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Generic stabilizers for simple algebraic groups acting on orthogonal and symplectic Grassmannians

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Abstract

We consider faithful actions of simple algebraic groups on self-dual irreducible modules and on the associated varieties of totally singular subspaces, under the assumption that the dimension of the group is at least as large as the dimension of the variety. We prove that in all but a finite list of cases, there is a dense open subset where the stabilizer of any point is conjugate to a fixed subgroup, called the generic stabilizer. We use these results to determine whether there exists a dense orbit. This in turn lets us complete the answer to the problem of determining all pairs of maximal connected subgroups of a classical group with a dense double coset.

Contents

1. Introduction and statement of results

Let G be a simple algebraic group over an algebraically closed field K of characteristic p, where we take $p = \infty$ if K has characteristic zero. Let V be a nontrivial irreducible KG-module of dimension d. For $1 \leq k \leq d$, the Grassmannian variety $\mathcal{G}_k(V)$ consists of all k-dimensional subspaces of V and is

isomorphic to the variety $SL(V)/P$, where P is a maximal parabolic subgroup of $SL(V)$ stabilising a k-dimensional subspace. Assume that the module V is self-dual. Then the group G preserves a nondegenerate bilinear form, which is either symmetric or alternating (unless $p = 2$ when it is both). If the form is symmetric and G preserves an associated quadratic form, we say that the module V is *orthogonal*, and we say it is *symplectic* otherwise. For $1 \le k \le \frac{d}{2}$, we denote by $\mathcal{S}_k(V)$ the variety of totally singular k-dimensional subspaces of V. Any such variety is irreducible unless $k = \frac{d}{2}$ and V is orthogonal, in which case the two $SO(V)$-orbits on $\mathcal{S}_k(V)$ are its irreducible components, which we shall denote by $\mathcal{S}_{k'}(V)$ and $\mathcal{S}_{k''}(V)$, or $\mathcal{S}'_k(V)$ and $\mathcal{S}''_k(V)$. Note that there is no intrinsic way of choosing which of the two orbits is labeled $\mathcal{S}_{k'}(V)$, and therefore, such choice is arbitrary and usually simply dependent on the order of consideration. As V is orthogonal or symplectic, each such irreducible variety is isomorphic to $SO(V)/P$ or $Sp(V)/P$, where P is a parabolic subgroup of $SO(V)$ or $Sp(V)$ (maximal unless V is orthogonal and $k = \frac{d}{2} - 1$), and the elements of the variety are *orthogonal Grassmannians* or *symplectic Grassmannians*.

If G acts faithfully on a variety X, we say that the action has *generic stabilizer S* if there exists a nonempty open subset $U \subseteq X$ such that the stabilizer G_u is conjugate to S for all $u \in U$.

We say that the action has *semi-generic stabilizer S* if there exists a nonempty open subset $U \subseteq X$ such that G_u is isomorphic to S for all $u \in U$.

In general, we say that G acting on X (not necessarily faithfully) has a (semi-)generic stabilizer if G/G_X, where G_X denotes the kernel of the action, has a (semi-)generic stabilizer for its faithful action on X. Note that when $X = \mathcal{G}_k(V)$ or $X = \mathcal{S}_k(V)$, the kernel G_X is precisely the center of G.

In characteristic zero, generic stabilizers exist under mild hypotheses. In this case, it follows from [1] that the generic stabilizer is positive dimensional if and only if $\dim G \ge \dim X$. However, even in characteristic 0, there are examples of actions with no generic stabilizers (see [25]).

If $p < \infty$, there are even more instances where generic stabilizers do not exist, as there is no analogue of Richardson's result. For example, in [21, Example 8.3], we find a construction for an $SL_2(K)$-action on an affine variety in positive characteristic, with no generic stabilizer.

Again, let G be a simple algebraic group over an algebraically closed field K of characteristic p and V be a nontrivial irreducible KG-module. Recently, Guralnick and Lawther have solved the generic stabilizer problem for the action on $X = \mathcal{G}_k(V)$. In [10], they proved that if $X = V$ or $X = \mathcal{G}_k(V)$, then the action of G on X has a generic stabilizer, unless $G = B_3$, $p = 2$, V is the spin module for G, and $k = 4$, in which case the action has a semi-generic stabilizer but not a generic stabilizer. They showed that for such actions, the generic stabilizer is in general trivial, and they otherwise determined all nontrivial (semi-)generic stabilizers explicitly.

In this paper, we treat the action of G on the orthogonal and symplectic Grassmannians of self-dual irreducible G-modules (i.e., $X = \mathcal{S}_k(V)$). We only deal with the case $\dim G \ge \dim X$. The reason for this is two-fold. We will be interested in applications to questions about the existence of dense orbits and dense double cosets, for which we only need to be concerned with the cases where $\dim G \ge \dim X$. Secondly, as shown in [10], the strategy for dealing with the situation $\dim G < \dim X$ presents entirely different challenges. The case $\dim G < \dim X$ shall be the subject of future work. Note that (semi-)generic stabilizers for the action on $\mathcal{S}_k(V)$ are generally going to be radically different from the ones for the action on $\mathcal{G}_k(V)$. Indeed, it is often going to be the case that the generic stabilizer for the action on $\mathcal{G}_k(V)$ is finite, while the generic stabilizer for the action on $\mathcal{S}_k(V)$ is positive-dimensional. We shall now state our first result. The modules V are denoted by their highest weight, the groups G by their Dynkin diagram.

Theorem 1. *Let G be a simple algebraic group over an algebraically closed field of characteristic p, and V a self-dual nontrivial irreducible G-module of dimension d and highest weight λ. For $1 \le k \le \frac{d}{2}$ such that $\dim G \ge \dim \mathcal{S}_k(V)$, if the action of G on $\mathcal{S}_k(V)$ has no generic stabilizer, then $p = 2$ and (G, λ, k) is one of the following:*

(i) $(E_7, \lambda_7, 2)$;
(ii) $(D_6, \lambda_6, 2)$;

(iii) $(A_5, \lambda_3, 2)$;
(iv) $(B_4, \lambda_4, 8)$.

In the first three cases, the action of G on $\mathcal{S}_k(V)$ has no generic stabilizer but does have a semi-generic one. In the last case, the action of G on $\mathcal{S}'_k(V)$ has a generic stabilizer, but the action of G on $\mathcal{S}''_k(V)$ only has a semi-generic one.

While Theorem 1 guarantees the existence, in Theorem 2, we shall explicitly determine the structure of every (semi-)generic stabilizer. The proof of Theorem 1 involves a quick reduction to a finite list of families of cases to be considered, followed by a case-by-case analysis using many of the methods adopted in [10]. This case-by-case analysis is the subject of the majority of this paper.

Before stating our remaining results, we shall set up some more notation. If G is semisimple, let T be a fixed maximal torus, and Φ the root system for G with respect to T, described by its Dynkin diagram. The root system has positive roots Φ^+ and a base $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ of simple roots. For the simple algebraic groups, the ordering of the simple roots is taken according to Bourbaki [4]. For a KG-module V and a weight μ of G, we write V_μ for the μ-weight space of V. If G acts on a set X, for $x \in X$, we denote by G_x the stabilizer in G of x. We use P to denote a parabolic subgroup containing a Borel subgroup $B \geq T$ and P_k to denote the maximal parabolic subgroup obtained by deleting the k-th node of the Dynkin diagram for G. Similarly, we use $P_{i,j}$ to denote the parabolic subgroup obtained by deleting the i-th and j-th nodes of the Dynkin diagram for G. We use T_i to denote an i-dimensional torus, $Sym(n)$ and $Alt(n)$ to denote the symmetric group and the alternating group on a set of size n, and $Dih(2n)$ for a dihedral group of order $2n$. Throughout the paper, we work modulo field twists and exceptional isogenies. So, for example, we only treat one of C_n or B_n in characteristic 2, and we always assume that the highest weight λ is not a multiple of p.

We now set up further notation to better encapsulate the exact setup we will adopt. As in [10], we define a *quadruple* to be a 4-tuple of the form (G, λ, p, k) with the following properties:

(i) G is a simple algebraic group over an algebraically closed field of characteristic p;
(ii) $V = V_G(\lambda)$ is an irreducible G-module;
(iii) $1 \leq k \leq \frac{\dim V}{2}$.

We say that a quadruple (G, λ, p, k) is *small* if

$$\dim G \geq \dim \mathcal{G}_k(V).$$

We say that a quadruple (G, λ, p, k) is self-dual if $V_G(\lambda)$ is self-dual, in which case the quadruple is *ts-small* if

$$\dim G \geq \dim \mathcal{S}_k(V).$$

We say that the quadruple (G, λ, p, k) has a (semi-)generic stabilizer if the action of G on $X = \mathcal{G}_k(V)$ has a (semi-)generic stabilizer. For a self-dual quadruple (G, λ, p, k), we say it has a *(semi-)generic ts-stabilizer* if the action of G on $X = \mathcal{S}_k(V)$ has a (semi-)generic stabilizer. In this paper, we classify (semi-)generic ts-stabilizers of ts-small quadruples.

Our main result will be given in a single table (Table 1). In the first column, we have the type of our simple algebraic group G; in the second column, the highest weight of the irreducible G-module V; in the third column, we list the rank ℓ of G; in the fourth column, we have conditions on p; and in the fifth column, we have the particular k which specifies which variety $\mathcal{S}_k(V)$ we are acting on. We then have a column listing the (semi-)generic stabilizer for the action, denoted by $C_{\mathcal{S}_k(V)}$ in which we use an asterisk to indicate whether the stabilizer is not generic but semi-generic. In the columns 'Orth?' and 'Dense?', we indicate whether the module V is orthogonal and whether there exists a dense orbit for the action on $\mathcal{S}_k(V)$. In the last column, we give the number of the proposition within the paper where the relevant information is obtained.

Table 1. *Generic ts-stabilizers for ts-small quadruples.*

G	λ	ℓ	p	k	$C_{\mathcal{S}_k(V)}$	Orth?	Dense?	Ref
A_ℓ	λ_1	1	any	1	P_1	no	yes	3.1
	$\lambda_1 + p^i\lambda_1$	1	$< \infty$	1	T_1	yes	yes	3.3
	$\lambda_1 + p^i\lambda_1$	1	$< \infty$	2	$U_1 T_1$	yes	yes	5.1
	$3\lambda_1$	1	> 3	1	$Sym(3)$	no	yes	3.2
	$3\lambda_1$	1	> 3	2	$Alt(4)$	no	yes	5.1
	$4\lambda_1$	1	> 3	1	$Alt(4)$	yes	yes	5.1
	$4\lambda_1$	1	> 3	2	$Sym(3)$	yes	yes	5.1
	$\lambda_1 + \lambda_2$	2	3	1	$U_2 T_1$	yes	yes	3.4
	$\lambda_1 + \lambda_2$	2	3	2	U_1	yes	yes	5.1
	$\lambda_1 + \lambda_2$	2	3	3	$T_2.\mathbb{Z}_3$	yes	yes	5.9
	$\lambda_1 + \lambda_2$	2	$\neq 3$	1	$T_2.\mathbb{Z}_3$	yes	yes	4.3
	$\lambda_1 + \lambda_2$	2	$\neq 3$	$4', 4''$	$T_2.\mathbb{Z}_3$	yes	yes	5.10
	$\lambda_1 + \lambda_3$	3	2	1	$T_3.Alt(4)$	yes	yes	4.3
	$\lambda_1 + \lambda_\ell$	≥ 3	$\neq 2$	1	T_ℓ	yes	no	4.1
	$\lambda_1 + \lambda_\ell$	$\geq 4, \not\equiv 1 \bmod 4$	2	1	T_ℓ	yes	no	4.1
	$\lambda_1 + \lambda_\ell$	$\geq 4, \equiv 1 \bmod 4$	2	1	T_ℓ	no	no	4.1
	λ_3	5	2	1	$U_8 A_2 T_1$	yes	yes	3.5
	λ_3	5	$\neq 2$	1	$A_2^2.\mathbb{Z}_2$	no	yes	3.2
	λ_3	5	2	2	$T_2.U_1.\mathbb{Z}_2(*)$	yes	no	5.6
	λ_3	5	$\neq 2$	2	$T_2.\mathbb{Z}_2.\mathbb{Z}_2$	no	no	5.6
B_ℓ	λ_1	≥ 2	$\neq 2$	any	P_k	yes	yes	3.1
	$2\lambda_2$	2	$\neq 2$	1	$T_2.\mathbb{Z}_4$	yes	yes	4.3
	$2\lambda_2$	2	$\neq 2, 5$	$5', 5''$	$\mathbb{Z}_5.\mathbb{Z}_4$	yes	yes	5.26
	$2\lambda_2$	2	5	$5'$	$\mathbb{Z}_5.\mathbb{Z}_4$	yes	yes	5.26
	$2\lambda_2$	2	5	$5''$	$\mathbb{Z}_4$	yes	yes	5.19
	λ_2	≥ 3	$\neq 2$	1	$T_\ell.\mathbb{Z}_2$	yes	no	4.1
	λ_3	3	any	1	$U_6 A_2 T_1$	yes	yes	3.6
	λ_3	3	any	$4', 4''$	$U_6 A_2 T_1$	yes	yes	5.11
	λ_3	3	any	2	$U_5 A_1 A_1 T_1$	yes	yes	3.7
	λ_3	3	any	3	$U_3 A_2 T_1$	yes	yes	3.7
	λ_4	4	any	1	$U_7 G_2 T_1$	yes	yes	3.8
	λ_4	4	2	2	$U_5 A_1 A_1$	yes	yes	5.1
	λ_4	4	$\neq 2$	2	$A_1(A_2.\mathbb{Z}_2)$	yes	yes	5.1
	λ_4	4	any	3	A_1	yes	yes	5.12
	λ_4	4	any	$8'$	$A_2.\mathbb{Z}_2$	yes	yes	5.15
	λ_4	4	$\neq 2$	$8''$	A_1^3	yes	no	5.17
	λ_4	4	2	$8''$	$A_1^3(*)$	yes	no	5.18
	λ_4	4	any	7	$T_2.\mathbb{Z}_2$	yes	no	5.16
	λ_5	5	2	1	$U_{14} B_2 T_1$	yes	yes	3.8
	λ_5	5	$\neq 2$	1	$A_4.\mathbb{Z}_2$	no	yes	3.2
	λ_6	6	2	1	$A_2^2.\mathbb{Z}_2$	no	no	5.1
	λ_6	6	2	1	$(A_2.\mathbb{Z}_2)^2.\mathbb{Z}_2$	yes	yes	5.1
C_ℓ	λ_1	≥ 3	any	any	P_k	no	yes	3.1
	$2\lambda_1$	≥ 3	$\neq 2$	1	$T_\ell.\mathbb{Z}_2$	yes	yes	4.1
	λ_2	3	3	1	$U_6 A_1 T_1$	yes	yes	3.9
	λ_2	3	3	2	$U_1 T_1.\mathbb{Z}_2$	yes	yes	5.1
	λ_2	3	$\neq 3$	1	$A_1^3.\mathbb{Z}_3$	yes	yes	4.4
	λ_2	3	$\neq 3, 7$	$7', 7''$	$\mathbb{Z}_7.\mathbb{Z}_6$	yes	yes	5.31
	λ_2	3	7	$7'$	$\mathbb{Z}_7.\mathbb{Z}_6$	yes	yes	5.31
	λ_2	3	7	$7''$	$\mathbb{Z}_6$	yes	yes	5.27
	λ_2	4	2	1	$A_1^4.Alt(4)$	yes	yes	4.4
	λ_2	≥ 4	$\neq 2$	1	A_1^ℓ	yes	no	4.2
	λ_2	$\geq 5, \not\equiv 2 \bmod 4$	2	1	A_1^ℓ	yes	no	4.2
	λ_2	$\geq 5, \equiv 2 \bmod 4$	2	1	A_1^ℓ	no	no	4.2
	λ_2	3	$\neq 3$	2	$T_1.Sym(3)$	yes	no	5.8
	λ_3	3	$\neq 2$	1	$A_2.\mathbb{Z}_2$	no	yes	3.2
D_ℓ	λ_1	≥ 4	any	$\neq \ell - 1$	P_k	yes	yes	3.1
	λ_1	≥ 4	any	$\ell - 1$	$P_{\ell-1,\ell}$	yes	yes	3.1
	λ_2	4	2	1	$T_4.(\mathbb{Z}_2^3.Alt(4))$	yes	yes	4.4
	λ_2	≥ 4	$\neq 2$	1	$T_\ell.\mathbb{Z}_{(2,\ell)}$	yes	no	4.1
	λ_2	$\geq 5, \not\equiv 2 \bmod 4$	2	1	$T_\ell.(\mathbb{Z}_2)^{\ell-1}$	yes	no	4.1
	λ_2	$\geq 5, \equiv 2 \bmod 4$	2	1	$T_\ell.(\mathbb{Z}_2)^{\ell-1}$	no	no	4.1

