral T is regular. When the underlying measure is finite or special discrete we show further that every T_p is regular for every T in the algebra. Algebraic techniques together with interpolation results are then used to get relationships between the spectrum and the order spectrum of the associated T_p's.

If an operator T belongs to $B(L^p)$ for more than one value of p, and $\sigma(T_p)$ denotes the spectrum of T in the Banach algebra $B(L^p)$, one can ask how $\sigma(T_p)$ varies with p. Typically, we are interested in the situation when p takes on all values in some closed subinterval $[s, t]$ of $[1, \infty]$. The well known example of Boyd (in [6]) shows that even for a reasonable operator T, the function $p \longrightarrow \sigma(T_p)$ is a nonconstant function of p. Several authors have studied the function $p \longrightarrow \sigma(T_p)$. It is interesting to investigate the continuity properties of this function and bounds for its range in terms of the 'endpoint' spectra $\sigma(T_s)$ and $\sigma(T_t)$. Recent related work appears in [3], [8], [10] and [14].

In this paper we investigate the spectral theory of operators that are defined on every L^p-space, $1 \leq p \leq \infty$. These operators form a Banach algebra which we denote by $\mathcal{B}_{1,\infty}$. We introduce its subalgebra of 'integral' operators and show that this set is in fact a right ideal of $\mathcal{B}_{1,\infty}$. Many of these operators are regular operators (in the sense of Schaefer, see [13]). We show that each of the integral operators of $\mathcal{B}_{1,\infty}$ is always regular and, when the underlying space has finite measure, that every operator of $\mathcal{B}_{1,\infty}$ is regular. For these cases, we give conditions that imply equality of the (polynomially convex hull of the) spectrum and the (polynomially convex hull of the) order spectrum. The problem of characterizing classes of regular operators with equal spectrum and order spectrum and the more general study of the 'pure' order spectrum (the set of complex numbers belonging to the order spectrum but not the ordinary spectrum) are addressed in [1], [12] and [15].

1. **The algebras $\mathcal{B}_{1,\infty}$ and $\mathcal{A}_{1,\infty}$.** Throughout this paper Ω will denote a measure space equipped with a positive, σ-finite, measure μ. The measure μ will be called *special discrete* if it is defined on the σ-algebra of all subsets of $\Omega = \{1, 2, \cdots\}$ and the

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set $\{\mu(\{n\})\}_{n=1}^{\infty}$ is bounded away from zero. The usual Lebesgue space on Ω will be denoted by L^p. $B(L^p)$ will denote the Banach algebra of all bounded linear operators on L^p with the usual operator norm. For fixed values of s and t, $1 \leq s < t \leq \infty$, $\mathcal{B}_{s,t}$ will denote the algebra of all linear operators $T: L^s \cap L^t \longrightarrow L^s \cap L^t$ that are continuous with respect to both the s-norm and the t-norm. The 'interpolation' algebra $\mathcal{B}_{s,t}$ is a Banach algebra which was introduced and first investigated in [3]. If $T \in \mathcal{B}_{s,t}$ then it follows from the Riesz-Thorin Theorem that T has a unique extension to an operator $T_p \in B(L^p)$ for each value of p in $[s,t]$. If $t = \infty$ then T_∞ is not in general defined on all of L^∞ but is instead defined on L_0^∞, the closure of $L^s \cap L^t$ with respect to the ∞-norm. Notice that if the underlying measure μ is finite then $L_0^\infty = L^\infty$ and if μ is special discrete then $L_0^\infty = c_0$.

A *kernel* shall simply mean a measurable function $k : \Omega \times \Omega \longrightarrow \mathbb{C}$. Two kernels are considered equal if they are equal almost everywhere. Let $M(\Omega)$ denote the set of all measurable and almost everywhere finite functions on Ω. Let X denote a subspace of $M(\Omega)$ that is a Banach space. A bounded linear operator T on X is called *integral* if it is induced by a kernel k via the formula

$$Tf(x) = \int_\Omega k(x,y)f(y)\ d\mu(y) \tag{1}$$

for $f \in X$ and almost all $x \in \Omega$. T is called *absolutely integral* if the kernel $|k|(x,y) = |k(x,y)|$ also defines a bounded linear operator on X. Recall that on a Banach lattice X a linear operator is called *regular* if it is a linear combination of positive operators. Every regular operator $T \in B(L^p)$ has a *modulus*, $|T|$, given by the formula

$$|T|(f) = \sup\{\,|Tg| : g \in L^p,\ |g| \leq f\}$$

for all real-valued $f \geq 0$ (this formula for the modulus actually works on any order complete Banach lattice). Background material on Banach lattices and operators with order preserving properties can be found in [13]. We will make immediate use of the following proposition. Surprisingly, it was not even proved on $L^2((0,1))$ until 1971. For a proof and discussion see [7, page 63] or [13, page 295].

PROPOSITION 1.1. *Let $K \in B(L^p)$ be defined by the formula*

$$Kf(x) = \int_\Omega k(x,y)f(y)\, d\mu(y), \quad f \in L^p,$$

for some kernel k. Then K is regular if and only if $|k|$ induces a bounded linear operator on L^p. In this case, the modulus of K is given by

$$|K|(f)(x) = \int_\Omega |k(x,y)| f(y)\, d\mu(y), \quad f \in L^p.$$

Let $\mathcal{A}_{1,\infty}$ denote the set of all kernels k such that

$$\|k\| \equiv \max\{\text{ess sup}_x \int_\Omega |k(x,y)|\ d\mu(y),\ \text{ess sup}_y \int_\Omega |k(x,y)|\ d\mu(x)\} < \infty. \tag{2}$$

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With pointwise linear operations and multiplication given by

$$(k * j)(x, y) = \int_{\Omega} k(x, z) j(z, y)\, d\mu(z), \quad j, k \in \mathcal{A}_{1,\infty},$$

$\mathcal{A}_{1,\infty}$is an algebra. Endowed with the norm in (2), it is a Banach algebra. From (2) it follows that each $k \in \mathcal{A}_{1,\infty}$ defines, via formula (1), a bounded operator K_p on L^p for $p = 1$ and $p = \infty$. For convenience we will sometimes use the notation $\mathrm{Int}_p(k)$ in place of K_p. It now follows from the Riesz-Thorin Theorem that every $k \in \mathcal{A}_{1,\infty}$ defines a bounded linear operator on L^p for each value of $p \in [1, \infty]$. The following characterization of $\mathcal{A}_{1,\infty}$shows that the converse is also true. We use the notation T' to denote the adjoint of the operator T.

