INTEGRALS INVOLVING E-FUNCTIONS AND MODIFIED BESSEL FUNCTIONS OF THE SECOND KIND

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(Received 4th September, 1953)

§ 1. *Introductory.* The two following formulae are to be established.

If $R(m \pm n) > 0$, $|\operatorname{amp} z| < \pi$,

$$\int_0^\infty \lambda^{m-1} K_n(\lambda) E(p\,;\ a_r:\ q\,;\ \rho_s:\ z/\lambda)\, d\lambda$$

$$= 2^{\alpha_1+\alpha_2+\dots+\alpha_p-\rho_1-\dots-\rho_q+q-p+m-2}\pi^{\frac{1}{2}(q-p+1)}$$

$$\times \left[E\left\{ \begin{matrix} \frac{m+n}{2}, \frac{m-n}{2}, \frac{\alpha_1}{2}, \frac{\alpha_1+1}{2}, \dots, \frac{\alpha_p}{2}, \frac{\alpha_p+1}{2} : e^{\pm i\pi} 4^{q-p} z^2 \\ \frac{1}{2}, \frac{\rho_1}{2}, \frac{\rho_1+1}{2}, \dots, \frac{\rho_q}{2}, \frac{\rho_q+1}{2} \end{matrix} \right\} - \frac{2^{p-q}}{z} E\left\{ \begin{matrix} \frac{m+n+1}{2}, \frac{m-n+1}{2}, \frac{\alpha_1+1}{2}, \frac{\alpha_1+2}{2}, \dots, \frac{\alpha_p+1}{2}, \frac{\alpha_p+2}{2} : e^{\pm i\pi} 4^{q-p} z^2 \\ \frac{3}{2}, \frac{\rho_1+1}{2}, \frac{\rho_1+2}{2}, \dots, \frac{\rho_q+1}{2}, \frac{\rho_q+2}{2} \end{matrix} \right\} \right] \quad \dots\dots(1)$$

If $p \geqq q+1$, $R(k \pm n + 2\alpha_r) > 0$, $r = 1, 2, \dots, p$, $|\operatorname{amp} z| < \pi$,

$$\int_0^\infty K_n(\lambda) \lambda^{k-1} E(p\,;\ \alpha_r : q\,;\ \rho_s : \lambda^2 z)\, d\lambda$$

$$= 2^{k-2} \frac{\pi^2}{\sin\left(\frac{k+n}{2}\pi\right) \sin\left(\frac{k-n}{2}\pi\right)} E\left(p\,; \alpha_r : 1 - \frac{k+n}{2}, 1 - \frac{k-n}{2}, \rho_1, \dots, \rho_q : 4z\right)$$

$$+ \sum_{n,-n} \frac{\pi^2 2^{-n-2}}{\sin\left(\frac{k+n}{2}\pi\right) \sin(n\pi)} z^{-(k+n)/2} E\left(\begin{matrix} \alpha_1 + \frac{k+n}{2}, \dots, \alpha_p + \frac{k+n}{2} \\ 1 + \frac{k+n}{2}, n+1, \rho_1 + \frac{k+n}{2}, \dots, \rho_q + \frac{k+n}{2} : 4z \end{matrix} \right) \quad \dots\dots\dots\dots\dots(2)$$

For other values of p and q the formula holds if the integral is convergent.

In § 2 these formulae are proved; in § 3 integrals of products of Bessel Functions are evaluated by means of (2).

§ 2. *Proofs.* For the first formula, consider the integral

$$\int_0^\infty \lambda^{m-1} K_n(\lambda) E(:q\,;\ \rho_s : z/\lambda),$$

https://doi.org/10.1017/S2040618500033001 Published online by Cambridge University Press

where $R(m \pm n) > 0$. It can be written

$$\int_0^\infty \lambda^{m-1} K_n(\lambda) \frac{1}{\Gamma(\rho_1) \dots \Gamma(\rho_q)} F(\,;\ \rho_1, \dots, \rho_q;\ -\lambda/z)\, d\lambda$$

$$= \int_0^\infty \lambda^{m-1} K_n(\lambda)$$

$$\times \left[\begin{array}{l} \dfrac{1}{\Gamma(\rho_1) \dots \Gamma(\rho_q)} F\left(\,;\ \dfrac{1}{2}, \dfrac{\rho_1}{2}, \dfrac{\rho_1+1}{2}, \dots, \dfrac{\rho_q}{2}, \dfrac{\rho_q+1}{2};\ \lambda^2 z^{-2} 4^{-q-1}\right) \\ - \dfrac{1}{\Gamma(\rho_1+1) \dots \Gamma(\rho_q+1)} \dfrac{\lambda}{z} F\left(\,;\ \dfrac{3}{2}, \dfrac{\rho_1+1}{2}, \dfrac{\rho_1+2}{2}, \dots, \dfrac{\rho_q+1}{2}, \dfrac{\rho_q+2}{2};\ \lambda^2 z^{-2} 4^{-q-1}\right) \end{array} \right] d\lambda.$$