Table 1. Continued

G	λ	ℓ	**p**	**k**	$C_{\mathcal{S}_k(V)}$	Orth?	Dense?	Ref
	λ_6	6	2	1	$U_{14}B_3T_1$	yes	yes	3.10
	λ_6	6	$\neq 2$	1	$A_5.\mathbb{Z}_2$	no	yes	3.2
	λ_6	6	2	2	$A_1^3.U_1.\mathbb{Z}_2(*)$	yes	no	5.6
	λ_6	6	$\neq 2$	2	$A_1^3.\mathbb{Z}_2.\mathbb{Z}_2$	no	no	5.6
G_2	λ_1	2	$\neq 2$	1	$U_5A_1T_1$	yes	yes	3.11
	λ_1	2	2	1	$U_5A_1T_1$	no	yes	3.2
	λ_1	2	2	2	$U_3A_1T_1$	no	yes	3.12
	λ_1	2	$\neq 2$	2	$U_3A_1T_1$	yes	yes	3.12
	λ_1	2	2	3	A_2	no	yes	3.13
	λ_1	2	$\neq 2$	3	A_2	yes	yes	5.3
	λ_2	2	$\neq 3$	1	$T_2.\mathbb{Z}_6$	yes	yes	4.3
F_4	λ_4	4	3	1	$U_{14}G_2T_1$	yes	yes	3.14
	λ_4	4	$\neq 3$	1	$D_4.\mathbb{Z}_3$	yes	yes	5.1
	λ_4	4	3	2	$U_1A_2.\mathbb{Z}_2$	yes	yes	5.1
	λ_4	4	$\neq 3$	2	$A_2.Sym(3)$	yes	yes	5.8
	λ_1	4	$\neq 2$	1	$T_4.\mathbb{Z}_2$	yes	no	4.1
E_6	λ_2	6	any	1	T_6	yes	no	4.1
E_7	λ_7	7	2	1	$U_{26}F_4T_1$	yes	yes	3.15
	λ_1	7	$\neq 2$	1	$T_7.\mathbb{Z}_2$	yes	no	4.1
	λ_1	7	2	1	$T_7.\mathbb{Z}_2$	no	no	4.1
	λ_7	7	2	2	$D_4.U_1.\mathbb{Z}_2(*)$	yes	no	5.6
	λ_7	7	$\neq 2$	2	$D_4.\mathbb{Z}_2.\mathbb{Z}_2$	no	no	5.6
E_8	λ_8	8	any	1	$T_8.\mathbb{Z}_2$	yes	no	4.1

Theorem 2. *The (semi-)generic ts-stabilizer for a ts-small quadruple is given in Table 1. In addition, the existence or nonexistence of a dense orbit is indicated.*

We are quickly able to determine whether a dense orbit on X exists, as the set of points in X that have stabilizers of minimal dimension is open in X (see, for example, [21, Lemma 2.1]). This means that the dimension of a (semi-)generic stabilizer is actually the minimal dimension of any stabilizer, and if this is larger than $\dim G - \dim X$, there is no dense orbit. There is extensive interest in the literature around the existence of a dense orbit. In particular, if G is a reductive algebraic group and V is an irreducible rational G-module such that G has a dense orbit on V, the pair (G, V) is called a *prehomogeneous vector space*, often shortened to *PV*-space. A classification of *PV*-spaces was determined in characteristic zero in [29] and extended to positive characteristic in [32][33]. The fact that this classification covers all semisimple algebraic groups implies that the density question is already understood for the action on $\mathcal{G}_k(V)$. Indeed, G has a dense orbit on $\mathcal{G}_k(V)$ if and only if $GL_k \otimes G$ has a dense orbit on $V_1 \otimes V$, where V_1 is the natural module for GL_k. The next result summarises the answer to the dense-orbit question for the action on orthogonal and symplectic Grassmannians.

Theorem 3. *Let G be a simple algebraic group over an algebraically closed field of characteristic p, and V a self-dual nontrivial irreducible G-module of dimension d and highest weight λ. For $1 \leq k \leq \frac{d}{2}$, the action of G on $\mathcal{S}_k(V)$ has a dense orbit if and only if $\dim G \geq \dim \mathcal{S}_k(V)$, the zero-weight space of V has dimension at most 2, and (G, λ, p, k) is not one of the following:*

(i) $(A_5, \lambda_3, any, 2)$;
(ii) $(B_4, \lambda_4, any, 7)$;
(iii) $(B_4, \lambda_4, any, 8'')$;
(iv) $(D_6, \lambda_6, any, 2)$;
(v) $(E_7, \lambda_7, any, 2)$.

Remark 1. The only *ts*-small quadruples (G, λ, p, k) with a zero-weight space of dimension at least 3 have $k = 1$, with either $V_G(\lambda)$ a composition factor of the adjoint module for G, or $G = C_\ell$ and $\lambda = \lambda_2$.

Denote by $Cl(V)$ a classical group with natural module V. Given that $X = \mathcal{G}_k(V)$ and $X = \mathcal{S}_k(V)$ are varieties of cosets of the form $Cl(V)/P$, where P is a parabolic subgroup of $Cl(V)$, the action of G on X has a dense orbit if and only if there is a dense (G, P)-double coset in $Cl(V)$. We now seek to complete the answer to the question of existence of a dense double coset in the following sense. In Theorem 5, we classify pairs (G, H) of closed maximal connected subgroups of $Cl(V)$ such that there exists a dense (G, H)-double coset in $Cl(V)$. In order to do this, we first need to determine which semisimple groups acting tensor-decomposably on an irreducible module V have a dense orbit on $\mathcal{S}_k(V)$. The possibilities for such subgroups are given by Proposition 2.2. They are $SO_n \otimes SO_m \leq SO_{mn}$, $Sp_{2n} \otimes SO_m \leq Sp_{2mn}$, $Sp_{2n} \otimes Sp_{2m} \leq SO_{4mn}$, with $p \neq 2$ if one of the factors is an orthogonal group acting on an odd-dimensional vector space. The strategy is similar to the one employed for the proof of Theorem 2, although we only care about the connected component of (semi-)generic stabilizers. This is achieved in the following theorem.

Theorem 4. *Let $G = Cl(V_1) \otimes Cl(V_2)$ be a maximal subgroup of either $SO(V)$ or $Sp(V)$ with $V = V_1 \otimes V_2$ and suppose $k \leq \frac{1}{2} \dim V_1 \dim V_2$. Then the action of G on $\mathcal{S}_k(V)$ has a dense orbit if and only if (G, k) is one of the following:*

  (i) $(Sp(V_1) \otimes SO(V_2), 1)$ *and* (K^*G, V) *is a prehomogeneous vector space, as classified in [32][33];*
 (ii) $(Sp_2 \otimes Sp_{2n}, 1)$ *with $n \geq 1$;*
(iii) $(Sp_2 \otimes Sp_{2n}, 2)$ *with $n \geq 2$;*
(iv) $(Sp_2 \otimes Sp_{2n}, 3)$ *with $n \geq 3$;*
 (v) $(Sp_2 \otimes Sp_{2n}, k)$ *with $k = (2n)'$ or $k = (2n)''$ and $1 \leq n \leq 3$;*
(vi) $(Sp_2 \otimes Sp_{2n}, 2n - 1)$ *with $1 \leq n \leq 3$;*
(vii) $(Sp_4 \otimes Sp_{2n}, 1)$ *with $n \geq 2$.*

We are now ready to classify pairs (G, H) of closed maximal connected subgroups of $Cl(V)$ such that there exists a dense (G, H)-double coset in $Cl(V)$. One particular class of such pairs (see case $(iii)(b)$ in the following theorem) arises from *spherical* subgroups. These are reductive subgroups G such that there is a dense (G, B)-double coset, where B is a Borel subgroup. Spherical subgroups have been classified by Krämer in characteristic 0 and by Knop and Röhrle in [14] for arbitrary characteristic.

Theorem 5. *Let Γ be a classical group $Cl(V)$. Let G, H be a pair of maximal connected subgroups of Γ. Then there exists a dense (G, H)-double coset in Γ if and only if one of the following holds:*

  (i) *G and H are both reductive and $\Gamma = GH$. Such factorizations were classified in [17].*
 (ii) *G and H are both parabolic subgroups.*
(iii) *After possibly interchanging G and H, we have that H is parabolic, G is reductive and one of the following holds:*
    (a) *$\Gamma = SL(V)$, $H = P_k$ (or $\Gamma = Sp(V)$, $H = P_k$ with $k = 1$) and $(GL_k \otimes G, K^k \otimes V)$ is a prehomogeneous vector space, as classified in [29][32][33];*
    (b) *G is the stabilizer of a subspace X of V and either X is nondegenerate or $p = 2$ and X is nonsingular of dimension 1 with V orthogonal. In this case G is spherical, so H is arbitrary.*
    (c) *$G = Cl(V_1) \otimes Cl(V_2)$, $V = V_1 \otimes V_2$, and G has a dense orbit on the variety of totally singular subspaces corresponding to Γ/H, as detailed in Theorem 4.*
    (d) *G is simple and irreducible on V and has a dense orbit on the variety of totally singular subspaces corresponding to Γ/H, as detailed in Theorem 3.*

Remark 2. Dropping the maximality assumption presents considerable challenges. Even the case where both G and H are reductive does not have a general solution, as the results in [6] require a technical condition on G and H.

Remark 3. The double coset density question remains open when Γ is an exceptional group. In case (iii) of Theorem 5, the only information available for Γ exceptional is when G is of maximal rank (see [8]). The complete classification will be the subject of forthcoming work.

Let us conclude with a related question. Given two maximal connected subgroups of $Cl(V)$, are there finitely many double cosets? Of course, the existence of finitely many double cosets implies the existence of a dense double coset, while the opposite need not be true. For example, $G = A_\ell$ for $\ell \geq 8$ has a dense orbit on $X = \mathcal{G}_2(V)$ where $V = V_G(\lambda_2)$, but also has infinitely many orbits ([11]). Therefore, there is a dense (G, P_2)-double coset in $SL(V)$, as well as infinitely many double cosets.

However, it was shown in [11] that a simple group G having finitely many orbits on $X = \mathcal{G}_1(V)$, where V is an irreducible G-module, is equivalent to G having a dense orbit on X. This result was replicated for self-dual modules when $X = \mathcal{S}_1(V)$ and $X = \mathcal{S}_2(V)$ in [26][27].

Strikingly, we shall conjecture that there is only one exception to this equivalence when X is an orthogonal or symplectic Grassmannian.

Conjecture 1. *Let G be a simple connected irreducible subgroup of either $SO(V)$ or $Sp(V)$. Then unless $(G, \lambda, p, k) = (C_3, \lambda_2, p, 7)$ with $p \neq 3$, the action of G on $\mathcal{S}_k(V)$ has a dense orbit if and only if G acts on $\mathcal{S}_k(V)$ with finitely many orbits.*

The paper will be structured as follows. In Section 2, we shall consolidate the notation and background material, develop the tools for our analysis and determine the complete list of ts-small quadruples, which we divide into three tables (see Proposition 2.17). We will devote a section to each table.

Given a ts-small quadruple, we then proceed to determine its (semi-)generic ts-stabilizer. In Section 3, we handle the cases where we already had finitely many orbits on all k-spaces. In Section 4, we deal with the cases having a large zero-weight space, and in Section 5, we handle the remaining cases. This completes the proof of Theorem 1 and Theorem 2.

We then shift our attention to the double coset density question. In Section 6, we prove Theorem 4. Finally, in Section 7, we use all of the previous results to prove Theorem 5.

2. Preliminary results

In this section, we gather some useful results, develop notation and conclude with a complete list of ts-small quadruples.

2.1. Bilinear forms

We start by fixing the notation for the action of an orthogonal group on its natural module V_{nat}. To do this, we use the standard notation for its root system: we take an orthonormal basis $\epsilon_1, \ldots, \epsilon_\ell$ of the ℓ-dimensional Euclidean space and take simple roots $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i < \ell$ and $\alpha_\ell = \epsilon_\ell$ or $\epsilon_{\ell-1} + \epsilon_\ell$ according as $G = B_\ell$ or D_ℓ.

If $G = D_\ell$, then V_{nat} has (hyperbolic) basis $e_1, f_1, \ldots, e_\ell, f_\ell$ on which root elements act by

$$
\begin{aligned}
x_{\epsilon_i - \epsilon_j}(t) &: e_j \mapsto e_j + te_i, \quad f_i \mapsto f_i - tf_j, \\
x_{-\epsilon_i + \epsilon_j}(t) &: e_i \mapsto e_i + te_j, \quad f_j \mapsto f_j - tf_i, \\
x_{\epsilon_i + \epsilon_j}(t) &: f_j \mapsto f_j + te_i, \quad f_i \mapsto f_i - te_j, \\
x_{-\epsilon_i - \epsilon_j}(t) &: e_j \mapsto e_j - tf_i, \quad e_i \mapsto e_i + tf_j,
\end{aligned}
$$

while fixing the basis vectors that are not listed.

If $G = B_\ell$, then V_{nat} has (hyperbolic) basis $v_0, e_1, f_1, \ldots, e_\ell, f_\ell$ on which root elements act by

$$
\begin{aligned}
x_{\epsilon_i - \epsilon_j}(t) &: e_j \mapsto e_j + te_i, \quad f_i \mapsto f_i - tf_j, \\
x_{-\epsilon_i + \epsilon_j}(t) &: e_i \mapsto e_i + te_j, \quad f_j \mapsto f_j - tf_i, \\
x_{\epsilon_i + \epsilon_j}(t) &: f_j \mapsto f_j + te_i, \quad f_i \mapsto f_i - te_j, \\
x_{-\epsilon_i - \epsilon_j}(t) &: e_j \mapsto e_j - tf_i, \quad e_i \mapsto e_i + tf_j,
\end{aligned}
$$

$$x_{\epsilon_i}(t) : v_0 \mapsto v_0 + 2te_i, \quad f_i \mapsto f_i - tv_0 - t^2 e_i,$$
$$x_{-\epsilon_i}(t) : v_0 \mapsto v_0 - 2tf_i, \quad e_i \mapsto e_i + tv_0 - t^2 f_i,$$

while fixing the basis vectors that are not listed.

The following two results describe the structure of maximal connected subgroups of classical groups. Recall that by $Cl(V)$, we denote a classical group with natural module V.

Theorem 2.1 [18]. *Let H be a closed connected subgroup of $G = Cl(V)$. Then one of the following holds:*

(i) $H \leq Stab_G(X)$ *with $X \leq V$ a proper nonzero subspace which is either totally singular or nondegenerate, or $p = 2$, $G = SO(V)$ and X is nonsingular of dimension 1;*

(ii) $V = V_1 \otimes V_2$ *and H lies in a subgroup of the form $Cl(V_1) \otimes Cl(V_2)$ acting naturally on $V_1 \otimes V_2$ with $\dim V_i \geq 2$ for $i = 1, 2$;*

(iii) H *is a simple algebraic group acting irreducibly on V and $V|_H$ is tensor indecomposable.*

The possibilities for the second case of Theorem 2.1 are given by the following proposition.

Proposition 2.2 [18, Prop 2.2]. *Suppose $V = V_1 \otimes V_2$ and f_i is a nondegenerate bilinear form on V_i.*

(i) *There is a unique nondegenerate bilinear form $f = f_1 \otimes f_2$ on V such that $f(u_1 \otimes u_2, v_1 \otimes v_2) = f_1(u_1, v_1) f_2(u_2, v_2)$ for all $u_i, v_i \in V_i$.*

(ii) *f is symmetric if f_1, f_2 are both alternating or both symmetric, and f is alternating otherwise.*

(iii) *f is preserved by $I(V_1) \circ I(V_2)$ acting naturally on the tensor product, where $I(V_i)$ is the stabilizer in $GL(V_i)$ of f_i.*

(iv) *If $p = 2$, then there is a unique quadratic form Q on V, with associated bilinear form f, such that $Q(v_1 \otimes v_2) = 0$ for all $v_i \in V_i$ and Q is preserved by $Sp(V_1) \otimes Sp(V_2)$.*

The following lemma gives the dimension of the symplectic and orthogonal Grassmannians we are acting on.