PROPOSITION 1.2. $\mathcal{A}_{1,\infty} = \{k : \Omega \times \Omega \longrightarrow \mathbb{C}$ measurable$|$ k induces $K_1 \in B(L^1)$ and $K_\infty \in B(L^\infty)\} = \{k : \Omega \times \Omega \longrightarrow \mathbb{C}$ measurable$|$ k induces $K_p \in B(L^p)$ for all $p \in [1, \infty]\}$.

Proof. Let k be a kernel such that equation (1) defines $K_1 \in B(L^1)$ and $K_\infty \in B(L^\infty)$. Since every operator in $B(L^1)$ or $B(L^\infty)$ is regular ([13], Theorem IV.1.5) it follows from Proposition 1.1 that K_1 and K_∞ are absolutely integral. By Proposition 1.1, $\mathrm{Int}_p(|k|) \in B(L^p)$ for $p = 1$ and $p = \infty$. Putting $f \equiv 1$ we have

$$\text{ess sup}_x \int_{\Omega} |k(x, y)|\, d\mu(y) = \|\mathrm{Int}_\infty(|k|)(f)\|_\infty < \infty.$$

Now, put $k'(x, y) \equiv k(y, x)$. Then $(\mathrm{Int}_1(k))' \in B(L^\infty)$ and $(\mathrm{Int}_1(k))' = \mathrm{Int}_\infty(k')$ on L^∞. So, again with $f \equiv 1$, we have

$$\begin{aligned} \text{ess sup}_x \int_{\Omega} |k(y, x)|\, d\mu(y) &= \text{ess sup}_x \int_{\Omega} |k'(x, y)|\, d\mu(y) \\ &= \|(\mathrm{Int}_\infty(|k'|))(f)\|_\infty = \|(\mathrm{Int}_1(|k|))'(f)\|_\infty < \infty. \end{aligned}$$

These two calculations show that $k \in \mathcal{A}_{1,\infty}$. ∎

We now give a few examples of the elements of $\mathcal{A}_{1,\infty}$. If the underlying space Ω is a unimodular locally compact group G and $g \in L^1(G)$ then the kernel defined by $k(x, y) \equiv g(xy^{-1})$ satisfies (2) and hence $\mathcal{A}_{1,\infty}$contains all L^1-convolution operators

$$K_p f(x) = (g * f)(x) = \int_{\Omega} g(xy^{-1}) f(y)\, d\mu(y), \quad f \in L^p.$$

As another example, if μ is counting measure on the set $\Omega = \{1, 2, \ldots\}$ then $\mathcal{A}_{1,\infty}$ contains the identity matrix and the unilateral shift matrix (in fact, we have $\mathcal{A}_{1,\infty} = \mathcal{B}_{1,\infty}$, see Theorem 1.4.). Both of these examples show that $\mathcal{A}_{1,\infty}$may contain non-compact operators.

LEMMA 1.3. *If $T \in \mathcal{B}_{1,\infty}$then T is regular as a map on $L^1 \cap L^\infty$. Furthermore, $|T| \in \mathcal{B}_{1,\infty}$.*

Proof. We first show that T is regular. Since $L^1 \cap L^\infty$ is order complete, it suffices to show that the set $E_f = \{|Tg| : g \in L^1 \cap L^\infty, |g| \leqq f\}$ is bounded above in $L^1 \cap L^\infty$ for each $f \in (L^1 \cap L^\infty)_+ \equiv \{g \in L^1 \cap L^\infty : g \geq 0\}$. If we restrict our attention to measurable functions then $|g| \leqq f \in L^1 \cap L^\infty$ automatically implies that $g \in L^1 \cap L^\infty$. Fix $f \in (L^1 \cap L^\infty)_+$. Since T is continuous in the ∞-norm on $L^1 \cap L^\infty$ there exists a constant $M \geq 0$ such that $\|Tg\|_\infty \leqq M\|g\|_\infty$, $g \in L^1 \cap L^\infty$. Therefore,

$$|Tg|(x) = |Tg(x)| \leqq \|Tg\|_\infty \leqq M\|g\|_\infty \leqq M\|f\|_\infty$$

whenever $|g| \leqq f$. The constant function $M\|f\|_\infty$ is in $M(\Omega)$ and therefore E_f is bounded above in $M(\Omega)$. Hence, the set $E_f \subseteq M(\Omega)$ has a supremum in $M(\Omega)$. Let h_f denote this supremum. Clearly $h_f \leqq M\|f\|_\infty \in L^\infty$ and hence $h_f \in L^\infty$. The function $|T_1|(f) \in M(\Omega)$ and is an upper bound for E_f; therefore $h_f \leqq |T_1|(f) \in L^1$ and hence $h_f \in L^1$. Now h_f is an element of $L^1 \cap L^\infty$ and is an upper bound of E_f in $L^1 \cap L^\infty$. This completes the proof that T is regular. Since $L^1 \cap L^\infty$ is order complete, the modulus of T is given by

$$|T|(f) = \sup_{L^1 \cap L^\infty} \{|Tg| : |g| \leqq f\}, \quad f \in (L^1 \cap L^\infty)_+.$$