On expanding term by term and applying the formula (**1**)

$$\int_0^\infty \lambda^{m-1} K_n(\lambda)\, d\lambda = 2^{m-2} \Gamma\left(\frac{m+n}{2}\right) \Gamma\left(\frac{m-n}{2}\right), \quad \dots\dots(3)$$

where $R(m \pm n) > 0$, the value of the integral is found to be

$$2^{m-2} \Gamma\left(\frac{m+n}{2}\right) \Gamma\left(\frac{m-n}{2}\right) \frac{\pi^{\frac{1}{2}}}{\Gamma(\frac{1}{2}) \Gamma(\rho_1) \dots \Gamma(\rho_q)} F\left(\begin{array}{l} \frac{1}{2}m + \frac{1}{2}n,\ \frac{1}{2}m - \frac{1}{2}n;\ \frac{1}{z^2 4^q} \\ \frac{1}{2}, \frac{\rho_1}{2}, \dots, \frac{\rho_q+1}{2} \end{array} \right)$$

$$2^{m-1} \Gamma\left(\frac{m+n+1}{2}\right) \Gamma\left(\frac{m-n+1}{2}\right) \frac{\pi^{\frac{1}{2}}}{\Gamma(\frac{3}{2}) \Gamma(\rho_1+1) \dots \Gamma(\rho_q+1)} \left(\frac{1}{2z}\right)$$

$$\times F\left(\begin{array}{l} \frac{1}{2}m + \frac{1}{2}n + \frac{1}{2},\ \frac{1}{2}m - \frac{1}{2}n + \frac{1}{2};\ \frac{1}{z^2 4^q} \\ \frac{3}{2}, \frac{\rho_1+1}{2}, \dots, \frac{\rho_q+2}{2} \end{array} \right)$$

$$= 2^{m-2-\rho_1-\dots-\rho_q+q} \pi^{\frac{1}{2}(q+1)} \left\{ \begin{array}{l} E\left(\dfrac{m+n}{2}, \dfrac{m-n}{2};\ \dfrac{1}{2}, \dfrac{\rho_1}{2}, \dots, \dfrac{\rho_q+1}{2} : e^{\pm i\pi} 4^q z^2\right) \\ -\dfrac{2^{-q}}{z} E\left(\dfrac{m+n+1}{2}, \dfrac{m-n+1}{2} : \dfrac{3}{2}, \dfrac{\rho_1+1}{2}, \dots, \dfrac{\rho_q+2}{2} : e^{\pm i\pi} 4^q z^2\right) \end{array} \right\}.$$

On making repeated applications of the formula, (**2**),

$$\int_0^\infty e^{-\lambda} \lambda^{k-1} E(p;\ \alpha_r : q;\ \rho_s : z/\lambda^m)\, d\lambda$$

$$= m^{k-\frac{1}{2}} (2\pi)^{\frac{1}{2}-\frac{1}{2}m} E(p+m;\ \alpha_r : q;\ \rho_s : z/m^m), \quad \dots\dots(4)$$

where m is a positive integer (1 and 2 in this case), $R(k) > 0$, $\alpha_{p+\nu+1} = (k+\nu)/m$, $\nu = 0, 1, \dots, m-1$, formula (1) is obtained.

Note. The method can be employed to express the integral as the sum of any number of E-functions.

In the proof of (2) the following formulae are required.

(**2**) If m is a positive integer and if $R(k \pm n) > 0$,

$$\int_0^\infty K_n(\lambda) \lambda^{k-1} E(p;\ \alpha_r : q;\ \rho_s : z/\lambda^{2m})\, d\lambda$$

$$= (2\pi)^{1-m} 2^{k-2} m^{k-1} E\{p+2m;\ \alpha_r : q;\ \rho_s : z/(2m)^{2m}\}, \quad \dots\dots(5)$$

where $\alpha_{p+\nu+1} = (k+n+2\nu)/(2m)$, $\alpha_{p+m+\nu+1} = (k-n+2\nu)/(2m)$, $\nu = 0, 1, 2, \dots, m-1$.