Lemma 2.3. *Let V be either a symplectic or orthogonal geometry of dimension d over an algebraically closed field. Then*

$$\dim \mathcal{S}_k(V) = kd - \frac{3k^2 + \epsilon_V k}{2},$$

where ϵ_V is 1 or -1 according as V is orthogonal or symplectic.

Proof. If V is orthogonal with $d = 2\ell$ and $k = \ell - 1$, we have $\dim \mathcal{S}_k(V) = \dim D_\ell - \dim P_{\ell-1,\ell}$. In all other cases, the dimension is simply given by $\dim Cl(V)/P_k = \dim Cl(V) - \dim P_k$ for $Cl(V) = B_{\frac{d-1}{2}}, C_{\frac{d}{2}}$, or $D_{\frac{d}{2}}$ as appropriate. $\quad\square$

Recall that a ts-small quadruple is a 4-tuple of the form (G, λ, p, k), with $V = V_G(\lambda)$ a self-dual irreducible G-module, and $\dim G \geq \dim \mathcal{S}_k(V)$. Lemma 2.3 gives the following dimension bound.

Lemma 2.4. *Let (G, λ, p, k) be a ts-small quadruple. Suppose that $V = V_G(\lambda)$ has dimension d. Then*

$$\dim G \geq kd - \frac{3k^2 + \epsilon_V k}{2},$$

where ϵ_V is 1 or -1 according as V is orthogonal or symplectic.

Proof. This follows directly from Lemma 2.3. $\quad\square$

A self-dual module is either orthogonal or symplectic. The following lemma provides a useful criterion in odd characteristic.

Lemma 2.5 [30, Lemma 78-79]. *Let G be a simple simply connected algebraic group and $V = V_G(\lambda)$ a self-dual G-module in characteristic $p \neq 2$. Then if $Z(G)$ has no element of order 2, the module V is orthogonal.*

Otherwise, let z be the only element of order 2 in $Z(G)$, except for the case $G = D_\ell$ with even ℓ, where z is the element of $Z(G)$ such that $G/\langle z \rangle \simeq SO_{2\ell}(k)$. Then the module V is orthogonal if $\lambda(z) = 1$, and symplectic otherwise.

Note that the value $\lambda(z)$ can be computed by [20, Appendix A.2]. If $p = 2$, for certain modules, we can determine whether V is symplectic or orthogonal thanks to [15]. More explicit descriptions of some of these forms can be found in [9] and [3].

When dealing with maximal totally singular subspaces of an orthogonal module V of dimension 2ℓ, we need to be able to distinguish between the two D_ℓ-orbits on $\mathcal{S}_\ell(V)$. The following lemma provides an easy way to do so.

Lemma 2.6. *Let V be an orthogonal module of even dimension and U, W two maximal totally singular subspaces of V. Then U and W are in the same $SO(V)$-orbit if and only if $\dim U - \dim U \cap W$ is even.*

Proof. This follows quickly from [2, 22.14(2)]. □

We conclude this section with a lemma concerning orthogonality of weight spaces in a self-dual irreducible module.

Lemma 2.7. *Let $V = V_G(\lambda)$ be a self-dual irreducible KG-module. Let $(\cdot, \cdot)$ be an irreducible symmetric or alternating bilinear form on V preserved by G. Then the following hold:*

(a) *For any two weights μ, ν such that $\mu \neq -\nu$, the weight spaces V_μ and V_ν are orthogonal to each other;*
(b) *V_μ is totally singular for all nonzero weights μ;*
(c) *V_0 is nondegenerate.*

Proof. Let v, u be two weight vectors in non-opposite weight spaces. Then there exists $t \in T$ such that $t.v = \kappa_1 v$ and $t.u = \kappa_2 u$ with $\kappa_1 \neq \kappa_2^{-1}$. Therefore, $(u, v) = \kappa_1 \kappa_2 (u, v)$, which implies $(u, v) = 0$, proving (a).

By part (a) with $\mu = \nu$, we have that V_μ is totally-isotropic. Similarly to the previous paragraph, if V is an orthogonal module, there exists $\kappa \neq \pm 1$ such that $Q(v) = \kappa^2 Q(v)$, implying $Q(v) = 0$ for all $v \in V_\mu$. This proves (b).

By (a), the zero weight space V_0 is orthogonal to all nonzero weight spaces. Therefore, any singular vector v in the radical of V_0 is a singular vector in the radical of V, which implies $v = 0$ since the form is nondegenerate. This proves that V_0 is itself nondegenerate. □

2.2. Clifford theory

Let $G \leq GL(V)$ be a subgroup acting completely reducibly and homogeneously on V. The following lemma shows that G must preserve a tensor product structure on V.

Lemma 2.8 [13, Lemma 4.4.3]. *Let V be an irreducible KG-module and suppose that $S \leq G$ acts completely reducibly and homogeneously on V, with $s \geq 2$ irreducible summands of dimension r. Then the following hold:*

(i) *there is a tensor decomposition $V = V_1 \otimes V_2$ (where $\dim V_1 = r$, $\dim V_2 = s$), such that $S \leq GL(V_1) \otimes 1$ and $C_{GL(V)}(S) = 1 \otimes GL(V_2)$;*
(ii) *$C_{GL(V)}(C_{GL(V)}(S)) = GL(V_1) \otimes 1$;*
(iii) *$N_{GL(V)}(S) = N_{GL(V_1)}(S) \otimes GL(V_2)$;*
(iv) *the irreducible KS-submodules of V are precisely the subspaces $V_1 \otimes v$, where $0 \neq v \in V_2$.*

The following lemma shows that no cyclic extension of G can act irreducibly on V.

Lemma 2.9. *Let V be an irreducible KG-module and suppose that $S \leq G$ acts completely reducibly and homogeneously on V, with $s \geq 2$ irreducible summands of dimension r. Let $S\langle\tau\rangle \leq G$ be a cyclic extension of S. Then $S\langle\tau\rangle$ does not act irreducibly on V.*

Proof. We apply Lemma 2.8 to get $V = V_1 \otimes V_2$ with $\dim V_1 = r$, $S \leq GL(V_1) \otimes 1$ and $\tau = \tau_1 \otimes \tau_2 \in GL(V_1) \otimes GL(V_2)$. Since τ_2 stabilises at least one 1-space of V_2, by part (iv) of Lemma 2.8, the element τ stabilises one irreducible KS-submodule of V. □

Lemma 2.10. *Let V be an irreducible KG-module and suppose that $S \leq G$ acts homogeneously on V as the sum of 2 irreducible summands. Let $S\langle\tau\rangle \leq G$ be a cyclic extension of S. Then $S\langle\tau\rangle$ fixes 1, 2, or all KS-submodules of V.*

Proof. From the proof of Lemma 2.9, we have $V = V_1 \otimes V_2$, with $\dim V_2 = 2$, $S \leq GL(V_1) \otimes 1$ and $\tau = \tau_1 \otimes \tau_2 \in GL(V_1) \otimes GL(V_2)$. Then τ_2 stabilises either 1, 2 or all 1-spaces of V_2, concluding. □

2.3. Generic stabilizers

In this section, we gather some of the essential lemmas that will allow us to determine (semi-)generic stabilizers. As mentioned in the introduction, a (semi-)generic stabilizer realises the minimum dimension of any stabilizer. This follows from the next two results.

Lemma 2.11 [22, Lemma 3.7]. *Let an algebraic group G act on a quasi-projective variety X. For any $t \in \mathbb{N} \cup \{0\}$, the set $\{x \in X | \dim G_x \geq t\}$ is closed.*

Corollary 2.12. *Let an algebraic group G act on an irreducible quasi-projective variety X with (semi-)generic stabilizer S. Then for all $x \in X$, we have $\dim G_x \geq \dim S$.*

Proof. By assumption, there exists an open set Y such that all elements of Y have stabilizer isomorphic to S. By Lemma 2.11, we then find that $\dim G_x \geq \dim S$ for all $x \in X$. □

Let us consider some of the methods used in [10]. In particular, we are interested in the *localization to a subvariety* approach [10, §4.1]. Let X be a variety on which a simple algebraic group G acts. Let Y be a subvariety of X and $x \in X$. The *transporter* in G of x into Y is

$$\text{Tran}_G(x, Y) = \{g \in G : g.x \in Y\}.$$

Let $\phi : G \times X \to X$ be the orbit map.

Lemma 2.13 [10, Lemma 4.1]. *For $y \in Y$, the following hold:*

(i) $\dim \text{Tran}_G(y, Y) = \dim \phi^{-1}(y)$;
(ii) $\text{codim} \, \text{Tran}_G(y, Y) = \dim(\overline{G.y}) - \dim(\overline{G.y \cap Y})$.

If Y is a subvariety of X, a point $y \in Y$ is called *Y-exact* if

$$\text{codim} \, \text{Tran}_G(y, Y) = \text{codim} \, Y.$$

Lemma 2.14 [10, Lemma 4.3]. *Let $\hat{Y}$ be a dense open subset of Y. Suppose that all points in $\hat{Y}$ are Y-exact. Then $\phi(G \times \hat{Y})$ contains a dense open subset of X.*

Such a set $\hat{Y}$ is sufficiently representative of the G-action on X, which leads to the following lemma.

Lemma 2.15 [10, Lemma 4.4]. *Let $\hat{Y}$ be a dense open subset of Y. Let C be a subgroup of G containing G_X. Suppose that for each $y \in \hat{Y}$, the following is true:*

(i) *y is Y-exact;*
(ii) *G_y is conjugate to C.*

Then C/G_X is the generic stabilizer in the action of G on X.

We similarly derive a criterion for proving that there does not exist a generic stabilizer.

Lemma 2.16. *Assume that Y is not finite, and let $\hat{Y}$ be a dense open subset of Y. Suppose that for each $y \in \hat{Y}$, the following is true:*

(i) *y is Y-exact;*

(ii) *for all $y' \in \hat{Y} \setminus \{y\}$, we have that G_y is not conjugate to $G_{y'}$.*

Then there is no generic stabilizer in the action of G on X.

Proof. By Lemma 2.14, there is a dense open subset U_1 of X, contained in $\phi(G \times \hat{Y})$ which by assumption is the union of G-orbits with pairwise non-conjugate stabilizers. Given any such orbit $G.y$ for some $y \in \hat{Y}$, we have

$$\dim(\overline{G.y}) = \dim(\overline{G.y}) - \dim(\overline{G.y \cap Y}) = \operatorname{codim} Y < \dim X,$$

since y is Y-exact combined with $\dim Y \geq 1$ and Lemma 2.13.

Assume that there is a generic stabilizer in the action of G on X. Then there is a dense open subset U_2 of X such that G_{x_1} and G_{x_2} are conjugate for all $x_1, x_2 \in U_2$. Taking the intersection of U_1 and U_2, we get an open dense subset U of X with the same property. Therefore, given any two $x_1, x_2 \in U$, we must have $G.x_1 = G.x_2$ (i.e., U consists of a single G-orbit). This implies that there is a dense orbit, contradicting $\dim(\overline{G.y}) < \dim X$. $\qquad\square$

2.4. Spin modules

We set up spin modules following [23]. Let $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}\} = \{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ be a standard basis for the $2n$-dimensional K-vector space $V = V_{2n}$ with quadratic form Q and bilinear form $(\cdot, \cdot)$, such that $\{e_i, e_{n+i}\} = \{e_i, f_i\}$ are hyperbolic pairs for $i \leq n$. Let L, M be the totally singular subspaces $\langle e_1, \ldots, e_n \rangle$ and $\langle f_1, \ldots, f_n \rangle$, respectively.

We denote by C the Clifford algebra of (V, Q). This is an associative algebra over K generated by V, in which $v^2 = Q(v)$ for every $v \in V$. It has the structure of a graded module over K. Let $\phi' : C \to C$, sending $x \mapsto x'$, be the involution of C fixing every element of V (i.e., the anti-automorphism sending a product $\prod_{i=1}^{k} v_i \in C$ to $\prod_{i=1}^{k} v_{n-i+1}$). We denote by C^+ and C^- the sums of homogeneous submodules of C of even and odd degrees, respectively. Then $C = C^+ \oplus C^-$. In particular, C^+ is a subalgebra of C invariant under ϕ'.

The *Clifford group* is $G^* = \{s \in C \mid s \text{ is invertible in } C \text{ and } sVs^{-1} = V\}$. The *even Clifford group* is $(G^*)^+ = G^* \cap C^+$. The *spin group* $Spin_{2n}$ is $\{s \in (G^*)^+ \mid ss' = 1\}$.

The *vector representation* of the Clifford group G^* is given by $\Theta : G^* \to Aut(V, Q)$, such that $\Theta(s) \cdot v = svs^{-1}$. The restriction of Θ to $Spin_{2n}$ is the natural representation of $Spin_{2n}$. The root subgroups of $Spin_{2n}$ are parametrised by pairs (i, j) with $i + j \neq 2n + 1$; the root subgroup parametrised by the pair (i, j) consists of elements of the form $1 + \lambda e_i e_j$, where $1 + \lambda e_i e_j$ acts on a vector $v \in V$ by $v \mapsto v + \lambda(e_j, v)e_i - \lambda(e_i, v)e_j$.

Put $e_L = e_1 e_2 \ldots e_n$ and $e_M = e_{n+1} e_{n+2} \ldots e_{2n}$. We denote by C_W the subalgebra of C generated by the elements of a subspace $W \subset V_{2n}$. Then $ce : M$ is a minimal left ideal in C, and the correspondence $x \mapsto xe_M$ generates an isomorphism $C_L \to ce : M$ of vector spaces. So for any $s \in C, x \in C_L$, there exists a unique element $y \in C_L$ for which $sxe_M = ye_M$. Setting $\rho(s) \cdot x = s \cdot x = y$ gives us the spinor representation ρ of the algebra C in C_L. Let $X = C_L \cap C^+$. Then restricting ρ to $Spin_{2n}$, we get the half-spin representation of $Spin_{2n}$ in X.

An element of X is called a *spinor*. The restriction to B_{n-1} is the spin representation for B_{n-1}.

2.5. List of ts-small quadruples

The following result lists all *ts*-small quadruples. We will then be able to prove Theorem 1 and Theorem 2 by proceeding case-by-case.

Table 2. ts-small quadruples with finitely many orbits on $\mathcal{G}_k(V)$.

G	λ	dim V	p	k	Orthogonal?
A_1	λ_1	2	any	1	no
A_1	$3\lambda_1$	4	> 3	1	no
A_1	$\lambda_1 + p^i\lambda_1$	4	$< \infty$	1	yes
A_2	$\lambda_1 + \lambda_2$	7	3	1	yes
A_5	λ_3	20	2	1	yes
A_5	λ_3	21	$\neq 2$	1	no
$B_\ell, \ell \geq 2$	λ_1	$2\ell + 1$	$\neq 2$	any	yes
B_3	λ_3	8	any	$1, 2, 3$	yes
B_4	λ_4	16	any	1	yes
B_5	λ_5	32	2	1	yes
B_5	λ_5	32	$\neq 2$	1	no
$C_\ell, \ell \geq 3$	λ_1	2ℓ	any	any	no
C_3	λ_2	13	3	1	yes
C_3	λ_3	14	$\neq 2$	1	no
$D_\ell, \ell \geq 4$	λ_1	2ℓ	any	any	yes
D_6	λ_6	32	2	1	yes
D_6	λ_6	32	$\neq 2$	1	no
G_2	λ_1	7	$\neq 2$	$1, 2$	yes
G_2	λ_1	6	2	$1, 2, 3$	no
F_4	λ_4	25	3	1	yes
E_7	λ_7	56	2	1	yes
E_7	λ_7	56	$\neq 2$	1	no

Table 3. Infinite families of ts-small quadruples.