Since this is the supremum in a sublattice of $M(\Omega)$, $|T|(f) \leqq h_f$. Since $h_f \in L^1 \cap L^\infty$, equality holds. The inequalities $h_f \leqq M\|f\|_\infty$ and $h_f \leqq |T_1|(f)$ show that $|T|$ is continuous in both the ∞-norm and the 1-norm. Hence, $|T| \in \mathcal{B}_{1,\infty}$. ∎

In the proof of Lemma 1.3 we saw that $|T|(f) \leqq |T_1|(f)$, for all $f \in (L^1 \cap L^\infty)_+$. The other inequality also holds,

$$\begin{aligned} |T_1|(f) &\equiv \sup_{L^1}\{|T_1 g| : g \in L^1, |g| \leqq f\} = \sup_{L^1}\{|Tg| : g \in L^1 \cap L^\infty, |g| \leqq f\} \\ &\leqq \sup_{L^1 \cap L^\infty}\{|Tg| : g \in L^1 \cap L^\infty, |g| \leqq f\} \equiv |T|(f) \quad f \in (L^1 \cap L^\infty)_+. \end{aligned}$$

Therefore, $|T|$ is the restriction of $|T_1|$ to $L^1 \cap L^\infty$.

The algebra $\mathcal{A}_{1,\infty}$ is identified with a subalgebra of $\mathcal{B}_{1,\infty}$ via $k \mapsto K$. We adopt the notation of Barnes (see [3]) and write T for an element of $\mathcal{B}_{s,t}$ and $T_{s,t}$ if we wish to consider it as an element of $B(L^s \cap L^t)$.

The next result, which is key to much of our spectral theory, is a consequence of a beautiful theorem of Schachermeyer ([11], Theorem 6.2).

THEOREM 1.4. *The algebra $\mathcal{A}_{1,\infty}$ is a right ideal of $\mathcal{B}_{1,\infty}$.*

Proof. Let K denote the element of $\mathcal{B}_{1,\infty}$ associated with the kernel $k \in \mathcal{A}_{1,\infty}$. Let $T \in \mathcal{B}_{1,\infty}$ be arbitrary. It follows from Schachermeyer's result that the operator $K_1 \circ T_1 \in B(L^1)$ is absolutely integral. Therefore, there exists a kernel j such that

$$(K_1 \circ T_1)(f)(x) = \int_\Omega j(x,y) f(y)\, d\mu(y), \quad f \in L^1,$$

and $|j|$ induces an operator in $B(L^1)$. By the remarks following Lemma 1.3, $K_{1,\infty} \circ T_{1,\infty}$ is regular and $|K_{1,\infty} \circ T_{1,\infty}|(f) = |K_1 \circ T_1|(f)$, $f \in L^1 \cap L^\infty$. From Proposition 1.1 it follows that

$$|K_1 \circ T_1|(f)(x) = |(K \circ T)_1|(f)(x) = \int_\Omega |j(x,y)| f(y)\, d\mu(y), \quad f \in L^1 \cap L^\infty.$$

In particular, the operator in $B(L^1 \cap L^\infty)$ that $|j|$ induces is continuous in the ∞-norm. Therefore, there exists a constant $M \geq 0$ such that

$$\operatorname{ess\,sup}_x\{ | \int_\Omega |j(x,y)| f(y)\, d\mu(y)| \} \leqq M\|f\|_\infty, \quad f \in L^1 \cap L^\infty.$$

Write $\Omega = \cup_{n=1}^\infty \Omega_n$ such that $\mu(\Omega_n) < \infty$ and $\Omega_1 \subseteq \Omega_2 \subseteq \cdots \subseteq \Omega$ and let $\chi_n \in L^1 \cap L^\infty$ be the characteristic function of Ω_n, $n = 1, 2, \ldots$. Then

$$\operatorname{ess\,sup}_x \int_\Omega |j(x,y)| \chi_n(y)\, d\mu(y) \leqq M, \quad n = 1, 2, \ldots.$$

By the Lebesgue Monotone Convergence Theorem,

$$(\dagger) \qquad \operatorname{ess\,sup}_x \int_\Omega |j(x,y)| \ d\mu(y) \leqq M.$$

By definition of j, $\mathrm{Int}_1(|j|) \in B(L^1)$. Therefore, $\mathrm{Int}_\infty(|j'|) = (\mathrm{Int}_1(|j|))' \in B(L^\infty)$. Put $f \equiv 1$. Then $f \in L^\infty$ and so $(\mathrm{Int}_\infty(|j'|))(f) \in L^\infty$. Therefore,

$$(\ddagger) \qquad \operatorname{ess\,sup}_x \int_\Omega |j(y,x)| \ d\mu(y) = \|(\mathrm{Int}_\infty(|j'|))(f)\|_\infty < \infty.$$

The inequalities $(\dagger)$ and $(\ddagger)$ imply that $j \in \mathcal{A}_{1,\infty}$. We have shown that $K \circ T \in \mathcal{B}_{1,\infty}$ is integral with kernel in $\mathcal{A}_{1,\infty}$; this completes the proof. ∎

2. **Order spectral theory of L^p-interpolation operators.** The first theorem of this section is, in its present form, due to Barnes ([3], Theorem 5.1). Similar results appear in [10] and [14]. Recall that the *polynomial convex hull* of a compact set $\Gamma \subseteq \mathbb{C}$, denoted by $\hat{\Gamma}$, is defined to be the complement of the (unique) unbounded connected component of Γ in $\mathbb{C}$. A *hole* of Γ is a bounded component of its complement.

