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RESEARCH ARTICLE

A note on almost Yamabe solitons

Wagner Oliveira Costa-Filho

Campus Arapiraca, Federal University of Alagoas, CEP 57309-005, Arapiraca, Alagoas, Brazil
Corresponding author: Wagner Oliveira Costa-Filho; Email: wagner.filho@arapiraca.ufal.br

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Abstract
In this paper, we present a sufficient condition for almost Yamabe solitons to have constant scalar curvature. Additionally, under some geometric scenarios, we provide some triviality and rigidity results for these structures.

1. Introduction

The celebrated Yamabe flow was introduced by R. Hamilton as an approach to solve the Yamabe problem. The importance of Yamabe flow lies in the fact that it is a geometric deformation of metrics into those with constant scalar curvature (see Brendle in [3] and references therein). The so-called Yamabe solitons are fixed points of the Yamabe flow. Furthermore, Yamabe solitons arise from the blowup procedure along the Yamabe flow.

In the last few years, the geometry of Yamabe-type solitons of an appropriate geometric flow has been extensively studied. More generally, under appropriate geometric and analytic conditions, the classification of these solitons has become a topic of growing interest. It is also worth mentioning that there exists a relationship between Yamabe solitons and Ricci–Bourguignon solitons, as discussed by Mi in [13], for instance.

An n-dimensional Riemannian manifold (M^n, g), $n \geq 2$, is an *almost Yamabe soliton* if it admits a complete vector field X and a smooth function $\lambda : M \to \mathbb{R}$ satisfying the equation

$$\frac{1}{2}\mathcal{L}_X g = (S - \lambda)g. \tag{1.1}$$

Here, $\mathcal{L}_X g$ represents the Lie derivative of the metric g in the direction of X, and S denotes the scalar curvature of g. If λ is a constant, the almost Yamabe soliton is referred to as a *Yamabe soliton*.

The concept of the almost Yamabe soliton was introduced by Barbosa and Ribeiro in [1] as a natural generalization of the self-similar conformal solutions of Hamilton's Yamabe flow. For more details on this subject, including some important examples of almost Yamabe solitons and interesting characterization results, we recommend the reader to consult [1].

An almost Yamabe soliton is called a *gradient almost Yamabe soliton* if $X = \nabla f$ is the gradient of a smooth real function f on M. The function f is referred to as a *potential function*. In this case, equation (1.1) becomes

$$\nabla^2 f = (S - \lambda)g,$$

where $\nabla^2 f$ denotes the Hessian of f. We trace the above equation to obtain

$$\Delta f = n(S - \lambda), \tag{1.2}$$

where Δ is the Laplace–Beltrami operator with respect to the metric tensor g.

An almost Yamabe soliton will be denoted by (M^n, g, X, λ). We would like to point out that in this case, X is a conformal vector field. When either the vector field X is trivial or the potential f is constant, an almost Yamabe soliton will be called trivial. Otherwise, it will be called nontrivial.

One of the interesting problems in this subject is to find conditions on a complete almost Yamabe soliton so that its metric has constant scalar curvature. It is worth noticing that the metric of any compact Yamabe soliton has constant scalar curvature (see Hsu in [10] and also Chow et al. in [5, Appendix B]). In [6], Cunha demonstrates that, under certain hypotheses, a complete Yamabe soliton with nonpositive Ricci curvature has constant scalar curvature. In [2], Turki, Chen, and Deshmukh find several sufficient constraints on the soliton vector fields of Yamabe solitons such that their metric has constant scalar curvature. More recently, Maeta [12] provided triviality results for generalized Yamabe solitons.

In what follows, we recall that a smooth vector field X on (M^n, g) is said to be a *closed conformal vector field* if there exists a smooth function $h : M \to \mathbb{R}$ such that

$$\bar{\nabla}_Y X = hY$$

for every smooth vector field Y, where $\bar{\nabla}$ denotes the Levi–Civita connection of M with respect to the metric $g = \langle , \rangle$. Closed conformal vector fields are also known as concircular vector fields in the mathematical literature. Riemannian and pseudo-Riemannian manifolds admitting closed conformal vector fields were intensively studied during the last decades (cf. Deshmukh in [8]). Moreover, they have significant applications in the geometric study of submanifolds and general relativity (see Chen in [4]).