G	λ	dim V	p	k	Orthogonal?
$A_\ell, \ell \geq 4$	$\lambda_1 + \lambda_\ell$	$\ell^2 + 2\ell - 1$	$\mid \ell + 1, \neq 2$	1	yes
$A_\ell, \ell \geq 2$	$\lambda_1 + \lambda_\ell$	$\ell^2 + 2\ell$	$\nmid \ell + 1$	1	yes
$A_\ell, \ell \equiv 3 \bmod 4$	$\lambda_1 + \lambda_\ell$	$\ell^2 + 2\ell - 1$	2	1	yes
$A_\ell, \ell \equiv 1 \bmod 4$	$\lambda_1 + \lambda_\ell$	$\ell^2 + 2\ell - 1$	2	1	no
B_ℓ	λ_2	$2\ell^2 + \ell$	$\neq 2$	1	yes
C_ℓ	$2\lambda_1$	$2\ell^2 + \ell$	$\neq 2$	1	yes
C_ℓ	λ_2	$2\ell^2 - \ell - 1$	$\nmid \ell, \neq 2$	1	yes
$C_\ell, \ell \geq 5$	λ_2	$2\ell^2 - \ell - 2$	$\mid \ell, \neq 2$	1	yes
$C_\ell, \ell \not\equiv 2 \pmod 4$	λ_2	$2\ell^2 - \ell - 1 - \gcd(\ell, 2)$	2	1	yes
$C_\ell, \ell \equiv 2 \pmod 4$	λ_2	$2\ell^2 - \ell - 2$	2	1	no
$D_\ell, \ell \geq 4$	λ_2	$2\ell^2 - \ell$	$\neq 2$	1	yes
D_ℓ, ℓ odd, $\ell \geq 4$	λ_2	$2\ell^2 - \ell - 1$	2	1	yes
$D_\ell, \ell \equiv 0 \bmod 4$	λ_2	$2\ell^2 - \ell - 2$	2	1	yes
$D_\ell, \ell \equiv 2 \bmod 4, \ell \geq 4$	λ_2	$2\ell^2 - \ell - 2$	2	1	no
G_2	λ_2	14	$\neq 3$	1	yes
F_4	λ_1	52	$\neq 2$	1	yes
E_6	λ_2	$78 - \delta_{p,3}$	any	1	yes
E_7	λ_1	133	$\neq 2$	1	yes
E_7	λ_1	132	2	1	no
E_8	λ_8	248	any	1	yes

Proposition 2.17. *Let (G, λ, p, k) be a ts-small quadruple and $V = V_G(\lambda)$. Then, precisely one of the following is true:*

(i) *G has finitely many orbits on $\mathcal{G}_k(V)$, and (G, λ, p, k) is in Table 2;*
(ii) *$k = 1$, and either V is a composition factor of $\mathrm{Lie}(G)$ or $G = C_\ell$ and $\lambda = \lambda_2$, as in Table 3;*
(iii) *(G, λ, p, k) is in Table 4.*

Proof. In [11], we find a complete list of modules with finitely many orbits on k-spaces, of which every self-dual one leads to a ts-small quadruple. Now assume that (G, λ, p, k) is a ts-small quadruple where G does not have finitely many orbits on $\mathcal{G}_k(V)$. In [26, Thm 3.1], we have a complete list of

Table 4. *Remaining ts-small quadruples.*

G	λ	dim V	p	k	Orthogonal?
A_1	$\lambda_1 + p^i \lambda_1$	4	any	2	yes
A_1	$3\lambda_1$	4	$\neq 2, 3$	2	no
A_1	$4\lambda_1$	5	$\neq 2, 3$	1, 2	yes
A_2	$\lambda_1 + \lambda_2$	7	3	2, 3	yes
A_2	$\lambda_1 + \lambda_2$	8	$\neq 3$	4	yes
A_5	λ_3	20	$\neq 2$	2	no
A_5	λ_3	20	2	2	yes
B_2	$2\lambda_2$	10	any	5	yes
B_3	λ_3	8	any	4	yes
B_4	λ_4	16	any	2, 3, 7, 8	yes
B_6	λ_6	64	2	1	yes
B_6	λ_6	64	$\neq 2$	1	no
C_3	λ_2	13	3	2	yes
C_3	λ_2	14	$\neq 3$	2	yes
C_3	λ_2	14	$\neq 3$	7	yes
D_6	λ_6	32	$\neq 2$	2	no
D_6	λ_6	32	2	2	yes
G_2	λ_1	7	$\neq 2$	3	yes
F_4	λ_4	25	3	2	yes
F_4	λ_4	26	$\neq 3$	1, 2	yes
E_7	λ_7	56	$\neq 2$	2	no
E_7	λ_7	56	2	2	yes

such quadruples with $k = 1$ and V orthogonal, while in [27, Prop. 4.1], we have a complete list of such quadruples for $k \geq 2$.

The proof follows from combining the three lists of ts-small quadruples, with the addition of the self-dual small quadruples where V is symplectic and $k = 1$. $\qquad\square$

3. Quadruples with finitely many orbits on $\mathcal{G}_k(V)$

In this section, we handle the ts-small quadruples where we already have finitely many orbits on all k-spaces (i.e., the ones listed in Table 2). It follows directly that there is a dense orbit for the action on totally singular k-spaces, and producing the generic stabilizer reduces to finding a stabilizer of appropriate dimension.

Proposition 3.1. *Let G be one of A_1, B_ℓ with $p \neq 2$, C_ℓ or D_ℓ. Let $\lambda = \lambda_1$. Then the ts-small quadruple (G, λ, p, k) has generic ts-stabilizer P_k, unless $k = \ell - 1$ and $G = D_\ell$, in which case it has generic ts-stabilizer $P_{\ell-1,\ell}$.*

Proof. Note that if $G = D_\ell$, by our convention, the values of k are $1, \ldots, \ell - 1, \ell', \ell''$. In all cases, in the statement of the proposition, the group G is transitive on $\mathcal{S}_k(V)$. It is well known that the maximal parabolic subgroups of a classical group are stabilizers of totally singular subspaces; therefore, these must be the generic stabilizers. Unless we are in type D and $k = \ell - 1$, the stabilizer of $y \in \mathcal{S}_k(V)$ is a conjugate of P_k. If $G = D_\ell$ and $k = \ell - 1$, the stabilizer of $y \in \mathcal{S}_k(V)$ is a conjugate of $P_{\ell-1,\ell}$. $\qquad\square$

Proposition 3.2. *The generic ts-stabilizers for the ts-small quadruples in Table 5 are as given.*

Proof. In each of these cases, $k = 1$ and the module V is symplectic. Therefore, $\mathcal{S}_k(V) = \mathcal{G}_k(V)$, and the result follows directly from [10]. In the last column of Table 5, we give a reference for each individual case. $\qquad\square$

Proposition 3.3. *Let $G = A_1$ and $\lambda = \lambda_1 + p^i \lambda_1$ with $p > 0$ and $i > 0$. Then the quadruple $(G, \lambda, p, 1)$ has generic ts-stabilizer T_1.*

Table 5. ts-small quadruples with $k = 1$ and V symplectic.

G	λ	p	k	$C_{\mathcal{S}_k(V)}$	Reference
A_1	$3\lambda_1$	> 3	1	$Sym(3)$	[10, Prop. 5.6]
A_5	λ_3	$\neq 2$	1	$A_2^2.\mathbb{Z}_2$	[10, Prop. 5.15]
C_3	λ_3	$\neq 2$	1	$A_2.\mathbb{Z}_2$	[10, Prop. 5.15]
B_5	λ_5	$\neq 2$	1	$A_4.\mathbb{Z}_2$	[10, Prop. 5.15]
D_6	λ_6	$\neq 2$	1	$A_5.\mathbb{Z}_2$	[10, Prop. 5.15]
G_2	λ_1	2	1	$U_5 A_1 T_1$	[10, Prop. 5.22]
E_7	λ_7	$\neq 2$	1	$E_6.\mathbb{Z}_2$	[10, Prop. 5.15]

Proof. Let $q = p^i$ and let $\sigma = \sigma_q$ be the standard Frobenius morphism acting on K as $t \mapsto t^\sigma = t^q$ and on G as $x_{\pm\alpha_1}(t) \mapsto x_{\pm\alpha_1}(t^q)$. Let $G = SL_2(K)$. We can view V as the space $M_{2\times 2}(K)$ of 2×2 matrices on which G acts by $g.v = gv(g^\sigma)^T$ for $v \in M_{2\times 2}(K)$ and $g \in G$. Since G preserves the determinant of v for all $v \in M_{2\times 2}(K)$, we can take the quadratic form $Q : V \to K$ as $Q(v) = \det v$. The singular 1-spaces of V are therefore the 1-spaces spanned by matrices with determinant 0. Let y be the singular 1-space spanned by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_y$ if and only if $b = c = 0$ and $d = a^{-1}$. Since $\dim G - \dim \mathcal{S}_1(V) = 1 = \dim G_y$, the element y is in a dense orbit for the G-action on $\mathcal{S}_1(V)$. Therefore, $C_{\mathcal{S}_1(V)} = T_1$. □

Proposition 3.4. *Let $G = A_2$ and $\lambda = \lambda_1 + \lambda_2$ with $p = 3$. Then the quadruple $(G, \lambda, p, 1)$ has generic ts-stabilizer $U_2 T_1$.*

Proof. By [27, Lemma 5.5], there is $y \in \mathcal{S}_1(V)$ with $G_y = U_2 T_1$. Since $\dim G - \dim \mathcal{S}_1(V) = 8 - 5 = \dim G_y$, the element y is in a dense orbit for the G-action on $\mathcal{S}_1(V)$. Therefore, $C_{\mathcal{S}_1(V)} = U_2 T_1$. □

Proposition 3.5. *Let $G = A_5$ and $\lambda = \lambda_3$ with $p = 2$. Then the quadruple $(G, \lambda, p, 1)$ has generic ts-stabilizer $U_8 A_2 T_1$.*

Proof. By [24, 2.3.1(II)], there is $y \in \mathcal{S}_1(V)$ with $G_y = U_8 A_2 T_1$. Since $\dim G - \dim \mathcal{S}_1(V) = 35 - 18 = \dim G_y$, the element y is in a dense orbit for the G-action on $\mathcal{S}_1(V)$. Therefore, $C_{\mathcal{S}_1(V)} = U_8 A_2 T_1$. □

Proposition 3.6. *Let $G = B_3$, $\lambda = \lambda_3$. Then the quadruple $(G, \lambda, p, 1)$ has generic ts-stabilizer $U_6 A_2 T_1$.*

Proof. By [17, Thm B], the group G is transitive on $\mathcal{S}_1(V)$. The generic stabilizer is the P_3-parabolic (i.e. $C_{\mathcal{S}_1(V)} = U_6 A_2 T_1$). □

Proposition 3.7. *Let $G = B_3$, $\lambda = \lambda_3$ with $k = 2$ or $k = 3$. Then the quadruple (G, λ, p, k) has generic ts-stabilizer $U_5 A_1 A_1 T_1$ if $k = 2$ and $U_3 A_2 T_1$ if $k = 3$.*

Proof. By [17, Thm B], the group G is transitive on $\mathcal{S}_4'(V)$. Let $W \in \mathcal{S}_4'(V)$. The group G is the group of fixed points of a triality automorphism of $D_4 = Cl(V)$. Therefore, G_W is isomorphic to the generic stabilizer for the action on $\mathcal{S}_1(V)$, i.e. $U_6 A_2 T_1$. Then it is easy to see ([11, Lemma 3.5]) that G_W acts on $\mathcal{G}_2(W)$ with two orbits, one with stabilizer $U_5 A_1 A_1 T_1$ and one with stabilizer $U_7 A_1 A_1 T_1$. Since every totally singular 2-space is contained in an element of $\mathcal{S}_4'(V)$, we conclude that there are at most two G-orbits on $\mathcal{S}_2(V)$. Since $\dim G - \dim \mathcal{S}_2(V) = 12$, there must be a 12-dimensional stabilizer for the G-action on $\mathcal{S}_2(V)$. The only possibility is therefore $C_{\mathcal{S}_2(V)} = U_5 A_1 A_1 T_1$.

Similarly, G_W acts on $\mathcal{G}_3(W)$ with two orbits, one with stabilizer $U_3 A_2 T_1$ and one with stabilizer $U_8 A_1 T_2$. Since $\dim G - \dim U_3 A_2 T_1 = \dim \mathcal{S}_3(V)$, we conclude that $C_{\mathcal{S}_3(V)} = U_3 A_2 T_1$. □

Proposition 3.8. *Let $G = B_\ell$, $\lambda = \lambda_\ell$ with $\ell = 4$ or $\ell = 5$, with $p = 2$ if $\ell = 5$. Then the quadruple $(G, \lambda, p, 1)$ has generic ts-stabilizer $U_7 G_2 T_1$ if $\ell = 4$ and $U_{14} B_2 T_1$ if $\ell = 5$.*

Proof. By [12, Prop. 5, Prop. 6] (when $p \neq 2$) and [11, Lemma 2.11] (when $p = 2$), there is only one orbit on $\mathcal{G}_1(V)$ with a 22-dimensional stabilizer when $\ell = 4$, with structure $U_7 G_2 T_1$, and only one orbit on

$\mathcal{G}_1(V)$ with 25-dimensional stabilizer when $\ell = 5$, with structure $U_{14}B_2T_1$. Since $\dim G - \dim \mathcal{S}_1(V) = 22$ when $\ell = 4$ and 25 when $\ell = 4$, we conclude that $C_{\mathcal{S}_1(V)} = U_7G_2T_1$ when $\ell = 4$ and $U_{14}B_2T_1$ when $\ell = 5$. □

Proposition 3.9. *Let $G = C_3$, $\lambda = \lambda_2$ with $p = 3$. Then the quadruple $(G, \lambda, p, 1)$ has generic ts-stabilizer $U_6A_1T_1$.*

Proof. By [27, Lemma 5.15], there is $y \in \mathcal{S}_1(V)$ with $G_y = U_6A_1T_1$. Since $\dim G - \dim \mathcal{S}_1(V) = 21 - 11 = \dim G_y$, the element y is in a dense orbit for the G-action on $\mathcal{S}_1(V)$. Therefore, $C_{\mathcal{S}_1(V)} = U_6A_1T_1$. □

Proposition 3.10. *Let $G = D_6$, $\lambda = \lambda_6$ with $p = 2$. Then the quadruple $(G, \lambda, p, 1)$ has generic ts-stabilizer $U_{14}B_3T_1$.*

Proof. By the proof of [11, Lemma 2.11], there is only one orbit on $\mathcal{G}_1(V)$ with a 36-dimensional stabilizer, with structure $U_{14}B_3T_1$. Since $\dim G - \dim \mathcal{S}_1(V) = 66 - 30 = \dim G_y$, the element y is in a dense orbit for the G-action on $\mathcal{S}_1(V)$. Therefore, $C_{\mathcal{S}_1(V)} = U_{14}B_3T_1$. □

Proposition 3.11. *Let $G = G_2$, $\lambda = \lambda_1$ with $p \neq 2$. Then the quadruple $(G, \lambda, p, 1)$ has generic ts-stabilizer $U_5A_1T_1$.*

Proof. By [17, Thm. A], G is transitive on singular 1-spaces of V. Therefore, a representative can be taken to be the 1-space spanned by the highest weight vector, with stabilizer $P_1 = U_5A_1T_1$. □

Proposition 3.12. *Let $G = G_2$, $\lambda = \lambda_1$. Then the quadruple $(G, \lambda, p, 2)$ has generic ts-stabilizer $U_3A_1T_1$.*

Proof. Let μ_1, μ_2, μ_3 be the positive weights $2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_1$, respectively. Given weight vectors $v_{\mu_1}, v_{-\mu_2}$, let $y := \langle v_{\mu_1}, v_{-\mu_2} \rangle$, an element of $\mathcal{S}_2(V)$. Since $n_2n_1n_2n_1n_2.\langle v_{\mu_1} \rangle = \langle v_{-\mu_2} \rangle$, we can easily determine that $G_{\langle v_{\mu_1} \rangle} \cap G_{\langle v_{-\mu_2} \rangle} = P_1 \cap P_1^{n_2n_1n_2n_1n_2} = U_3T_2$, by checking which root subgroups are in common between P_1 and $P_1^{n_2n_1n_2n_1n_2}$. Therefore, $G_y \leq U_3A_1T_1$. Since $\dim \mathcal{S}_2(V) = 7$ (in both cases $p \neq 2$ and $p = 2$), the minimum dimension of the stabilizer of any totally singular 2-space is 7. Therefore, $G_y = U_3A_1T_1 = C_{\mathcal{S}_2(V)}$. □

Proposition 3.13. *Let $G = G_2$, $\lambda = \lambda_1$ with $p = 2$. Then the quadruple $(G, \lambda, p, 3)$ has generic ts-stabilizer A_2.*

Proof. By the proof of [11, Lemma 3.4], there is an A_2-subgroup which is the stabilizer of $y \in \mathcal{S}_3(V)$. Since $\dim G - \dim \mathcal{S}_3(V) = 14 - 6 = \dim G_y$, the element y is in a dense orbit for the G-action on $\mathcal{S}_3(V)$. Therefore, $C_{\mathcal{S}_3(V)} = A_2$. □

Proposition 3.14. *Let $G = F_4$, $\lambda = \lambda_4$ with $p = 3$. Then the quadruple $(G, \lambda, p, 1)$ has generic ts-stabilizer $U_{14}G_2T_1$.*

Proof. By [6, Lemma 4.13], there is $y \in \mathcal{S}_1(V)$ such that $G_y = U_{14}G_2T_1$. Since $\dim G - \dim \mathcal{S}_1(V) = 29 = \dim G_y$, the element y is in a dense orbit for the G-action on $\mathcal{S}_1(V)$. Therefore, $C_{\mathcal{S}_1(V)} = U_{14}G_2T_1$. □

Proposition 3.15. *Let $G = E_7$, $\lambda = \lambda_7$ with $p = 2$. Then the quadruple $(G, \lambda, p, 1)$ has generic ts-stabilizer $U_{26}F_4T_1$.*

Proof. By [16, Lemma 4.3], there is only one orbit on $\mathcal{G}_1(V)$ with a 79-dimensional stabilizer, with structure $U_{26}F_4T_1$. Since $\dim G - \dim \mathcal{S}_1(V) = 133 - 54 = \dim U_{26}F_4T_1$, we must have $C_{\mathcal{S}_1(V)} = U_{26}F_4T_1$. □

4. Infinite families of quadruples

In this section, we handle the cases that appear in Table 3. The following two propositions provide a reduction to a finite list of cases.