THEOREM 2.1. *Assume that $T \in \mathcal{B}_{s,t}$ and let $\sigma(T)$ denote the spectrum of T in the Banach algebra $\mathcal{B}_{s,t}$. Then*

$$\sigma(T) = \sigma(T_s) \cup \sigma(T_t) \cup \sigma(T_{s,t}).$$

Furthermore,

$$\partial(\sigma(T)) \subseteq \sigma(T_s) \cup \sigma(T_t) \subseteq \sigma(T)$$

and hence

$$[\sigma(T)]\hat{} = [\sigma(T_s) \cup \sigma(T_t)]\hat{}.$$

If $t = \infty$, $\sigma(T_t)$ means $\sigma(T_\infty | L_0^\infty)$.

If $p \in [s,t]$, we can view $\mathcal{B}_{s,t}$ as a subalgebra of $B(L^p)$. Hence, $\sigma(T_p) \subseteq \sigma(T)$ and Theorem 2.1 tells us that $\sigma(T_p)$ is always contained in the polynomial convex hull of $\sigma(T_s) \cup \sigma(T_t)$. This has been noted by other authors as well as by Barnes. Barnes's statement of this result has an advantage in that it makes an attempt to describe the holes of $\sigma(T_s) \cup \sigma(T_t)$ that are needed. It says that the missing holes are contained in the spectrum of another operator, namely $T_{s,t}$. At this time our understanding of the operator $T_{s,t}$ on the Banach space $L^s \cap L^t$ and its spectrum is not satisfactory.

COROLLARY 2.2. *Assume that* $k \in \mathcal{A}_{1,\infty}$ *and let* $\sigma(k)$ *denote the spectrum of* k *in the Banach algebra* $\mathcal{A}_{1,\infty}$. *Then*

$$\sigma(k) = \sigma(K_1) \cup \sigma(K_\infty | L_0^\infty) \cup \sigma(K_{1,\infty}).$$

Furthermore,

$$\partial(\sigma(k)) \subseteq \sigma(K_1) \cup \sigma(K_\infty | L_0^\infty) \subseteq \sigma(k)$$

and hence

$$[\sigma(k)]\hat{} = [\sigma(K_1) \cup \sigma(K_\infty | L_0^\infty)]\hat{}.$$

Proof. With $k \in \mathcal{A}_{1,\infty}$ we associate the element $K \in \mathcal{B}_{1,\infty}$. So $\sigma(K)$ denotes the spectrum in $\mathcal{B}_{1,\infty}$ while $\sigma(k)$ denotes the spectrum of the same element when viewed in $\mathcal{A}_{1,\infty}$. Since $\mathcal{A}_{1,\infty}$ is a subalgebra of $\mathcal{B}_{1,\infty}$, we have $\sigma(K) \subseteq \sigma(k)$. To see the other inclusion assume that $\lambda \notin \sigma(K)$. Then there exists an element $S \in \mathcal{B}_{1,\infty}$ such that $(\lambda - K)S = I = S(\lambda - K)$ and hence $I - \lambda S - KS = 0$ and $I - \lambda S - SK = 0$. Note that K and S commute. If $\mathcal{A}_{1,\infty} = \mathcal{B}_{1,\infty}$ then the result is obvious from the preceeding theorem. If they are not equal then $\mathcal{A}_{1,\infty}$ is a proper right ideal (Theorem 1.4) and so $\lambda \neq 0$. Therefore,

$$\frac{1}{\lambda}K - KS - (\frac{1}{\lambda}K)KS = 0 \text{ and so } KS = (\frac{1}{\lambda}K)KS - \frac{1}{\lambda}K,$$

which shows that KS is in $\mathcal{A}_{1,\infty}$ and is a quasi-inverse for $\frac{1}{\lambda}K$. Thus $\sigma(K) = \sigma(k)$ and the result now follows immediately from Theorem 2.1. ∎

In the rest of this section it is our aim to study the relationship between the order spectrum and ordinary spectrum of these interpolation operators when they turn out to be regular. The notion of studying regular L^p-space operators in connection with interpolation theory is motivated by the fact that every bounded linear operator on either L^1 or L^∞ is regular (see [13] for a proof of this) while this is not necessarily true for $p \in (1, \infty)$. Hence, the idea is that those L^p-space operators which extend to L^1 and L^∞ operators will share some of the regularity conditions enjoyed by the elements of $B(L^1)$ and $B(L^\infty)$. Theorem 1.4 is an example of this phenomenon; Schachermeyer's result ([11], Theorem 6.2) shows that the 'integral' operators form a right ideal in the algebra of regular operators of L^p. Since every bounded linear operator on either L^1 or L^∞ is regular, his result implies that the 'integral' operators form a right ideal of $B(L^1)$ or $B(L^\infty)$. In the same paper, he gives examples which show that they do not form a right ideal of $B(L^2)$. Previous work on interpolation of regular operators appears in [15].

The algebra of all regular operators on L^p will be denoted by $B^r(L^p)$. The norm of a regular operator in this algebra is defined to be the operator norm of its modulus. It is a Banach algebra that is continuously embedded as a subalgebra of $B(L^p)$. Let $\sigma_0(T)$ denote the spectrum of a regular operator T in the Banach algebra $B^r(L^p)$. Then clearly $\sigma_0(T)$ always contains $\sigma(T)$ and, because the embedding is continuous, the two sets are equal whenever $\sigma_0(T)$ is totally disconnected (this fact was first proved in [12]). If $\sigma(T)$ is known to be totally disconnected then the same conclusion cannot be drawn; for an example of a positive compact operator with uncountable order spectrum see [2]. It is desirable to have information about the relationship between the two spectra when they are 'fatter' sets and to have conditions on the ordinary spectrum that imply equality of the two spectra. The 'interpolation' approach adopted here gives results in this direction. These results constitute the remainder of this section.