In this setting, inspired by the results established in [2], we have the following result, which shows that the scalar curvature is globally constant if it is constant along the integral curves of the conformal vector field under a geometrical condition concerning the angle between the gradients of S and the soliton function. Namely,

Theorem 1.1. *Let (M^n, g, X, λ) be an n-dimensional connected almost Yamabe soliton with a nontrivial closed conformal vector field X and scalar curvature S satisfying*

$$\langle \nabla S, \nabla \lambda \rangle \leq 0,$$

on M. If S is a constant along the integral curves of X, then M has constant scalar curvature.

As a direct consequence of the proof, we have the following corollaries:

Corollary 1.2. *Under the same assumptions as in Theorem 1.1, if $S - \lambda$ is constant along the integral curves of X, then M has constant scalar curvature.*

Corollary 1.3. *Under the same assumptions as in Theorem 1.1, if λ is constant along the integral curves of X, then M is a Yamabe soliton.*

The proof of Theorem 1.1 is presented in Section 3. In Section 2, dedicated to general aspects of almost Yamabe solitons, we also provide some results concerning triviality and rigidity. It is worth noting that some of these findings have already been explored by Maeta in [12]. Specifically, Proposition 2.2 is demonstrated in the proof of Proposition 2.1 in [12], and the second case of Theorem 2.4 corresponds to Theorem 1.5 (B) in [12].

2. Preliminaries results

In this section, we establish some properties regarding almost Yamabe solitons. All manifolds (M^n, g) are assumed to be connected and oriented. We begin stating the following fundamental equations (for a proof see [1], page 83).

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Lemma 2.1. *Let* $(M^n, g, \nabla f, \lambda)$ *be a gradient almost Yamabe soliton. Then,*

(a) $(n-1)(\nabla S - \nabla\lambda) + \mathrm{Ric}(\nabla f) = 0.$
(b) $(n-1)\Delta(S-\lambda) + \frac{1}{2}\langle\nabla S, \nabla f\rangle + S(S-\lambda) = 0.$

In this sequel, concerning a gradient almost Yamabe soliton, we prove the following result.

Proposition 2.2. *Let* $(M^n, g, \nabla f, \lambda)$ *be a closed gradient almost Yamabe soliton. Then,*

$$\int_M (\Delta f)^2\, dM = \frac{n}{n-1}\int_M \mathrm{Ric}(\nabla f, \nabla f)\, dM.$$

Proof. By using (1.2) and the divergence theorem, we have that

$$\int_M (\Delta f)^2\, dM = n\int_M (S-\lambda)\Delta f\, dM = n\int_M \langle\nabla\lambda - \nabla S, \nabla f\rangle\, dM.$$

Taking into account the first assertion of Lemma 2.1, we get

$$(n-1)\langle\nabla S - \nabla\lambda, \nabla f\rangle = -\mathrm{Ric}(\nabla f, \nabla f).$$

Thus,

$$\int_M (\Delta f)^2\, dM = n\int_M \langle\nabla\lambda - \nabla S, \nabla f\rangle\, dM = \frac{n}{n-1}\int_M \mathrm{Ric}(\nabla f, \nabla f)\, dM.$$

□

As an immediate consequence of this proposition, we have that a closed Riemannian manifold with nonpositive Ricci curvature cannot be a nontrivial gradient almost Yamabe soliton. Furthermore, applying the second assertion in Lemma 2.1, we obtain the following fact.

Proposition 2.3. *Let* $(M^n, g, \nabla f, \lambda)$*,* $n \geq 3$*, be a closed gradient almost Yamabe soliton. If the potential function* f *is the scalar curvature* S *of* g*, then the soliton is trivial.*

Proof. Using the hypothesis in conjunction with equation (1.2), we can write

$$(n-1)\Delta(S-\lambda) + \frac{1}{2}|\nabla S|^2 + \frac{1}{n}S\Delta S = 0.$$

So, integrating by parts the above condition, we have

$$\left(\frac{1}{2} - \frac{1}{n}\right)\int_M |\nabla S|^2\, dM = 0.$$

Thus, S is constant on M, completing the proof. □

Another proof of this proposition can also be obtained from Kazdan–Warner's equality (see Proposition 7 in [11]).