Proposition 4.1. *Let V be a composition factor of* $\mathrm{Lie}(G)$*, with* $p \neq 2$ *if* $G = B_\ell$ *or* $G = C_\ell$*, and assume that the* 0*-weight space* V_0 *is at least* 3*-dimensional. Then* $C_{\mathcal{S}_1(V)} = C_{\mathcal{G}_1(V)}$.

Proof. The composition factors of $\mathrm{Lie}(G)$ are listed in [19, Prop. 1.10]. By assumption on V, we have $V = \mathrm{Lie}(G)/Z$, where Z is the centre of $\mathrm{Lie}(G)$. If $p = 2$, and $G = A_\ell$ with $\ell \equiv 1 \bmod 4$, or $G = D_\ell$ with $\ell \equiv 2 \bmod 4$, or $G = E_7$, then the module V is symplectic and $\mathcal{S}_1(V) = \mathcal{G}_1(V)$. In all other cases, the module V is orthogonal. The proof closely mimics [10, Lemma 4.5(ii)], but in the interest of clarity, it is fully reproduced with the appropriate changes. Note that our setup corresponds to the specific case $\theta = 1$ in [10, Lemma 4.5(ii)], which in particular means $G = H$ in the proof of [10, Lemma 4.5(ii)]. Let

$$W^\ddagger = \{w \in W : \exists \xi \in K^*, \forall v \in V_0, w.v = \xi v\},$$

and let $N^\ddagger$ be the pre-image of $W^\ddagger$ under the quotient map $N \to W$. Let Y be $\mathcal{S}_1(V_0)$ and let $\hat{Y}_1$ be the subset of Y consisting of 1-spaces of V_0 spanned by images of regular semisimple elements in $\mathrm{Lie}(T)$. Since Z does not contain regular semisimple elements, the set $\hat{Y}_1$ is nonempty, and thus dense in Y. Since $\dim V_0 \geq 3$, the span of Y is the full V_0. Therefore, any element in N which fixes all $y \in Y$ must be in $N^\ddagger$. Thus, given $w \in W \setminus W^\ddagger$, take $n \in N$ with $nT = w$; the set of elements of Y fixed by n is a proper closed subvariety of Y. Let $\hat{Y}_2$ be the complement of the union of these subvarieties as w runs over $W \setminus W^\ddagger$. Set $\hat{Y} = \hat{Y}_1 \cap \hat{Y}_2$, a dense open subset of Y.

Let $y \in \hat{Y}$. By [11, Lemma 2.1], two elements of $\mathcal{G}_1(V_0)$ are in the same G-orbit if and only if they are in the same W-orbit. Therefore, $G.y \cap Y$ is finite and $\dim(\overline{G.y \cap Y}) = 0$. Also, since y is spanned by a regular semisimple element, we have $G_y^0 = T$, and therefore, $\dim(\overline{G.y}) = \dim G - \dim T$. Finally, note that since $V = \mathrm{Lie}(G)/Z$, we have $(C_G(V_0))^0 = T$, and therefore, $\dim V - \dim V_0 = \dim G - \dim T$ (see the proof of [11, Lemma 2.4]). Thus,

$$\dim \mathcal{S}_1(V) - \dim(\overline{G.y}) = \dim V - 2 - \dim(\overline{G.y}) =$$
$$= \dim G - \dim T + \dim V_0 - 2 - \dim(\overline{G.y}) =$$
$$= \dim Y - \dim(\overline{G.y \cap Y}) = \dim Y + \mathrm{codim}\,\mathrm{Tran}_G(y, Y) - \dim(\overline{G.y}),$$

where the last step uses Lemma 2.13. By definition, this proves that y is Y-exact. The conditions of Lemma 2.15 hold and $C_{\mathcal{S}_1(V)} = T.W^\ddagger = C_{\mathcal{G}_1(V)}$.   □

Proposition 4.2. *Let* $G = C_\ell$*,* $\lambda = \lambda_2$*. Furthermore, assume that the* 0*-weight space* V_0 *is at least* 3*-dimensional. Then* $C_{\mathcal{S}_1}(V) = C_{\mathcal{G}_1}(V) = A_1^\ell$.

Proof. If $p = 2$ and $\ell \equiv 2 \bmod 4$, then the module V is symplectic and $\mathcal{S}_1(V) = \mathcal{G}_1(V)$. Therefore, assume that when $p = 2$, we have $\ell \not\equiv 2 \bmod 4$, which implies that the module V is orthogonal. We use the setup of [10, Prop. 5.13] combined with the approach of [10, Lemma 4.5(ii)], which we saw in action in Proposition 4.1. Inside $\bigwedge^2 V_{nat}$, we have submodules $X_1 = \{\sum_{i<j} \rho_{ij} e_i \wedge e_j + \sum_{i<j} \sigma_{ij} f_i \wedge f_j + \sum_{i,j} \tau_{ij} e_i \wedge f_j : \sum_i \tau_{ii} = 0\}$ and $X_2 = \langle \sum_i e_i \wedge f_i \rangle$. If $p \nmid \ell$ then $V = X_1$, otherwise $X_2 < X_1$ and $V = X_1/X_2$. In all cases, $V = X_1/(X_1 \cap X_2)$. Let $x_i = e_i \wedge f_i$. The 0-weight space is $V_0 = \{\sum a_i x_i + (X_1 \cap X_2) : \sum a_i = 0\}$. Then G fixes a nondegenerate quadratic form on V such that $Q(a_i x_i + (X_1 \cap X_2)) = \sum a_i^2 + \sum_{i<j} a_i a_j$ (see [9, p. 8.1.2] when $p = 2$). Let $Y = \mathcal{S}_1(V_0)$ and let

$$\hat{Y}_1 = \left\{ \langle v \rangle \in Y, v = \sum a_i x_i + (X_1 \cap X_2) : a_i \neq a_j \text{ if } i \neq j \right\},$$

a dense subset of Y. Then $y \in \hat{Y}_1$ is fixed by $C = A_1^\ell = \bigcap_i G_{\langle e_i, f_i \rangle}$. Any minimal connected overgroup of C in G is isomorphic to $C_2 A_1^{\ell-2}$, which does not fix $y \in \hat{Y}_1$ because of the condition on the coefficients.

Therefore, for any $y \in \hat{Y}_1$, we have $(G_y)^0 = C$. Let $N = N_G(C)/C$, a group isomorphic to $Sym(\ell)$. Let

$$N^{\ddagger} = \{n \in N : \exists \xi \in K^*, \forall v \in V_0, n.v = \xi v\}.$$

Now assume that $n \in N$ fixes all $y \in Y$. Since $\dim V_0 \geq 3$, the span of Y is the full V_0. Therefore, any element in N which fixes all $y \in Y$ must be in $N^{\ddagger}$. Thus, the set of elements fixed by $n \in N \setminus N^{\ddagger}$ is a proper closed subvariety of Y. Let $\hat{Y}_2$ be the complement of the union of these subvarieties as n runs over $N \setminus N^{\ddagger}$. Set $\hat{Y} = \hat{Y}_1 \cap \hat{Y}_2$, a dense open subset of Y. Finally, note that the proof of [10, Prop. 5.13] shows that $N^{\ddagger} = 1$. Let $y \in \hat{Y}$. Two elements of $\mathcal{G}_1(V_0)$ are in the same G-orbit if and only if they are in the same N-orbit, and therefore, $\dim(\overline{G.y} \cap Y) = 0$. Also, $\dim V - \dim V_0 = \dim G - \dim C$, by the proof of [11, Lemma 2.4]. Therefore, as in Proposition 4.1, we get

$$\dim \mathcal{S}_1(V) - \dim(\overline{G.y}) = \dim V - 2 - \dim(\overline{G.y}) =$$
$$= \dim G - \dim C + \dim V_0 - 2 - \dim(\overline{G.y}) =$$
$$= \dim Y - \dim(\overline{G.y} \cap Y),$$

proving that y is Y-exact. The conditions of Lemma 2.15 hold and $C_{\mathcal{S}_1(V)} = C.N^{\ddagger} = C = A_1^{\ell}$. □

Proposition 4.3. *Let $G = B_2$ and $\lambda = 2\lambda_2$ with $p \neq 2$, or $G = A_2$ and $\lambda = \lambda_1 + \lambda_2$ with $p \neq 3$, or $G = A_3$ and $\lambda_1 + \lambda_3$ with $p = 2$, or $G = G_2$ and $\lambda = \lambda_2$ with $p \neq 3$. Then $C_{\mathcal{S}_1(V)}$ is respectively $T_2.\mathbb{Z}_4$, $T_2.\mathbb{Z}_3$, $T_3.Alt(4)$, $T_2.\mathbb{Z}_6$.*

Proof. In all of these cases, the module V is a composition factor of $\text{Lie}(G)$ and the 0-weight space is 2-dimensional, and we will show that the generic stabilizer is the stabilizer of one of the two singular 1-spaces of V_0.

Let $G = B_2$ and $\lambda = 2\lambda_2$ with $p \neq 2$. Then $V = \text{Lie}(G)$, and we can take $V_0 = \{\text{diag}(a, b, 0, -b, -a) : a, b \in K\}$. Since $p \neq 2$, the group G fixes the nondegenerate quadratic form Q induced by the Killing form. Let $v = \text{diag}(a, b, 0, -b, -a)$ be a singular element of V_0. Then since $Q(v) = 0$, we know that $a^2 + b^2 = 0$. Since v is regular semisimple, we must have $(G_{\langle v \rangle})^0 = T_2$. We then find that $W_{\langle v \rangle} = \langle w \rangle$, where w is an element of order 4 sending $\text{diag}(a, b, 0, -b, -a) \mapsto \text{diag}(b, -a, 0, a, -b)$. Since $\dim G - \dim \mathcal{S}_1(V) = 2 = \dim G_{\langle v \rangle}$, we conclude that $C_{\mathcal{S}_1(V)} = T_2.\mathbb{Z}_4$.

Let $G = A_2$ and $\lambda = \lambda_1 + \lambda_2$ with $p \neq 3$. Then $V = \text{Lie}(G)$, and we can take $V_0 = \{\text{diag}(a, b, -a - b) : a, b \in K\}$. If $p \neq 2$, a nondegenerate symmetric bilinear form preserved by G is given by the Killing form. If $p = 2$, we find an explicit description of a nondegenerate quadratic form preserved by G in [3, §5.1]. Let $v = \text{diag}(a, b, -a - b)$ be a singular element of V_0, which implies that $a^2 + b^2 + ab = 0$. Since v is regular semisimple, we must have $(G_{\langle v \rangle})^0 = T_2$. As $a^2 + b^2 + ab = 0$, we then find that $W_{\langle v \rangle} = \langle w \rangle$, where w is a 3-cycle in W. Since $\dim G - \dim \mathcal{S}_1(V) = 2 = \dim G_{\langle v \rangle}$, we conclude that $C_{\mathcal{S}_1(V)} = T_2.\mathbb{Z}_3$.

Let $G = A_3$ and $\lambda = \lambda_1 + \lambda_3$ with $p = 2$. Then $V = \mathfrak{sl}_4/\langle I \rangle$, where I is the identity 4×4 matrix, and we can take $V_0 = \{\text{diag}(a, b, a + b, 0) + \langle I \rangle : a, b \in K\}$. Let $v = \text{diag}(a, b, a + b, 0) + \langle I \rangle$ be a singular element of V_0, which implies that $a^2 + b^2 + ab = 0$ (again see [3, §5.1]). Since $\text{diag}(a, b, a+b, 0)$ is regular semisimple, we must have $(G_{\langle v \rangle})^0 = T_3$. As $a^2 + b^2 + ab = 0$, we then find that $W_{\langle v \rangle} = \langle \tau_1, \tau_2, w \rangle$, where w is a 3-cycle in W sending $\text{diag}(a, b, a + b, 0) \mapsto \text{diag}(a + b, a, b, 0)$, τ_1 is an element of order 2 sending $\text{diag}(a, b, a + b, 0) \mapsto \text{diag}(0, a + b, b, a)$ and τ_2 is an element of order 2 sending $\text{diag}(a, b, a + b, 0) \mapsto \text{diag}(a + b, 0, a, b)$. Since $\dim G - \dim \mathcal{S}_1(V) = 3 = \dim G_{\langle v \rangle}$, we conclude that $C_{\mathcal{S}_1(V)} = T_3.Alt(4)$.

Let $G = G_2$ and $\lambda = \lambda_2$ with $p \neq 3$. Then $V = \text{Lie}(G)$. We view G as a subgroup of B_3, so that we can take $V_0 = \{\text{diag}(a, b, -a - b, 0, -a, -b, a + b) : a, b \in K\}$. Let $v = \text{diag}(a, b, -a - b, 0, -a, -b, a + b)$ be a singular element of V_0, which implies that $a^2 + b^2 + ab = 0$ (see [3, §5.1]). Since v is regular semisimple, we must have $(G_{\langle v \rangle})^0 = T_2$. As $a^2 + b^2 + ab = 0$, we then find that $W_{\langle v \rangle} = \langle \tau, w \rangle$, where τ is an element of order 2 sending $v \mapsto -v$ and w is an element of order 3 sending $\text{diag}(a, b, -a - b, 0, a + b, -b, -a) \mapsto \text{diag}(-a - b, a, b, 0, -b, -a, a + b)$. Since $\dim G - \dim \mathcal{S}_1(V) = 2 = \dim G_{\langle v \rangle}$, we conclude that $C_{\mathcal{S}_1(V)} = T_2.\mathbb{Z}_6$. □

Proposition 4.4. *Let $G = C_3$ and $\lambda = \lambda_2$ with $p \neq 3$, or $G = C_4$ and $\lambda = \lambda_2$ with $p = 2$, or $G = D_4$ and $\lambda = \lambda_2$ with $p = 2$. Then $C_{\mathcal{S}_1(V)}$ is respectively $A_1^3.\mathbb{Z}_3$, $A_1^4.Alt(4)$, $T_4.(2^3.Alt(4))$.*

Proof. In all of these cases, the module V is a composition factor of $\wedge^2 V_{nat}$, the 0-weight space is 2-dimensional, and the generic stabilizer is the stabilizer of one of the two singular 1-spaces of V_0. Let X_1, X_2 be as in the proof of Proposition 4.2, so that $V = X_1/X_2$.