(3) If $p \geqq q+1$,

$$E(p\,;\ \alpha_r : q\,;\ \rho_s : z) = \sum_{r=1}^{p} \prod_{s=1}^{p}{}' \, \Gamma(\alpha_s - \alpha_r)\{\prod_{t=1}^{q} \Gamma(\rho_t - \alpha_r)\}^{-1}\Gamma(\alpha_r)$$

$$\times z^{\alpha_r} F\left\{\begin{matrix} q+1\,;\ \alpha_r, \alpha_r - \rho_1 + 1, \ldots, \alpha_r - \rho_q + 1 : (-1)^{p-q}z \\ p-1\,;\ \alpha_r - \alpha_1 + 1, \ldots * \ldots, \alpha_r - \alpha_p + 1 \end{matrix}\right\}, \quad \text{........(6a)}$$

and if $p \leqq q$,

$$E(p\,;\ \alpha_r : q\,;\ \rho_s : z) = \frac{\Gamma(\alpha_1) \ldots \Gamma(\alpha_p)}{\Gamma(\rho_1) \ldots \Gamma(\rho_q)} F\left(p\,;\ \alpha_r : q\,;\ \rho_s : -\frac{1}{z}\right). \quad \text{...........(6b)}$$

When $p=1$, $q=0$, the integral in (2) becomes

$$\int_0^\infty K_n(\lambda)\lambda^{k-1} E(\alpha_1 :: \lambda^2 z)\, d\lambda$$

$$= z^{\alpha_1} \int_0^\infty K_n(\lambda)\lambda^{k+2\alpha_1-1} E\{\alpha_1 :: 1/(z\lambda^2)\}\, d\lambda$$

$$= 2^{k+2\alpha_1-2} z^{\alpha_1} E\left(\alpha_1, \alpha_1 + \frac{k+n}{2}, \alpha_1 + \frac{k-n}{2} :: \frac{1}{4z}\right), \quad \text{by (5) with } m=1,$$

$$= 2^{k-2}\,\Gamma\left(\frac{k+n}{2}\right)\Gamma\left(\frac{k-n}{2}\right)\Gamma(\alpha_1)\, F\left(\alpha_1\,;\ 1-\frac{k+n}{2}, 1-\frac{k-n}{2}\,;\ -\frac{1}{4z}\right)$$

$$+ \sum_{n,-n} 2^{-n-2}\,\Gamma\left(\frac{-k-n}{2}\right)\Gamma(-n)\,\Gamma\left(\alpha_1 + \frac{k+n}{2}\right) z^{-\frac{1}{2}k-\frac{1}{2}n} F\left(\alpha_1 + \frac{k+n}{2}\,;\ 1+\frac{k+n}{2}, n+1\,;\ -\frac{1}{4z}\right),$$

by (6a),

$$= 2^{k-2} \frac{\pi^2}{\sin\left(\frac{k+n}{2}\pi\right)\sin\left(\frac{k-n}{2}\pi\right)} E\left(\alpha_1 : 1-\frac{k+n}{2}, 1-\frac{k-n}{2} : 4z\right)$$

$$+ \sum_{n,-n} \frac{\pi^2 2^{-n-2}}{\sin\left(\frac{k+n}{2}\pi\right)\sin(n\pi)} z^{-\frac{1}{2}k-\frac{1}{2}n} E\left(\alpha_1 + \frac{k+n}{2} : 1+\frac{k+n}{2}, n+1 : 4z\right), \quad \text{by (6b).}$$

From this (2) can be derived in the usual way.

§ 3. *Some Bessel Function Integrals.* In (2) take $p=0$, $q=1$, replace z by $4/z^2$, n by m and put $\rho_1 = n+1$; then, on applying the formula

$$E(\,: n+1 : 4\lambda^2/z^2) = (2\lambda/z)^n J_n(z/\lambda), \quad \text{...............................(7)}$$

it is found that, if z is real and positive, $R(k+n\pm m) > -\frac{3}{2}$,

$$(2/z)^n \int_0^\infty K_m(\lambda)\lambda^{k+n-1} J_n(z/\lambda)\, d\lambda$$

$$= \frac{2^{k-2}\pi^2}{\sin\left(\frac{k+m}{2}\pi\right)\sin\left(\frac{k-m}{2}\pi\right)} E\left(\ : 1-\frac{k+m}{2}, 1-\frac{k-m}{2}, n+1 : 16/z^2\right)$$

$$+ \sum_{m,-m} \frac{2^{-m-2}\pi^2}{\sin\left(\frac{k+m}{2}\pi\right)\sin(m\pi)} \left(\frac{z}{2}\right)^{k+m} E\left(\ : 1+\frac{k+m}{2}, m+1, 1+n+\frac{k+m}{2} : 16/z^2\right).$$