Now, let us highlight the following identity, which can be regarded as the almost Yamabe soliton version of Lemma 2.3 in [9]. Consider a gradient almost Yamabe soliton $(M^n, g, \nabla f, \lambda)$. Using the classical Bochner formula, we have

$$\begin{aligned}\frac{1}{2}\Delta|\nabla f|^2 &= |\nabla^2 f|^2 + \langle\nabla(\Delta f), \nabla f\rangle + \mathrm{Ric}(\nabla f, \nabla f)\\ &= |\nabla^2 f|^2 + n\langle\nabla(S-\lambda), \nabla f\rangle + \mathrm{Ric}(\nabla f, \nabla f)\\ &= |\nabla^2 f|^2 - \frac{1}{n-1}\mathrm{Ric}(\nabla f, \nabla f).\end{aligned} \tag{2.1}$$

The next result characterizes triviality for a gradient almost Yamabe soliton under certain integrality assumptions, consonant with a recent work due to Cunha in [6, Theorem 2]. At this point, we recall that a Riemannian manifold M is said to be *parabolic* (with respect to the Laplacian operator) if the constant functions are the only subharmonic functions on M which are bounded from above.

Theorem 2.4. *Let $(M^n, g, \nabla f, \lambda)$ be a nontrivial complete gradient almost Yamabe soliton satisfying* $\mathrm{Ric}(\nabla f, \nabla f) \leq 0$. *If either M is parabolic and $|\nabla f| \in L^{\infty}(M)$ or $|\nabla f| \in L^p(M)$ with $p > 1$, then $S = \lambda$.*

Considering equality (2.1), the proof follows the same steps as those in Theorem 2 of [6] and will be omitted.

In the context of an almost Yamabe soliton M^n isometrically immersed into a space form $N^{n+1}(c)$ of constant sectional curvature c and based on paper by Cunha and de Lima [7], we present the following result. Here, H is the mean curvature of M^n associated with second fundamental form A of the hypersurface.

Theorem 2.5. *Let $x : (M^n, g, X, \lambda) \to N^{n+1}(c)$ be a closed almost Yamabe soliton immersed into a space form of curvature c. If the function λ satisfies $\lambda \geq n(n-1)c + n^2H^2$, then M is isometric to a round sphere.*

Proof. Observe that the (non-normalized) scalar curvature S is expressed via the Gauss equation by

$$S = n(n-1)c + n^2H^2 - |A|^2,$$

in terms of the squared norm of the shape operator of M^n.

On the other hand, it follows from (1.1) that

$$\mathrm{div}(X) = n(S - \lambda).$$

Therefore, from our hypothesis, we obtain $\mathrm{div}(X) \leq -n|A|^2$. Hence, by divergence theorem one gets that M^n is totally geodesic and $S = \lambda = n(n-1)c$. Moreover, under compactness assumption, the ambient space is (necessarily) a sphere and M^n is isometric to a Euclidean sphere $\mathbb{S}^n$, as desired. $\square$

With respect to the above theorem, by supposing that M^n is complete and noncompact, if $|X| \in L^1(M^n)$ has integrable norm on M^n, then reasoning as in the proof of the previous result, we may use Theorem 1.3 of [1] to conclude that X is a Killing vector field.

3. Proof of the main result

As before, let (M^n, g) be a Riemannian manifold and let $|\cdot|$ denote the corresponding norm of g. In general, if X is a closed conformal vector field on M with conformal potential function h, given that the Hessian of $|X|^2$ is a symmetric 2-tensor, we deduce that:

$$Y(h)\langle X, Z\rangle = Z(h)\langle X, Y\rangle,$$

for all smooth vector fields Y, Z on M. It is also important to recall that the set of the zeros of X is a discrete set (see Ros and Urbano [14], page 207).

We now turn our attention to the case of examining an almost Yamabe soliton, as outlined in Theorem 1.1, and proceed to establish our central theorem.

Proof. Initially, it is easy to check that

$$\bar{\nabla}_Y X = (R - \lambda)Y.$$

Thus,

$$Y(S - \lambda)\langle X, Z\rangle = Z(S - \lambda)\langle X, Y\rangle.$$

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Using a local orthonormal frame $\{e_1, \ldots, e_n\}$ for TM and replacing Z by e_i, $i = 1, \ldots, n$, the above identity gives

$$Y(S - \lambda)X = \langle X, Y\rangle \nabla(S - \lambda).$$

Hence, taking the inner product with ∇S and inserting $Y = X$ in the last expression, we have

$$|X|^2(|\nabla S|^2 - \langle \nabla S, \nabla\lambda\rangle) = X(S - \lambda)X(S).$$

As the scalar curvature S is a constant along the integral curves of X, it follows that $X(S) = 0$. So, we obtain that

$$|X|^2|\nabla S|^2 = |X|^2\langle \nabla S, \nabla\lambda\rangle.$$

Therefore, by hypothesis, we arrive at $|X|^2|\nabla S|^2 = 0$ on M. Finally, since X is nontrivial, this condition implies that S is a constant.

This completes the proof of Theorem 1.1. □

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