Let $G = C_3$ and $\lambda = \lambda_2$ with $p \neq 3$. Then $X_2 = 0$, and we can take

$$V_0 = \{ae_1 \wedge f_1 + be_2 \wedge f_2 - (a+b)e_3 \wedge f_3 : a, b, \in K\}.$$

Let $v = ae_1 \wedge f_1 + be_2 \wedge f_2 - (a+b)e_3 \wedge f_3$ be a singular element of V_0. Then since $Q(v) = 0$, we know that $a^2 + b^2 + ab = 0$. Since $a \neq b$, we must have $(G_{\langle v \rangle})^0 = A_1^3$ as in the proof of Proposition 4.2. We then find that $W_{\langle v \rangle} = \langle w \rangle$, where w is a 3-cycle in W. Since $\dim G - \dim \mathcal{S}_1(V) = 9 = \dim G_{\langle v \rangle}$, we conclude that $C_{\mathcal{S}_1(V)} = A_1^3.\mathbb{Z}_3$.

The remaining two cases are entirely similar, with the result for D_4 being derived from C_4, since $V_{D_4}(\lambda_2) = V_{C_4}(\lambda_2) \downarrow D_4$ when $p = 2$. □

5. Remaining quadruples

This section is where we deal with the remaining cases (i.e., all the possibilities listed in Table 4).

Proposition 5.1. *The generic ts-stabilizers for the ts-small quadruples in Table 6 are as given.*

Proof. In each of these cases, except for $(B_6, \lambda_6, p, 1)$ with $p \neq 2$, it was proven in [26] or [27] that G has a dense orbit on $\mathcal{S}_k(V)$. The structure of the generic stabilizer is as given in [26][27], with the appropriate reference listed in Table 6. Note that for the case $(B_6, \lambda_6, 2, 1)$, Proposition 5.2 clarifies the lackluster explanation found in [26].

Finally, in the case $(B_6, \lambda_6, p, 1)$ with $p \neq 2$, the module V is symplectic, and we have $\mathcal{S}_k(V) = \mathcal{G}_k(V)$, with the result following from [10]. □

Proposition 5.2. *Let $G = B_6$, $\lambda = \lambda_6$ with $p = 2$. Then the quadruple $(G, \lambda, p, 1)$ has generic ts-stabilizer $(U_5 A_1)^2.\mathbb{Z}_2$.*

Proof. This result is already listed in [26, Thm. 1], albeit without a full explanation. The quadruple $(D_5, \lambda_5, 2, 1)$ has generic stabilizer $(G_2 G_2).\mathbb{Z}_2$ by [10]. By [26, Lemma 5.17], we deduce that $C_{\mathcal{S}_1(V)}$ is isomorphic to the generic stabilizer of the action of $(G_2 G_2).\mathbb{Z}_2$ on $V_{G_2}(\lambda_1) \oplus V_{G_2}(\lambda_1)$. Since G_2 is transitive on nonzero vectors of $V_{G_2}(\lambda_1)$, the generic stabilizer for this action is easily seen to be $(U_5 A_1)^2.\mathbb{Z}_2$. □

Table 6. *Remaining ts-small quadruples with known generic ts-stabilizer.*

G	λ	p	k	$C_{\mathcal{S}_k(V)}$	Reference
A_1	$\lambda_1 + p^i \lambda_1$	$< \infty$	2	$U_1 T_1$	[27, Prop. 5.1]
A_1	$3\lambda_1$	> 3	2	$Alt(4)$	[27, Prop. 5.2]
A_1	$4\lambda_1$	> 3	1	$Alt(4)$	[26, Prop. 4.1]
A_1	$4\lambda_1$	> 3	2	$Sym(3)$	[27, Prop. 5.2]
A_2	$\lambda_1 + \lambda_2$	3	2	U_1	[27, Prop. 5.3]
B_4	λ_4	2	2	$U_5 A_1 A_1$	[27, Prop. 5.55]
B_4	λ_4	$\neq 2$	2	$A_1(A_2.\mathbb{Z}_2)$	[27, Prop. 5.55]
B_6	λ_6	$\neq 2$	1	$A_2^2.\mathbb{Z}_2$	[10, Prop. 5.17]
B_6	λ_6	2	1	$(U_5 A_1)^2.\mathbb{Z}_2$	Proposition 5.2
C_3	λ_2	3	2	$U_1(T_1.\mathbb{Z}_2)$	[27, Prop. 5.14]
F_4	λ_4	3	2	$U_1(A_2.\mathbb{Z}_2)$	[27, Prop. 5.29]
F_4	λ_4	$\neq 3$	1	$D_4.\mathbb{Z}_3$	[26, Prop. 6.6]

Proposition 5.3. *Let $G = G_2$, $\lambda = \lambda_1$ with $p \neq 2$. Then the quadruple $(G, \lambda, p, 3)$ has generic ts-stabilizer A_2.*

Proof. As in Proposition 3.13, we know that there is $y \in \mathcal{S}_3(V)$ with $G_y = A_2$. Since $\dim G - \dim A_2 = 6 = \dim \mathcal{S}_3(V)$, we conclude that $C_{\mathcal{S}_3(V)} = A_2$. □

We now relax slightly the condition that the group acting should be simple: we allow a product of isomorphic simple groups, possibly extended by a graph automorphism. If the connected group is of the form $G_1 G_2 G_3$, we write $\lambda = \mu_1 \otimes \mu_2 \otimes \mu_3$ to mean $V_G(\lambda) = V_{G_1}(\mu_1) \otimes V_{G_2}(\mu_2) \otimes V_{G_3}(\mu_3)$, where each μ_i is a dominant weight for G_i.

Lemma 5.4. *Let $G = A_1^3$, $\lambda = \lambda_1 \otimes \lambda_1 \otimes \lambda_1$ with $p = 2$. Then there is an open dense subset $\hat{Y}$ of $\mathcal{S}_2(V)$ such that for all $y_1 \neq y_2 \in \hat{Y}$, we have $G_{y_1} \simeq G_{y_2} = U_1.\mathbb{Z}_2$, and G_{y_1} is not conjugate to G_{y_2}.*

Proof. This is proved in [27, Lemma 5.69]. □

Lemma 5.5. *Let $G = A_1^3$, $\lambda = \lambda_1 \otimes \lambda_1 \otimes \lambda_1$ with $p \neq 2$. The quadruple $(G, \lambda, p, 2)$ has generic ts-stabilizer $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. We can recover this result by a slight change of the proof of [10, Prop. 6.7]. Let $G = SL_2(K)^3$ with basis e, f for V_{nat}, so that $V = V_{nat} \otimes V_{nat} \otimes V_{nat}$. Then, like in the proof of [10, Prop. 6.7], given $\mathbf{a} = (a_1, a_2, a_3, a_4)$, let

$$v^{(1)} = a_1 e \otimes e \otimes e + a_2 e \otimes f \otimes f + a_3 f \otimes e \otimes f + a_4 f \otimes f \otimes e,$$
$$v^{(2)} = a_1 f \otimes f \otimes f + a_2 f \otimes e \otimes e + a_3 e \otimes f \otimes e + a_4 e \otimes e \otimes f.$$

Then $y_\mathbf{a} := \langle v^{(1)}, v^{(2)} \rangle$ is totally singular if and only if $a_1^2 + a_2^2 + a_3^2 + a_4^2 = 0$. In the proof of [10, Prop. 6.7], they then proceed to define $\hat{Y}$ as a dense subset of $Y := \{y_\mathbf{a} : \mathbf{a} \neq (0,0,0,0)\}$ by requiring certain polynomials on the coefficients to be nonzero. One of these conditions is that $a_1^2 + a_2^2 + a_3^2 + a_4^2 \neq 0$. However, this is later only used for the case $p = 2$. Therefore, we can modify the definition of $\hat{Y}$ in the proof of [10, Prop. 6.7] and still end up with an open dense subset of $\mathcal{G}_2(V)$ where all stabilizers are conjugate to $\mathbb{Z}_2.\mathbb{Z}_2$. The difference is that now this set will also contain totally singular 2-spaces, and therefore, $C_{\mathcal{S}_2(V)} = C_{\mathcal{G}_2(V)} = \mathbb{Z}_2.\mathbb{Z}_2$. □

Proposition 5.6. *Let $G = E_7$ and $\lambda = \lambda_7$, or $G = D_6$ and $\lambda = \lambda_6$, or $G = A_5$ and $\lambda = \lambda_3$. Then the quadruple $(G, \lambda, p, 2)$ has no generic ts-stabilizer if $p = 2$ but has a semi-generic ts-stabilizer. If $p \neq 2$, we have $C_{\mathcal{S}_2(V)} = C_{\mathcal{G}_2(V)}$.*

Proof. We shall describe how to use the proof of [10, Proposition 6.28] to reach the conclusion. In [10, Proposition 6.28], the authors determine the generic stabilizer for the G-action on all 2-spaces. They do so in the following manner. They define a certain 8-space $V_{[0]}$ spanned by pairs of opposite weight vectors. This 8-space is the fixed point space of a subgroup A of G, where $A = D_4, A_1^3, T_2$ according to whether $G = E_7, D_6, A_5$, respectively. They define a dense subset $\hat{Y}_1$ of $Y := \mathcal{G}_2(V_{[0]})$ with the property that for any $y \in Y$, we have $\text{Tran}_G(y, Y) = AA_1^3.Sym(3)$, where A_1^3 acts on $V_{[0]}$ as $\lambda_1 \otimes \lambda_1 \otimes \lambda_1$. The set $\hat{Y}_1$ is defined by requiring certain expressions in terms of the coefficients of the given $V_{[0]}$ basis to be nonzero. Here, the key observation is that these conditions do not exclude all totally singular 2-spaces of $V_{[0]}$, and therefore, $\hat{Y}_1^{\mathcal{S}} := \hat{Y}_1 \cap \mathcal{S}_2(V_{[0]})$ is a dense subset of $Y^{\mathcal{S}} := \mathcal{S}_2(V_{[0]})$. Given $y \in \hat{Y}_1^{\mathcal{S}}$, since $\text{Tran}_G(y, Y) = AA_1^3.Sym(3)$, we also have $\text{Tran}_G(y, Y^{\mathcal{S}}) = AA_1^3.Sym(3)$. Furthermore, they show that $G_y = A(A_1^3)_y$ for all $y \in \hat{Y}_1$.

Assume $p \neq 2$. Then by Lemma 5.5, there exists a dense open subset $\hat{Y}_2^{\mathcal{S}}$ of $Y^{\mathcal{S}}$ such that every stabilizer is A_1^3-conjugate to $\mathbb{Z}_2.\mathbb{Z}_2$. Taking the intersection with $\hat{Y}_1^{\mathcal{S}}$, we get an open dense subset $\hat{Y}^{\mathcal{S}}$ of $Y^{\mathcal{S}}$. For all $y \in \hat{Y}^{\mathcal{S}}$, we know that G_y is conjugate to $A.\mathbb{Z}_2.\mathbb{Z}_2$. In each case, the codimension of the

transporter of $y \in \hat{Y}^{\mathcal{S}}$ into $Y^{\mathcal{S}}$ is equal to the codimension of $Y^{\mathcal{S}}$ in $\mathcal{S}_2(V)$. Therefore, every $y \in \hat{Y}^{\mathcal{S}}$ is $Y^{\mathcal{S}}$-exact. By Lemma 2.15, we conclude that $C_{\mathcal{S}_2(V)}$ is $D_4.\mathbb{Z}_2.\mathbb{Z}_2$, $A_1^3.\mathbb{Z}_2.\mathbb{Z}_2$, $T_2.\mathbb{Z}_2.\mathbb{Z}_2$ according to whether $G = E_7, D_6, A_5$, respectively. These generic stabilizers are the same as for the action on all 2-spaces.

Now assume that $p = 2$. By Lemma 5.4, there exists a dense open subset $\hat{Y}_2^{\mathcal{S}}$ of $Y^{\mathcal{S}}$ such that every stabilizer has a 1-dimensional connected component and stabilizers are pairwise non-conjugate. Taking the intersection with $\hat{Y}_1^{\mathcal{S}}$, we get a dense open subset $\hat{Y}_3^{\mathcal{S}}$ of $Y^{\mathcal{S}}$. Let $y_1, y_2 \in \hat{Y}_3^{\mathcal{S}}$ and assume that $x.y_1 = y_2$. Then $x \in N_G(A)$, and since V_0 is the fixed space of A, we must have $x \in \text{Tran}_G(y_1, Y^{\mathcal{S}}) = AA_1^3.Sym(3)$. Therefore, y_1 and y_2 must be in the same $A_1^3.Sym(3)$-orbit. However by construction, y_1 and y_2 are not in the same A_1^3-orbit, and therefore, there exists a dense open subset $\hat{Y}^{\mathcal{S}}$ of $Y^{\mathcal{S}}$, contained in $\hat{Y}_3^{\mathcal{S}}$, such that any two distinct elements have non-conjugate stabilizers. In each case, the codimension of the transporter of $y \in \hat{Y}^{\mathcal{S}}$ into $Y^{\mathcal{S}}$ is equal to the codimension of $Y^{\mathcal{S}}$ in $\mathcal{S}_2(V)$. Therefore, every $y \in \hat{Y}^{\mathcal{S}}$ is $Y^{\mathcal{S}}$-exact. By Lemma 2.16, there is no generic stabilizer in the action of G on $\mathcal{S}_2(V)$. In fact, Lemma 2.14 shows that we have semi-generic stabilizers $D_4.U_1.\mathbb{Z}_2$, $A_1^3.U_1.\mathbb{Z}_2$, $T_2.U_1.\mathbb{Z}_2$ according to whether $G = E_7, D_6, A_5$, respectively. $\qquad\square$

Lemma 5.7. *Let $G = A_2$, $\lambda = \lambda_1 + \lambda_2$ with $p \neq 3$. Let τ be a graph automorphism of G. Then the quadruple $(G\langle\tau\rangle, \lambda, p, 2)$ has generic ts-stabilizer $Sym(3)$.*

Proof. Take $G = SL_3(K)$ acting on $V = \mathfrak{sl}_3(K)$ by conjugation. Here, $Z(G) = \langle\text{diag}(\omega, \omega, \omega)\rangle$, where ω is a nontrivial third-root of unity. Note that $Z(G)$ acts trivially on V. Let τ be the graph automorphism acting on G as $g \mapsto g^{-T}$ and on V as $v \mapsto -v^T$. We have that G fixes a nondegenerate quadratic form on V given by

$$Q\big((m_{ij})_{ij}\big) = m_{11}^2 + m_{22}^2 + m_{11}m_{22} + \sum_{i<j} m_{ij}m_{ji}.$$

For $1 \leq i, j \leq 3$, let e_{ij} denote a 3×3 matrix with a 1 in position (i, j) and zeroes everywhere else. There are three G-orbits on $\mathcal{S}_1(V)$, which we label as Δ_1, Δ_2 and Δ_3, respectively with representatives $\langle e_{13}\rangle$, $\langle e_{12} + e_{23}\rangle$, $\langle e_{11} + \omega e_{22} + \omega^2 e_{33}\rangle$. This follows directly from considering the Jordan Canonical Form of elements in V. The stabilizers are respectively $B = U_3T_2$, $Z(G).U_2T_1$ and $T_2.\mathbb{Z}_3$.