We know that we can view $\mathcal{A}_{1,\infty}$as a subalgebra of $B(L^p)$ for each p, $1 \leq p \leq \infty$. Since we can decompose any $k \in\mathcal{A}_{1,\infty}$as a linear combination of four positive, real valued functions on $\Omega \times \Omega$, $\mathcal{A}_{1,\infty}$can actually be viewed as a subalgebra of the smaller algebra $B^r(L^p)$. Therefore, for each $k \in\mathcal{A}_{1,\infty}$, $\sigma(K_p) \subseteq \sigma_0(K_p) \subseteq \sigma(k)$ for p, $1 \leq p \leq \infty$. Thus,

$$\bigcup_{p\in[1,\infty]} \sigma(K_p) \subseteq \bigcup_{p\in[1,\infty]} \sigma_0(K_p) \subseteq \sigma(k) \tag{3}$$

for each $k \in\mathcal{A}_{1,\infty}$.

COROLLARY 2.3. *Assume that* $k \in\mathcal{A}_{1,\infty}$. *Then*

$$\bigcup_{p\in[1,\infty]} \sigma(K_p) = \bigcup_{p\in[1,\infty]} \sigma_0(K_p) = \sigma(k) = \sigma(K_1) \cup \sigma(K_\infty)$$

whenever one of the following conditions is satisfied:

(a) μ *is finite, or*

(b) μ *is special discrete and* $K_\infty(\ell^\infty(\mu)) \subseteq c_0(\mu)$.

Proof. The only thing that remains to be seen is that $\sigma(K_\infty) = \sigma(K_\infty | L_0^\infty)$ when μ is special discrete and $K_\infty(\ell^\infty(\mu)) \subseteq c_0(\mu)$. This follows from [5], Theorem 4. ∎

Boyd's example ([6]) gives a kernel k (necessarily not in $\mathcal{A}_{1,\infty}$) defining a bounded linear operator K_p on each $L^p((0\infty))$, $1 < p \leq \infty$, such that $\sigma(K_p) \not\subseteq \sigma(K_s) \cup \sigma(K_t)$, $1 < s < p < t \leq \infty$. We do not know of a kernel in $\mathcal{A}_{1,\infty}$(μ necessarily infinite) with this property.

It is perhaps worth pointing out that the conditions on μ required in Corollary 2.3 are exactly the conditions under which the Lebesgue spaces form a chain; if μ is finite then $L^\infty \subseteq \cdots \subseteq L^2 \subseteq L^1$ and if μ is special discrete then $\ell^1 \subseteq \ell^2 \subseteq \cdots \subseteq c_0 \subseteq \ell^\infty$.

COROLLARY 2.4. *Assume that* $k \in\mathcal{A}_{1,\infty}$. *Then*

$$[\bigcup_{p\in[1,\infty]} \sigma(K_p) \cup \{0\}\,]\hat{} = [\bigcup_{p\in[1,\infty]} \sigma_0(K_p) \cup \{0\}\,]\hat{}$$

whenever $K_\infty(L^\infty) \subseteq L_0^\infty$.

Proof. Corollary 2.2 and [5], Theorem 4. ∎

When $K_\infty(L^\infty) \subseteq L_0^\infty$, $\sigma(K_\infty)$ and $\sigma(K_\infty | L_0^\infty)$ can only differ by $\{0\}$ (see [5]). This accounts for the hypothesis $K_\infty(L^\infty) \subseteq L_0^\infty$ and for the extra $\{0\}$ in the conclusion of the preceeding corollary (and for them in the sequel). If $K_\infty(L^\infty)$ is not contained in L_0^∞, the results of this section hold with $\sigma(K_\infty | L_0^\infty)$ in place of $\sigma(K_\infty)$.

COROLLARY 2.5. *Assume that* $k \in\mathcal{A}_{1,\infty}$*and that* $K_\infty(L^\infty) \subseteq L_0^\infty$ *(note that this is not a restriction on* $k \in\mathcal{A}_{1,\infty}$*when* μ *is finite). If* $\sigma(K_1)$ *and* $\sigma(K_\infty)$ *are both totally*

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disconnected then $\sigma(K_1) = \sigma(K_p) = \sigma_0(K_p)$ for all p, $1 \leqq p \leqq \infty$. In particular, if K_1 and K_∞ are compact, these equalities hold.

Proof. From Corollary 2.2 it follows that $\sigma(k)$ is totally disconnected whenever $\sigma(K_1)$ and $\sigma(K_\infty)$ are both totally disconnected. Since $\sigma_0(K_p) \subseteq \sigma(k)$ for all p, $1 \leqq p \leqq \infty$, $\sigma_0(K_p)$ is also totally disconnected for all p, $1 \leqq p \leqq \infty$. Now $\sigma(K_p) = \sigma_0(K_p)$ for all p, $1 \leqq p \leqq \infty$, follows from Schaefer's result. That this totally disconnected set is independent of p follows from the fact that for any $1 \leqq s \leqq t \leqq \infty$, L^t is continuously embedded in L^s (or see [1], Proposition 2.2). ∎

COROLLARY 2.6. *Assume that $k \in \mathcal{A}_{1,\infty}$ and that $\sigma(K_p)$ is independent of p, $1 \leqq p \leqq \infty$.*

(a) If μ is finite or special discrete then $\sigma(K_p) = \sigma_0(K_p)$, $1 \leqq p \leqq \infty$.

(b) If μ is arbitrary and $K_\infty(L^\infty) \subseteq L_0^\infty$ then $\sigma(K_p) \subseteq \sigma_0(K_p) \subseteq [\sigma(K_p)]\hat{}$, $1 \leqq p \leqq \infty$.

Proof. These results are immediate from Corollaries 2.3 and 2.4. ∎

We recall that $\mathcal{A}_{1,\infty} = \mathcal{B}_{1,\infty}$ whenever the underlying measure is special discrete. Hence, the results above, with hypothesis 'μ is special discrete', are true for every interpolation operator $T \in \mathcal{B}_{1,\infty}$. In the next section we show that the results of this section are also true for every $T \in \mathcal{B}_{1,\infty}$ whenever the underlying measure is finite. For non-special discrete measures we do not know whether the results of this section remain valid. In particular, we do not know if there exists a measure and a $T \in \mathcal{B}_{1,\infty}$ such that T_p is not regular for some $p > 1$. If this can happen then theorem 3.2 implies the existence of a non-regular bounded operator of L_0^∞.