Here replace k by $k-n$; then, if z is real and positive, $R(k\pm m)>-\frac{3}{2}$,

$$\int_0^\infty \lambda^{k-1}K_m(\lambda)\,J_n(z/\lambda)\,d\lambda$$

$$=\frac{\Gamma\left(\frac{k+m-n}{2}\right)\Gamma\left(\frac{k-m-n}{2}\right)}{2^{2n-k+2}\,\Gamma(n+1)}\,z^n\,F\left(\;;\;1-\frac{k+m-n}{2},\,1-\frac{k-m-n}{2},\,n+1;\;-\frac{z^2}{16}\right)$$

$$+\sum_{m,-m}\frac{\Gamma\left(\frac{-k-m+n}{2}\right)\Gamma(-m)}{\Gamma\left(1+\frac{k+m+n}{2}\right)}\,\frac{z^{k+m}}{2^{k+2m+2}}\,F\left(\;;\;1+\frac{k+m-n}{2},\,1+\frac{k+m+n}{2},\,m+1;\;-\frac{z^2}{16}\right).\ (8)$$

On applying the formula

$$G_n(z)=\frac{\pi}{2\sin n\pi}\{J_{-n}(z)-e^{-in\pi}J_n(z)\},\ \ldots\ldots\ldots\ldots(9)$$

it follows that, if $0\leqq\operatorname{amp} z\leqq\pi$, $R(k\pm m)>-\frac{3}{2}$,

$$i^n\int_0^\infty \lambda^{k-1}\,K_m(\lambda)\,G_n(z/\lambda)\,d\lambda$$

$$=\sum_{n,-n}2^{k+2n-3}\,\Gamma\left(\frac{k+m+n}{2}\right)\Gamma\left(\frac{k-m+n}{2}\right)\Gamma(n)\left(\frac{i}{z}\right)^n$$

$$\times F\left(\;;\;1-\frac{k+m+n}{2},\,1-\frac{k-m+n}{2},\,1-n;\;-\frac{z^2}{16}\right)$$

$$+\sum_{m,-m}\frac{\pi}{2\sin n\pi}\,\Gamma\left(\frac{-k-m-n}{2}\right)\Gamma\left(\frac{-k-m+n}{2}\right)\Gamma(-m)\,\frac{z^{k+m}}{2^{k+2m+2}}$$

$$\times F\left(\;;\;1+\frac{k+m+n}{2},\,1+\frac{k+m-n}{2},\,m+1;\;-\frac{z^2}{16}\right)$$

$$\times\frac{1}{\pi}\left[-\sin\left(\frac{k+m-n}{2}\,\pi\right)e^{in\pi/2}+\sin\left(\frac{k+m+n}{2}\,\pi\right)e^{-in\pi/2}\right].$$

Now the expression in the bracket is equal to

$$\sin n\pi\, i^{-k-m}.$$

Hence, on replacing z by iz, and noting that

$$G_n(iz)=i^{-n}K_n(z),\ \ldots\ldots\ldots\ldots(10)$$

the formula becomes **(4)**

$$\int_0^\infty \lambda^{k-1}K_m(\lambda)\,K_n(z/\lambda)\,d\lambda$$

$$=\sum_{n,-n}2^{k+2n-3}\,\Gamma\left(\frac{k+m+n}{2}\right)\Gamma\left(\frac{k-m+n}{2}\right)\Gamma(n)\,z^{-n}$$

$$\times F\left(\;;\;1-\frac{k+m+n}{2},\,1-\frac{k-m+n}{2},\,1-n;\;\frac{z^2}{16}\right)$$

$$+\sum_{m,-m}2^{-k-2m-3}\,\Gamma\left(\frac{-k-m-n}{2}\right)\Gamma\left(\frac{-k-m+n}{2}\right)\Gamma(-m)\,z^{k+m}$$

$$\times F\left(\;;\;1+\frac{k+m+n}{2},\,1+\frac{k+m-n}{2},\,m+1;\;\frac{z^2}{16}\right),\ \ldots\ldots\ldots\ldots(11)$$

where $R(z)>0$.

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