Let

$$u_{bc} = \begin{pmatrix} & 1 & \\ & & b \\ c & & \end{pmatrix}, \quad v_{ad} = \begin{pmatrix} & & 1 \\ a & & \\ & d & \end{pmatrix};$$

$$Y = \{\langle u_{bc}, v_{ad}\rangle : a + c + bd = 0\}.$$

The set Y is a 3-dimensional subvariety of $\mathcal{S}_2(V)$. Let

$$\hat{Y} = \left\{\langle u_{bc}, v_{ad}\rangle \in Y : abcd \neq 0, \frac{(bd - c)^2}{bcd} \neq 0, -\frac{3}{2}, -3\right\},$$

where we disregard the expression $\frac{(bd-c)^2}{bcd} \neq -\frac{3}{2}$ if $p = 2$. Then $\hat{Y}$ is a dense subset of Y. Let $y = \langle u_{bc}, v_{ad}\rangle \in \hat{Y}$. Then $\langle u_{bc}\rangle$ and $\langle v_{ad}\rangle$ are in Δ_3 since they are spanned by rank-3 matrices. Now consider $v = u_{bc} + \lambda v_{ad}$. We have $\det v = bc + ad\lambda^3$, which implies that there are precisely three 1-spaces of y not belonging to Δ_3. It is clear that none of these have rank 1, and therefore, all three of these 1-spaces belong to Δ_2. Let $\lambda_1, \lambda_2, \lambda_3$ be the three distinct roots of $q(x) = bc + adx^3$, so that $\langle u_{bc} + \lambda_i v_{ad}\rangle \in \Delta_2$. Then

$$G_y \leq (G_{\langle u_{bc} + \lambda_1 v_{ad}\rangle} \cap G_{\langle u_{bc} + \lambda_2 v_{ad}\rangle}).Sym(3).$$

Let $g^* = \operatorname{diag}(1, \omega, \omega^2) \in G$. Then $g^*.u_{bc} = \omega^2 u_{bc}$ and $g^*.v_{ad} = \omega v_{ad}$, implying $g^* \in G_y$ and $Z(G)\langle g^* \rangle \simeq Z(G).\mathbb{Z}_3 \le G_y$. Take $\mu, \nu \in K$ with $\mu^3 = \frac{1}{ac}$ and $\nu^3 = \frac{ab}{d}$, and let

$$g^{\ddagger} = \operatorname{diag}(\mu, \nu, (\mu\nu)^{-1})\tau,$$

an element of $N_{G\langle\tau\rangle}(\langle g^*\rangle)$ fixing y. All elements of the form $h\tau$ with $h \in T$ are conjugate under T. Therefore, for any $y_1, y_2 \in \hat{Y}$, we know that $(G\langle\tau\rangle)_{y_1}$ and $(G\langle\tau\rangle)_{y_2}$ contain a subgroup conjugate to $Z(G).\langle g^*\rangle\langle g^{\ddagger}\rangle \simeq Z(G).Sym(3)$.

We now proceed in the following way. We show that the stabilizer in $G_{\langle u_{bc} + \lambda_1 v_{ad}\rangle}$ of y is $Z(G)$. This in turn implies that $G_y = Z(G).\mathbb{Z}_3$. Since $\langle u_{bc} + \lambda_1 v_{ad}\rangle \in \Delta_2$, we are able to find an element of G sending $u_{bc} + \lambda_1 v_{ad} \mapsto e_{12} + e_{23}$. This is achieved by a scalar multiple of

$$R = \begin{pmatrix} 0 & 0 & 1 \\ c & \lambda_1 d & 0 \\ -\frac{bc}{\lambda_1} & c & -\lambda_1 a \end{pmatrix}.$$

We now have

$$R.y = \left\langle \begin{pmatrix} a_1 & 0 & 1 \\ a_2 & a_1 & 0 \\ a_3 & -a_2 & -2a_1 \end{pmatrix}, e_{12} + e_{23} \right\rangle,$$

where $a_1 = \lambda_1 a$, $a_2 = \frac{a}{d}(bd - c) = \frac{c^2}{d} - b^2 d$, and $a_3 = \frac{3abc}{\lambda_1 d} = -3a_1^2$. By assumption, $a_1 a_2 a_3 \ne 0$. Now let $g \in G_{\langle e_{12}+e_{23}\rangle}$. Multiplying by an element of $Z(G)$, we can assume that $g = ns$, where $s = \operatorname{diag}(\frac{1}{t}, 1, t)$ and $n = \begin{pmatrix} 1 & n_1 & n_2 \\ 0 & 1 & n_1 \\ 0 & 0 & 1 \end{pmatrix}$. Then

$$(m_{ij})_{ij} := g.\begin{pmatrix} a_1 & 0 & 1 \\ a_2 & a_1 & 0 \\ -3a_1^2 & -a_2 & -2a_1 \end{pmatrix} =$$

$$= \begin{pmatrix} a_1 + a_2 t n_1 - 3a_1^2 t^2 n_2 & m_{12} & m_{13} \\ t(a_2 - 3a_1^2 t n_1) & a_1 - 2a_2 t n_1 + 3a_1^2 t^2 n_1^2 & m_{23} \\ -3a_1^2 t^2 & -t(a_2 - 3a_1^2 t n_1) & -2a_1 + a_2 t n_1 - 3a_1^2 t^2 (n_1^2 - n_2) \end{pmatrix},$$

where

$$m_{12} = t(a_2(-n_1^2 - n_2) + 3a_1^2 t n_1 n_2),$$

$$m_{13} = \frac{1}{t^2} + a_2 t n_1^3 - 3a_1 n_2 - 3a_1^2 t^2 n_2(n_1^2 - n_2),$$

$$m_{23} = -3a_1 n_1 - 3a_1^2 t^2 (n_1^2 - n_2) + a_2 t(n_1^2 - n_2).$$

Assume that g fixes $R.y$. Since $m_{31} = -3a_1^2 t^2$, we must have $(m_{ij})_{ij} = t^2 R.y + \alpha(e_{12} + e_{23})$ for some $\alpha \in K$. Therefore, $m_{21} = t^2 a_2$ which implies $n_1 = \frac{a_2(1-t)}{3a_1^2 t}$. Similarly, we must have $m_{11} = m_{22}$ which implies $n_2 = \frac{a_2 n_1}{a_1^2 t} - n_1^2$. Then $m_{11} = t^2 a_1$ implies that either $t = \pm 1$ or $a_2^2 = 3a_1^3$. Assume that $a_2^2 = 3a_1^3$. Since $a_2 = \frac{a}{d}(bd - c)$ and $a_1 = \lambda_1 a$, the equation $a_2^2 = 3a_1^3$ implies $(bd - c)^2 = -3bcd$, contrary to the definition of $\hat{Y}$. Therefore, $t = \pm 1$. If $t = 1$, we immediately get $n_1 = n_2 = 0$, concluding. Assume therefore that $t = -1$ and $p \ne 2$. Then $m_{12} = m_{23}$ forces $2a_2^2 = 3a_1^3$, which implies $2(bd - c)^2 = -3bcd$, which is impossible by assumption on $\hat{Y}$.

This concludes the proof that $G\langle\tau\rangle_y = Z(G)\langle g^*\rangle\langle g^{\ddagger}\rangle$. Now for any $y \in \hat{Y}$, since $G_y = Z(G)\langle g^*\rangle$, any element in $\mathrm{Tran}_G(y, Y)$ must be in $N(\langle g^*\rangle)$. We know that $N(\langle g^*\rangle) = T_2.\mathbb{Z}_3$, and it is easy to check that $T_2 \in \mathrm{Tran}_G(y, Y)$. Therefore, $\dim \mathrm{Tran}_G(y, Y) = 2$, and then since $\dim \mathcal{S}_2(V) - \dim G = 1$, the set $\hat{Y}$ is Y-exact. By Lemma 2.15, we conclude that the quadruple $(G\langle\tau\rangle, \lambda, p, 2)$ has generic ts-stabilizer $Sym(3)$.   □

Proposition 5.8. *Let $G = F_4$ and $\lambda = \lambda_4$, or $G = C_3$ and $\lambda = \lambda_2$, with $p \neq 3$. Then the quadruple $(G, \lambda, p, 2)$ has generic ts-stabilizer $A_2.Sym(3)$ or $T_1.Sym(3)$, respectively.*

Proof. This is entirely similar to the proof of Proposition 5.6 and relies on the construction used in [10, Prop. 6.26]. All we have to observe is that the set $\hat{Y}_1$ defined in the proof of [10, Prop. 6.26] does indeed contain totally singular 2-spaces. This follows from the observation in the proof of [10, Prop. 6.25], where the authors need to show that the set $\hat{Y}_1$ is nonempty. They do so by saying that $v^{(1)} = a_{33}e_{\gamma_{33}} + a_{12}e_{\gamma_{12}} + a_{21}e_{\gamma_{21}}$, $v^{(2)} = b_{11}e_{\gamma_{11}} + b_{23}e_{\gamma_{23}} + b_{32}e_{\gamma_{32}}$, and $v^{(3)} = e_{\gamma_{22}} + e_{\gamma_{31}} + e_{\gamma_{13}}$ span a 3-space in $\hat{Y}_1$ if $(a_{12}b_{23} - a_{33}b_{11})(a_{21}b_{32} - a_{12}b_{23})(a_{33}b_{11} - a_{21}b_{32}) \neq 0$. Clearly, there are totally singular 2-spaces $\langle v^{(1)}, v^{(2)}\rangle$ with coefficients satisfying this condition, and therefore, the set $\hat{Y}_1$ defined in the proof of [10, Prop. 6.26] does contain totally singular 2-spaces.

Once we understand this, the generic stabilizer is, respectively, $A_2.X$, $T_1.X$, where X is the generic stabilizer for the action of $A_2.\mathbb{Z}_2$ on $\mathcal{S}_2(\lambda_1 + \lambda_2)$. By Lemma 5.7, we conclude.   □

Proposition 5.9. *Let $G = A_2$ and $\lambda = \lambda_1 + \lambda_2$ with $p = 3$. Then $C_{\mathcal{S}_3}(V) = T_2.\mathbb{Z}_3$.*

Proof. Let α_1, α_2 be the fundamental roots for A_2 and let $\alpha_3 = \alpha_1 + \alpha_2$. The adjoint module $\mathrm{Lie}(G)$ has the Chevalley basis $e_{\alpha_3}, e_{\alpha_2}, e_{\alpha_1}, h_{\alpha_1}, h_{\alpha_2}, e_{-\alpha_1}, e_{-\alpha_2}, e_{-\alpha_3}$. We write v_1v_2 for the Lie product of vectors $v_1, v_2 \in \mathrm{Lie}(G)$. We assume that the structure constants are as described by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \end{pmatrix},$$

where the rows and columns are in the order $\alpha_3, \alpha_2, \alpha_1, -\alpha_1 - \alpha_2, -\alpha_3$. By [27, Lemma 5.4], we can explicitly construct our highest weight irreducible module as

$$V_G(\lambda_1 + \lambda_2) = \mathrm{Lie}(G)/\langle h_{\alpha_1} - h_{\alpha_2}\rangle.$$

In a slight abuse of notation, we omit writing the quotient, so that v actually stands for $v + \langle h_{\alpha_1} - h_{\alpha_2}\rangle$. We order the basis for $V_G(\lambda_1 + \lambda_2)$ as $e_{\alpha_3}, e_{\alpha_2}, e_{\alpha_1}, h_{\alpha_1}, e_{-\alpha_1}, e_{-\alpha_2}, e_{-\alpha_3}$. With respect to this ordering, using standard formulas found in [7, §4.4], we find the matrices denoting the transformations $x_{\pm\alpha_1}(t), x_{\pm\alpha_2}(t), x_{\pm\alpha_3}(t)$, as well as $h_{\alpha_1}(\kappa)$ and $h_{\alpha_2}(\kappa)$. These are straightforward calculations, and we therefore only state the results.

$$x_{\alpha_1}(t) = \begin{pmatrix} 1 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & t & -t^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad x_{-\alpha_1}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ t & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -t & 1 & 0 & 0 & 0 \\ 0 & 0 & -t^2 & -t & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -t & 1 \end{pmatrix},$$

$$x_{\alpha_2}(t) = \begin{pmatrix} 1 & 0 & -t & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & t & 0 & -t^2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & t \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad x_{-\alpha_2}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -t & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -t & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -t^2 & 0 & -t & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & t & 0 & 1 \end{pmatrix},$$

$$x_{\alpha_3}(t) = \begin{pmatrix} 1 & 0 & 0 & -t & 0 & 0 & -t^2 \\ 0 & 1 & 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -t \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad x_{-\alpha_3}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ t & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -t & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 1 & 0 \\ -t^2 & 0 & 0 & t & 0 & 0 & 1 \end{pmatrix},$$

$h_{\alpha_1}(\kappa) = \operatorname{diag}(\kappa, \kappa^{-1}, \kappa^2, 1, \kappa^{-2}, \kappa, \kappa^{-1})$ and $h_{\alpha_2}(\kappa) = \operatorname{diag}(\kappa, \kappa^2, \kappa^{-1}, 1, \kappa, \kappa^{-2}, \kappa^{-1})$.

Let $(\cdot, \cdot) : V \times V \to K$ be the nondegenerate symmetric bilinear form given by $(e_{\alpha_i}, e_{-\alpha_j}) = \delta_{ij}$, $(e_{\alpha_i}, e_{\alpha_j}) = (e_{-\alpha_i}, e_{-\alpha_j}) = 0$, $(h_{\alpha_1}, e_{\pm\alpha_i}) = 0$ and $(h_{\alpha_1}, h_{\alpha_1}) = -1$, where $1 \le i, j \le 3$. Then G fixes this form, as can be seen by just checking the action of the generators.

We need some information about the action of G on singular 1-spaces. Let $T = \langle h_{\alpha_1}(\kappa), h_{\alpha_2}(\kappa) : \kappa \in K^* \rangle$ be the standard maximal torus and $B = \langle T, x_{\alpha_1}(t), x_{\alpha_2}(t) : t \in K \rangle$ a Borel subgroup. Then by [27, Lemma 5.5], the group G has 2 orbits on singular vectors in V, with representatives $x = e_{\alpha_3}$ and $y = e_{\alpha_1} + e_{\alpha_2}$. Furthermore, $G_x = U_3 T_1$, $G_y = U_2$, $G_{\langle x \rangle} = U_3 T_2 = B$ and $G_{\langle y \rangle} = U_2 T_1 \le B$.

We now define a totally singular 3-space that we will show has a 2-dimensional stabilizer, therefore belonging to a dense orbit. Let

$$W_3 = \langle e_{\alpha_1}, e_{\alpha_2}, e_{-\alpha_3} \rangle.$$

Let Δ be the G-orbit with representative $\langle x \rangle$, where $x = e_{\alpha_3}$. We start by observing that

$$\mathcal{G}_1(W_3) \cap \Delta = \{\langle e_{\alpha_1} \rangle, \langle e_{\alpha_2} \rangle, \langle e_{-\alpha_3} \rangle\}.$$

It suffices to show that every other 1-space of W_3 is in the same orbit as $\langle y \rangle$, where $y = e_{\alpha_1} + e_{\alpha_2}$. This is clear for $\langle e_{\alpha_1} + \lambda e_{\alpha_2} \rangle$ when $\lambda \ne 0$. The element $x_{-\alpha_1}(t)x_{-\alpha_2}(-t)x_{-\alpha_3}(t^2)$ sends $y \mapsto y + t^3 e_{-\alpha_3}$; therefore, $\langle e_{\alpha_1} + \lambda e_{\alpha_2} + \gamma e_{-\alpha_3} \rangle$ is in the orbit of $\langle y \rangle$ for $\lambda, \gamma \ne 0$. Finally, $n_{\alpha_2}(1)n_{\alpha_1}(1).\langle y \rangle = \langle e_{\alpha_1} + e_{-\alpha_3} \rangle$ and $n_{\alpha_1}(-1)n_{\alpha_2}(-1).\langle y \rangle = \langle e_{\alpha_2} + e_{-\alpha_3} \rangle$, concluding.