3. **Interpolation of regular operators and the finite measure case..**

In this section we show that all of $\mathcal{B}_{1,\infty}$ actually sits inside of $B^r(L^p)$ when the underlying space has finite measure. The main theorem of this section, Theorem 3.2, is a much more general result. It states that the interpolation of regular operators is again regular. The first lemma is a generalization of Lemma 1.3.

LEMMA 3.1. *If $T \in \mathcal{B}_{s,t}$, $1 \leq s \leq t \leq \infty$, has regular extensions $T_s \in B^r(L^s)$ and $T_t \in B^r(L^t)$ then T is regular. Furthermore, $|T|$ maps $L^s \cap L^t$ into $L^s \cap L^t$ and is continuous in both the s-norm and the t-norm and hence $|T| \in \mathcal{B}_{s,t}$.*

Proof. It suffices to show that T is regular as a map on $L^s \cap L^t$ and that its modulus satisfies $|T|(f) = |T_s|(f) = |T_t|(f)$, for all $f \in (L^s \cap L^t)_+$. Since $L^s \cap L^t$ is order complete, T will be regular if for each $f \in (L^s \cap L^t)_+$, the set $E_f \equiv \{|Tg| : g \in L^s \cap L^t, |g| \leqq f\}$ is bounded above in $L^s \cap L^t$. If we restrict our attention to measurable functions then $|g| \leqq f \in L^s \cap L^t$ automatically implies that $g \in L^s \cap L^t$. Fix $f \in (L^s \cap L^t)_+$. Since $E_f \subseteq M(\Omega)$ is bounded above by $|T_s|(f) \in M(\Omega)$, $h_f \equiv \sup(E_f)$ exists in $M(\Omega)$. Since $|T_s|(f) \in M(\Omega)$ is an upper bound, $h_f \leqq |T_s|(f) \in L^s$ and hence $h_f \in L^s$. The same argument shows that $h_f \in L^t$. This completes the proof that $T \in B(L^s \cap L^t)$ is regular. Since $h_f \in L^s \cap L^t$, $h_f = |T|(f)$ and therefore $|T|(f) \leqq |T_s|(f)$. Conversely,

$$|T_s|(f) \equiv \sup_{L^s}\{|T_s g| : g \in L^s, |g| \leqq f\} = \sup_{L^s}\{|Tg| : g \in L^s \cap L^t, |g| \leqq f\}$$
$$\leqq \sup_{L^s \cap L^t}\{|Tg| : g \in L^s \cap L^t, |g| \leqq f\} \equiv |T|(f).$$

Therefore, $|T|(f) = |T_s|(f)$. Similarly, $|T|(f) = |T_t|(f)$. Since $f \in (L^s \cap L^t)_+$ was arbitrary, the proof is complete. ∎

THEOREM 3.2. *If $T \in \mathcal{B}_{s,t}$, $1 \leq s \leq t \leq \infty$, has regular extensions $T_s \in B^r(L^s)$ and $T_t \in B^r(L^t)$ then T_p is regular for each $s \leqq p \leqq t$ and $|T|_p = |T_p|$.*

Proof. By Lemma 3.1, T is regular with modulus $|T| \in \mathcal{B}_{s,t}$. We prove the result when the underlying field is $\mathbb{R}$. The proof can be extended to the complexification in the usual way. Since T is regular, there exists positive operators T_1 and T_2 in $B(L^s \cap L^t)$ such that $T = T_1 - T_2$. For $f \in (L^s \cap L^t)_+$, $T_1 f \leqq |T|(f)$ and therefore

$$\|T_1 f\|_s \leqq \|\, |T|(f)\,\|_s \leqq \|\, |T|\, \| \cdot \|f\|_s.$$

Hence, T_1 is continuous in the s-norm on $(L^s \cap L^t)_+$ and hence on $L^s \cap L^t$. Similarly, T_1 is continuous in the t-norm on $L^s \cap L^t$. The same argument shows that T_2 has the same properties. This shows that T_1 and T_2 are in $\mathcal{B}_{s,t}$. Therefore they have extensions $(T_1)_p$ and $(T_2)_p$ in $B(L^p)$ for p, $s \leqq p \leqq t$. Since these extensions act positively on a dense subset of $(L^p)_+$ and are continuous in the p-norm, they are positive on L^p. Further, for $f \in L^s \cap L^t$, $(T_1)_p(f) - (T_2)_p(f) = T_1 f - T_2 f = Tf = T_p f$. By uniqueness of extensions, $(T_1)_p - (T_2)_p = T_p$ on L^p. In particular, T_p is the difference of two positive operators and so $T_p \in B^r(L^p)$. ∎

Now, if $T \in \mathcal{B}_{1,\infty}$then T has extensions to T_1 on L^1 and to T_∞ on L_0^∞. The operator T_1 is always regular. In addition, if μ is finite, then $L_0^\infty = L^\infty$ and hence T_∞ is also regular. By the last theorem, each induced operator T_p is regular and so we may view $\mathcal{B}_{1,\infty}$as a subalgebra of $B^r(L^p)$, $1 \leqq p \leqq \infty$, when μ is finite. The conclusions of Corollary 2.3, 2.4, 2.5 and 2.6(a) thus hold for all $T \in \mathcal{B}_{1,\infty}$(not just the ones associated to $k \in \mathcal{A}_{1,\infty}$) when μ is finite. In the case that $\mu(\Omega) = \infty$ we do not know if the same thing is true. It would be interesting to construct an operator (or to know that such an operator does not exist) which is bounded but not regular for some value of $1 < p < \infty$ which is 1-continuous and ∞-continuous.

4. **Application to integral operators on continuous function spaces..** The purpose of this section is to apply some of our results to a problem of Jörgens.

Assume now that our underlying measure space Ω is a locally compact and σ-compact topological space. The terminology of this section is all taken from [9], section 12, although some of the notation is changed. We begin with a few definitions.

We let $C(\Omega)$ denote the set of bounded, continuous functions $\Omega \longrightarrow \mathbb{C}$ with the supremum norm $\|\cdot\|_\infty$. A sequence $\{f_n\}_{n=1}^\infty \subseteq C(\Omega)$ is said to *converge locally* if it is bounded in $C(\Omega)$ and there exists an $f \in C(\Omega)$ such that $f_n(x) \longrightarrow f(x)$ for all $x \in \Omega$. This convergence will be denoted by $f_n \xrightarrow{\text{loc}} f$. An operator T mapping $C(\Omega)$ into itself is said to be *locally continuous* if $Tf_n \xrightarrow{\text{loc}} Tf$ whenever $f_n \xrightarrow{\text{loc}} f$ and *locally compact* if every bounded sequence $\{f_n\}_{n=1}^\infty \subseteq C(\Omega)$ contains a subsequence $\{f_{n_k}\}_{k=1}^\infty$ such that $\{Tf_{n_k}\}_{k=1}^\infty$ converges uniformly on compact sets.

Put $C_1 \equiv C(\Omega) \cap L^1(\Omega)$. Endowed with the norm $\|f\| \equiv \max\{\|f\|_\infty, \|f\|_1\}$ C_1 is a Banach space, and $\langle C(\Omega), C_1\rangle$ is a dual system with respect to the form

$$\langle f, g\rangle \equiv \int_\Omega f(x) g(x)\, d\mu(x), \quad f \in C(\Omega),\ g \in C_1.$$

Given a $T \in B(C(\Omega))$ it is not necessarily the case that T has an adjoint with respect to this form. That is, there may or may not exist an operator $T' \in B(C(\Omega))$ such that $T'(C_1) \subseteq C_1$ and $\langle Tf, g\rangle = \langle f, T'g\rangle$ for all $f \in C(\Omega)$, $g \in C_1$. We define $\mathcal{D} \equiv \mathcal{D}(\Omega, \mu)$ to be the set of all $T \in B(C(\Omega))$ such that $T(C_1) \subseteq C_1$ and there exists an adjoint $T' \in B(C(\Omega))$ of T with $T'(C_1) \subseteq C_1$. Endowed with the norm $|||\, T \,||| \equiv \max\{\, \|T\|, \, \|T'\|\,\}$ $\mathcal{D}$ is a Banach algebra (here $\|\cdot\|$ denotes the operator norm in $B(C(\Omega))$).

Put

$$\mathcal{L} \equiv \mathcal{L}(\Omega) \equiv \{\, T \in \mathcal{D} : \ T \text{ and } T' \text{ are locally compact}\,\}.$$

This is a closed subalgebra of $\mathcal{D}$ ([9], Theorem 12.5). Put $k_1(x) \equiv k(x, \cdot)$ and $k_2(x) \equiv k(\cdot, x)$. If $T \in \mathcal{L}$ then there exists a measurable function $k : \Omega \times \Omega \longrightarrow \mathbb{C}$ such that k_1 and k_2 are continuous and bounded from Ω into L^1 with

$$Tf(x) = \int_\Omega k(x, y) f(y)\, d\mu(y) \text{ and } T'f(x) = \int_\Omega k(y, x) f(y)\, d\mu(y)$$

for all $x \in \Omega$, $f \in C(\Omega)$ ([9], Theorem 12.5). The restriction that k_1 is bounded from Ω into L^1 means that

$$\operatorname{ess\,sup}_x \int_\Omega |k(x, y)|\, d\mu(y) < \infty.$$

The restriction on k_2 is the same, hence

$$\operatorname{ess\,sup}_x \int_\Omega |k(y, x)|\, d\mu(y) < \infty.$$

Therefore, if $T \in \mathcal{L}$ it follows that $T = \operatorname{Int}(k)$ for some kernel $k \in \mathcal{A} = \mathcal{A}_{1,\infty}$. We now have the following characterization of $\mathcal{L}$:

THEOREM 4.1. $\mathcal{L} = \{\, k \in \mathcal{A} : \ k_1$ and k_2 are continuous maps from Ω into $L^1\,\}$.

Proof. The discussion above shows that

$$\mathcal{L} \subseteq \{\, k \in \mathcal{A} : \ k_1 \text{ and } k_2 \text{ are continuous maps from } \Omega \text{ into } L^1\,\}.$$

The other inclusion is part of [9], Theorem 12.2. ∎

From this theorem, we see that each element of $\mathcal{L}$ may be thought of as a kernel $k \in \mathcal{A}$ with its associated operators $K_p \in B(L^p)$, $1 \le p \le \infty$. Jörgens gives several examples of kernels in $\mathcal{L}$ such that

$$\bigcup_{p \in [1,\infty]} \sigma(K_p) = \sigma_{\mathcal{D}}(k),$$

yet $\sigma(K_p)$ vary with p. It is always true that the inclusion $\bigcup_{p\in[1,\infty]} \sigma(K_p) \subseteq \sigma_{\mathcal{D}}(k)$ holds; Jörgens asks whether the equality always holds. At this stage, we know that $\sigma_{\mathcal{L}}(k)$ contains both $\sigma_{\mathcal{D}}(k)$ and $\sigma_{\mathcal{A}}(k)$ since $\mathcal{L}$ may be viewed as a subalgebra of either $\mathcal{D}$ or $\mathcal{A}$. In fact, all three of these spectra are equal. We prove that $\sigma_{\mathcal{L}}(k) = \sigma_{\mathcal{A}}(k)$ and then give a partial answer to Jörgens' question.

LEMMA 4.2. *There exists a right ideal $\mathcal{L}_r$ of $\mathcal{A}$ and a left ideal $\mathcal{L}_l$ of $\mathcal{A}$ such that $\mathcal{L} = \mathcal{L}_r \cap \mathcal{L}_l$. Hence, $\sigma_{\mathcal{L}}(k) = \sigma_{\mathcal{A}}(k)$ for each $k \in \mathcal{L}$.*

Proof. Consider the subspaces $\mathcal{L}_r \equiv \{k \in \mathcal{A} : k_1 \text{ is continuous}\}$ and $\mathcal{L}_l \equiv \{k \in \mathcal{A} : k_2 \text{ is continuous}\}$ of $\mathcal{A}$ Choose $k \in \mathcal{L}_r$, $j \in \mathcal{A}$, $\epsilon > 0$ and consider $x \in \Omega$. Then, since k_1 is continuous, there exists a neighborhood U of x such that

$$\int_\Omega |k(u,y) - k(x,y)|\ d\mu(y) = \|k_1(u) - k_1(x)\|_1 < \frac{\epsilon}{\|j\|}$$

whenever $u \in U$. Therefore, for $u \in U$,

$$\begin{aligned}\|(k*j)_1(u) - (k*j)_1(x)\|_1 &= \int_\Omega |(k*j)(u,y) - (k*j)(x,y)|\ d\mu(y) \\ &= \int_\Omega \Big| \int_\Omega [k(u,z)j(z,y) - k(x,z)j(z,y)]\ d\mu(z)\Big|\ d\mu(y) \\ &\leqq \int_\Omega |k(u,z) - k(x,z)| \Big[\int_\Omega |j(z,y)|\ d\mu(y)\Big]\ d\mu(z) \\ &\leqq \|j\| \cdot \int_\Omega |k(u,z) - k(x,z)|\ d\mu(z) < \epsilon.\end{aligned}$$

Therefore, $(k*j)_1$ is continuous and hence $k*j \in \mathcal{L}_r$. This shows that $\mathcal{L}_r$ is a right ideal of $\mathcal{A}$. Similarly, $\mathcal{L}_l$ is a left ideal of $\mathcal{A}$. We have now shown that $\mathcal{L}$ is the intersection of a right ideal and a left ideal. To complete the proof suppose that $\lambda \neq 0$ is not in $\sigma_{\mathcal{A}}(k)$. Then there exists $j \in \mathcal{A}$ such that

$$\frac{1}{\lambda}k + j - \frac{1}{\lambda}k*j = 0 = \frac{1}{\lambda}k + j - \frac{1}{\lambda}j*k.$$

Since $k \in \mathcal{L} \subseteq \mathcal{L}_r$, $j = \frac{1}{\lambda}k*j - \frac{1}{\lambda}k \in \mathcal{L}_r$. Since $k \in \mathcal{L} \subseteq \mathcal{L}_l$, $j = \frac{1}{\lambda}j*k - \frac{1}{\lambda}k \in \mathcal{L}_l$. Thus $j \in \mathcal{L}$ and hence λ is not contained in $\sigma_{\mathcal{L}}(k)$. This shows one inclusion; the other is obvious. ∎

Since $\mathcal{L}$ is a subalgebra of $\mathcal{D}$, $\sigma_{\mathcal{D}}(k) \subseteq \sigma_{\mathcal{L}}(k)$ holds for all $k \in \mathcal{L}$. It follows from ([9], 12.8) that

$$\bigcup_{p\in[1,\infty]} \sigma(K_p) \subseteq \sigma_{\mathcal{D}}(k) \subseteq \sigma_{\mathcal{L}}(k) = \sigma_{\mathcal{A}}(k), \quad k \in \mathcal{L}. \tag{4}$$

THEOREM 4.3. *Assume that $k \in \mathcal{L}$. Then*

$$\bigcup_{p\in[1,\infty]} \sigma(K_p) = \sigma_{\mathcal{D}}(k) = \sigma(K_1) \cup \sigma(K_\infty)$$

whenever one of the following conditions is satisfied:

(a) μ is finite, or

(b) μ is special discrete and $K_\infty(\ell^\infty(\mu)) \subseteq c_0(\mu)$.

Proof. Corollary 2.3 and (4). ∎

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THEOREM 4.4. *Assume that $k \in \mathcal{L}$. If $K_\infty(L^\infty) \subseteq L_0^\infty$ then*

$$[\sigma_{\mathcal{D}}(k) \cup \{0\}]\hat{} = [\sigma(K_1) \cup \sigma(K_\infty) \cup \{0\}]\hat{} = [\bigcup_{p \in [1,\infty]} \sigma(K_p) \cup \{0\}]\hat{}.$$

Proof. By Corollary 2.2 and (3) we have that

$$\begin{aligned}[\sigma_{\mathcal{A}}(k) \cup \{0\}]\hat{} &= [\sigma(K_1) \cup \sigma(K_\infty | L_0^\infty) \cup \{0\}]\hat{} \\ &= [\sigma(K_1) \cup \sigma(K_\infty) \cup \{0\}]\hat{} \subseteq [\bigcup_{p \in [1,\infty]} \sigma(K_p) \cup \{0\}]\hat{} \\ &\subseteq [\sigma_{\mathcal{A}}(k) \cup \{0\}]\hat{},\end{aligned}$$

and so all are equal. The result now follows from (4). ∎

The following is a direct consequence of Theorem 4.4.

COROLLARY 4.5. *Assume that $k \in \mathcal{L}$. If $K_\infty(L^\infty) \subseteq L_0^\infty$ and $\sigma(K_1) \cup \sigma(K_\infty)$ is polynomially convex then $\bigcup_{p \in [1,\infty]} \sigma(K_p) \cup \{0\} = \sigma_{\mathcal{D}}(k) \cup \{0\}$.*

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