Since $\dim \mathcal{S}_3(V) = 6$, it suffices to prove that $G_{W_3} = T.\mathbb{Z}_3$. Since y only contains three 1-spaces that are in the orbit Δ, the connected component of G_{W_3} is simply

$$G_{\langle e_{\alpha_1} \rangle} \cap G_{\langle e_{\alpha_2} \rangle} \cap G_{\langle e_{-\alpha_3} \rangle},$$

which is easily seen to be the maximal torus T. Finally, one checks that the elements of order 2 of $W = N_G(T)/T$ do not fix W_3, while the elements of order 3 do. Therefore, $G_{W_3} \simeq T_2.\mathbb{Z}_3$, and indeed, $C_{\mathcal{S}_3}(V) = T_2.\mathbb{Z}_3$. □

Proposition 5.10. *Let $G = A_2$ and $\lambda = \lambda_1 + \lambda_2$ with $p \ne 3$. Then $C_{\mathcal{S}_4'(V)} = C_{\mathcal{S}_4''(V)} = T_2.\mathbb{Z}_3$.*

Proof. The group G is stable under a triality automorphism of D_4; therefore, the (G, P_3)-double cosets in D_4 are in bijection with the (G, P_1)-double cosets in D_4, as are the (G, P_4)-double cosets. By Proposition 4.3, the group G acts on $\mathcal{S}_1(V)$ with generic stabilizer $T_2.\mathbb{Z}_3$, concluding. □

Proposition 5.11. *Let $G = B_3$ and $\lambda = \lambda_3$. Then $C_{\mathcal{S}_4'(V)} = C_{\mathcal{S}_4''(V)} = C_{\mathcal{S}_1(V)} = U_6 A_2 T_1$.*

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Proof. Similarly to Proposition 5.10, the (G, P_3)-double cosets in D_4 are in bijection with the (G, P_1)-double cosets in D_4, as are the (G, P_4)-double cosets. Therefore, $C_{S'_4(V)} = C_{S''_4(V)} = C_{S_1(V)} = P_3(G)$, as claimed. □

Proposition 5.12. *Let $G = B_4$ and $\lambda = \lambda_4$. Then $C_{S_3(V)} = A_1$.*

Proof. By [10, Prop. 6.21], we already know that D_5 has a dense orbit on all 3-spaces of $V_{D_5}(\lambda_5)$. We construct the module $V_{D_5}(\lambda_5)$ in the same way as in [10, Prop. 6.21] and then consider the restriction to G. Let $\beta_1, \ldots, \beta_6$ be the simple roots of a group of type E_6 and let $D_5 < E_6$ have simple roots $\alpha_1 = \beta_1, \alpha_2 = \beta_3, \alpha_3 = \beta_4, \alpha_4 = \beta_5, \alpha_5 = \beta_2$. Then we may take

$$V_{D_5}(\lambda_5) = \left\langle e_\alpha : \alpha = \sum m_i \beta_i, m_6 = 1 \right\rangle < \mathrm{Lie}(E_6).$$

With this notation, [10, Prop. 6.21] shows that if we write

$$\gamma_2 = 101111, \gamma_3 = 011111, \gamma_4 = 111111, \gamma_5 = 011211, \gamma_6 = 111211, \gamma_7 = 011221,$$

then

$$W_3 := \langle e_{\gamma_2} + e_{\gamma_3}, e_{\gamma_4} + e_{\gamma_5}, e_{\gamma_6} + e_{\gamma_7} \rangle$$

has stabilizer $A_1 A_1$ in $D_5/Z(D_5)$. The generators for the first A_1 are simply $X_{\pm\rho}$, where ρ is the longest root of D_5, while the second A_1 is generated by

$$x(t) = x_{\beta_1}(-t) x_{\beta_4}(2t) x_{\beta_5}(t) x_{\beta_2}(3t) x_{\beta_4+\beta_5}(-t^2) x_{\beta_2+\beta_4}(3t^2) x_{\beta_2+\beta_4+\beta_5}(4t^3) \text{ for } t \in K;$$
$$T_1 = \{h_{\beta_1}(\kappa) h_{\beta_2}(\kappa^3) h_{\beta_4}(\kappa^4) h_{\beta_5}(\kappa^3) : \kappa \in K^*\};$$
$$n = n_{\beta_1} n_{\beta_4} n_{\beta_2\beta_4\beta_5}^{-1}.$$

Let $\{e_1, \ldots, e_5, f_5, \ldots, f_1\}$ be the standard basis for the natural D_5-module, and let $G = (D_5)_{\langle e_2+f_2 \rangle}$. We will show that W_3 is totally singular and has stabilizer A_1 in $G/Z(G)$. Let $T = \langle h_{\beta_i}(\kappa_i) : 1 \leq i \leq 5, \kappa_i \in K^* \rangle$ be the standard maximal torus of D_5. Then $T_G := T \cap G = \langle h_{\beta_i}(\kappa_i) : 1 \leq i \leq 5, \kappa_i \in K^*, \kappa_1 = \kappa_3 \rangle$, since $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_3$. Then the T_G-weights on $\langle e_{\gamma_i} \rangle_{2 \leq i \leq 7}$ are respectively given by $\frac{\kappa_1}{\kappa_2}, \frac{\kappa_2}{\kappa_4}, \frac{\kappa_1\kappa_2}{\kappa_4}, \frac{\kappa_4}{\kappa_1\kappa_5}, \frac{\kappa_4}{\kappa_5}, \frac{\kappa_5}{\kappa_1}$. No two such weights form a pair of opposite weights, and therefore by Lemma 2.7, the subspace W_3 is totally singular.

Finally, we find that the diagonal subgroup of $A_1 A_1$ with positive root subgroup $x(t)x_\rho(t)$ fixes $\langle e_2 + f_2 \rangle$. By maximality of this diagonal A_1 in $A_1 A_1$, this means that $(A_1 A_1) \cap G = A_1$. Therefore, $\dim G - \dim S_3(V) = 3 = \dim G_{W_3}$, which implies that $C_{S_3(V)} = A_1$. □

In what follows, we refer the reader back to Section 2.4 for the relevant notation on spin modules. For the following propositions, let $G = B_4 = (D_5)_{e_5-f_5}$, $\lambda = \lambda_4$ and order the basis of V as

$v_1 = 1,$	$v_5 = e_1 e_5,$	$v_9 = e_1 e_2 e_3 e_5,$	$v_{13} = e_1 e_4,$
$v_2 = e_1 e_2,$	$v_6 = e_2 e_5,$	$v_{10} = e_1 e_2 e_4 e_5,$	$v_{14} = e_2 e_4,$
$v_3 = e_1 e_3,$	$v_7 = e_3 e_5,$	$v_{11} = e_1 e_3 e_4 e_5,$	$v_{15} = e_3 e_4,$
$v_4 = e_2 e_3,$	$v_8 = e_4 e_5,$	$v_{12} = e_2 e_3 e_4 e_5,$	$v_{16} = e_1 e_2 e_3 e_4.$

Lemma 5.13. *The quadratic form given by the matrix* antidiag$(1, -1, 1, -1, 1, -1, 1, -1, 0, 0, 0, 0, 0, 0, 0, 0)$ *defines a nondegenerate quadratic form on $V_{B_4}(\lambda_4)$ fixed by the B_4-action.*

Proof. The pairs v_i, v_{17-i} are pairs of opposite weight vectors. We can assume that $Q(v_1 + v_{16}) = 1$ and use the B_4-action to determine whether $Q(v_i + v_{17-i})$ is 1 or -1. Let $g = 1 + e_1 e_2 \in B_4$. Then $g.v_1 = 1 + e_1 e_2 = v_1 + v_2$ and $g.v_{15} = e_3 e_4 + e_1 e_2 e_3 e_4 = v_{15} + v_{16}$. Therefore, $0 = Q(v_1 + v_{15}) = $

$Q(v_1 + v_2 + v_{15} + v_{16}) = 1 + Q(v_2 + v_{15})$, as claimed. The same approach shows that $Q(v_3 + v_{14}) = 1$ and $Q(v_4 + v_{13}) = -1$. To conclude, let $g = (1 + e_1 f_5)(1 + e_1 e_5)$ which is an element of B_4 since it fixes $e_5 - f_5$. Then $0 = Q(v_1 + v_{12}) = Q(v_1 + v_5 + v_{12} - v_{16}) = -1 + Q(v_5 + v_{12})$. The remaining cases follow similarly. □

Lemma 5.14. *For $\alpha \in \Phi(B_4)$, the root elements $x_\alpha(t)$ are written in terms of spinors as follows:*

$$x_{\epsilon_i - \epsilon_j}(t) = 1 + t e_i f_j,$$
$$x_{-\epsilon_i + \epsilon_j}(t) = 1 + t e_j f_i,$$
$$x_{\epsilon_i + \epsilon_j}(t) = 1 + t e_i e_j,$$
$$x_{-\epsilon_i - \epsilon_j}(t) = 1 - t f_i f_j,$$
$$x_{\epsilon_i}(t) = (1 + t e_i e_5)(1 + t e_i f_5),$$
$$x_{-\epsilon_i}(t) = (1 - t f_i e_5)(1 - t f_i f_5).$$

Proof. The action on V_{nat} of the elements on the left-hand sides of each equation is as described at the beginning of Section 2.1. The action on V_{nat} of the elements on the right-hand sides of each equation is described in Section 2.4. The result follows by comparing the two actions. □

Proposition 5.15. *Let $G = B_4$, $\lambda = \lambda_4$. Then $C_{\mathcal{S}_8'(V)} = A_2.\mathbb{Z}_2$.*

Proof. Suppose that $p \neq 3$. Let $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$, $\beta_3 = \alpha_3$, $\beta_4 = -\alpha_0$, where α_0 is the longest root in $\Phi^+(B_4)$. Then $\{\beta_i\}_i$ is the base of a root system of type D_4. Let D be the corresponding D_4-subgroup of B_4. Then $V \downarrow D = \lambda_3 + \lambda_4 = V_8 + V_8'$. Let ω be a nontrivial third-root of unity. Let τ be the composition of $h_{\beta_2}(\omega)$ with the triality automorphism of D sending $x_{\beta_i}(t) \mapsto x_{\beta_{\sigma.i}}(t)$ for $i = 1, 3, 4$ and $\sigma = (134)$. Then the fixed points in D under the triality automorphism τ form an irreducible A_2-subgroup of D. Using the structure constants inherited from B_4, let A be the irreducible A_2-subgroup of D given by

$$A = \langle x_{\beta_1}(t) x_{\beta_3}(\omega^2 t) x_{\beta_4}(\omega t),$$
$$x_{-\beta_1}(t) x_{-\beta_3}(\omega t) x_{-\beta_4}(\omega^2 t),$$
$$x_{\beta_1 + \beta_2}(t) x_{\beta_2 + \beta_3}(-\omega t) x_{\beta_2 + \beta_4}(\omega^2 t),$$
$$x_{-\beta_1 - \beta_2}(t) x_{-\beta_2 - \beta_3}(-\omega^2 t) x_{-\beta_2 - \beta_4}(\omega t) : t \in K \rangle.$$

By Lemma 5.14, this is the same as

$$A = \langle (1 + t e_1 f_2)(1 + \omega^2 t e_3 f_4)(1 - \omega t f_1 f_2),$$
$$(1 + t e_2 f_1)(1 + \omega t e_4 f_3)(1 + \omega^2 t e_1 e_2),$$
$$(1 + t e_1 f_3)(1 - \omega t e_2 f_4)(1 - \omega^2 t f_1 f_3),$$
$$(1 + t e_3 f_1)(1 - \omega^2 t e_4 f_2)(1 + \omega t e_1 e_3) : t \in K \rangle.$$

With this setup, we have $V_8 \downarrow A \simeq V_8' \downarrow A \simeq V_{A_2}(\lambda_1 + \lambda_2)$. Then A fixes all 8-spaces of the form $\{v + \lambda\phi(v) : v \in V_8\}$ where ϕ is an A-module isomorphism between V_8 and V_8'. Given our explicit generators for A, it is easy to verify that we can take ϕ acting as

$e_1 e_2 e_3 e_4 \mapsto e_2 e_3 e_4 e_5,$	$e_2 e_3 \mapsto e_4 e_5,$	$e_3 e_4 \mapsto \omega^2 e_1 e_3 e_4 e_5,$
$e_1 e_2 \mapsto \omega e_2 e_5,$	$e_1 e_4 \mapsto e_1 e_2 e_3 e_5,$	$1 \mapsto e_1 e_5,$
$e_1 e_3 \mapsto \omega^2 e_3 e_5,$	$e_2 e_4 \mapsto \omega e_1 e_2 e_4 e_5.$	

When $\lambda \neq 0$, the group A must be the connected component of the stabilizer of $\{v + \lambda\phi(v) : v \in V_8\}$, since the only minimal connected overgroup of A in G is D, which only fixes the 8-spaces V_8 and V_8'. Also, $N_G(A) = N_{D.\mathbb{Z}_2}(A) = Z(G).A.\mathbb{Z}_2 = Z(G).A\langle \tau_2 \rangle$, where τ_2 acts as a graph automorphism on

D_4 and A_2, swapping V_8 and V_8'. Explicit calculations show that we can take $\tau_2 = h_{\alpha_1}(-1)n$, where $n = n_1 n_2 n_1 n_3 n_4 n_3 n_2 n_1$, for $n_i = n_{\alpha_i}$. One then checks that τ_2 fixes $\{v + \lambda\phi(v) : v \in V_8\}$ when $\lambda^2 = -1$ (i.e., when $\{v + \lambda\phi(v) : v \in V_8\}$ is totally singular). Since $\dim G - \dim \mathcal{S}_8'(V) = 8 = \dim A$, we conclude that $C_{\mathcal{S}_8'(V)} = A_2.\mathbb{Z}_2$.

Now assume that $p = 3$. This time let A be the A_2-subgroup of G obtained via $V_{A_2}(\lambda_1) \otimes V_{A_2}(\lambda_2)$. The subgroup A acts indecomposably on the natural module for G, as $1/7/1$. More concretely, we can realise A as the subgroup generated by

$$
\begin{aligned}
x_{\gamma_1}(t) &:= x_{\alpha_2}(t)x_{\alpha_4}(-t) = (1 + te_2 f_3)(1 - te_4 e_5)(1 - te_4 f_5), \\
x_{-\gamma_1}(t) &:= x_{-\alpha_2}(t)x_{-\alpha_4}(-t) = (1 + te_3 f_2)(1 + tf_4 e_5)(1 + tf_4 f_5), \\
x_{\gamma_2}(t) &:= x_{\alpha_2+\alpha_3}(t)x_{\alpha_3+\alpha_4}(t)x_{\alpha_0-\alpha_2}(t) = (1 + te_2 f_4)(1 + te_3 e_5)(1 + te_3 f_5)(1 + te_1 e_3), \\
x_{-\gamma_2}(t) &:= x_{-\alpha_2-\alpha_3}(t)x_{-\alpha_3-\alpha_4}(t)x_{\alpha_1+\alpha_2}(-t) = (1 + te_4 f_2)(1 - tf_3 e_5)(1 - tf_3 f_5)(1 - te_1 f_3),
\end{aligned}
$$

as t varies over K. Now let W be the 8-space spanned by vectors

$$
\begin{array}{lll}
e_3 e_5, & e_2 e_5 - e_3 e_4, & e_2 e_4, \\
1 - e_1 e_5, & e_1 e_2 e_3 e_4 + e_2 e_3 e_4 e_5, & e_2 e_3 + e_1 e_2 e_3 e_5, \\
e_1 e_2 + e_1 e_3 e_4 e_5, & e_1 e_4 - e_4 e_5. &
\end{array}
$$

By Lemma 5.13, the subspace W is totally singular. Simple calculations show that $A \leq G_W$. Furthermore, let i be a square root of -1 and

$$
\tau = h_{\alpha_1}(-1)h_{\alpha_2}(-1)h_{\alpha_3}(-1)h_{\alpha_4}(i)n_{0122}.
$$

One checks that $\tau \in N_G(A_2)$ and $\tau \in G_W$. Now let M be a minimal connected overgroup of A that fixes W. Since there are no irreducible subgroups of G containing A, we must have $A \leq M \leq P_1 = G_{\langle e_1 \rangle} = U_7 B_3 T_1$. Since the projection of A on B_3 is an irreducible $A_2 < B_3$, the projection of M on B_3 is either A_2 or G_2. As there is only one conjugacy class of G_2's in P_1, corresponding to the $G_2 \leq B_3$, the last case is not possible. Therefore, $M \leq U_7 A_2 T_1$, and as the A_2 is acting irreducibly on the U_7, we must have $M = U_7 A_2$. It is, however, straightforward to check that $U_7 \not\leq G_W$. This proves that $A = (G_W)^0$. The final step is to show that $C_G(A) = Z(G)$. One way to do this is to consider the centralizer C_1 of $T \cap A = \langle h_{\alpha_2}(\kappa)h_{\alpha_4}(\kappa), h_{\alpha_3}(\kappa) \rangle_{\kappa \in K^*}$, a maximal torus of A. We find that $C_1 = \langle T, X_{\pm 1111}, n_0 \rangle \simeq A_1 T_3.\mathbb{Z}_2$, where n_0 is an element of $N_G(T)$ sending each root to its negative. The centralizer of A must be contained in P_1, as $\langle e_1 \rangle$ is the only 1-space stabilised by A. We have $C_1 \cap P_1 = \langle T, X_{1111} \rangle$, and at this point, it is easy to see that $C_G(A) = C_{C_1 \cap P_1}(A) = Z(G)$. Therefore, $G_W = Z(G).A.\mathbb{Z}_2$, and by dimensional considerations, $A_2.\mathbb{Z}_2 \leq G/Z(G)$ is the generic stabilizer for the G-action on $\mathcal{S}_8'(V)$. $\qquad\square$

Proposition 5.16. *Let $G = B_4$, $\lambda = \lambda_4$. Then $C_{\mathcal{S}_7(V)} = T_2.\mathbb{Z}_2$.*

Proof. By Proposition 5.15, there is a dense G-orbit on $\mathcal{S}_8'(V)$, with stabilizer $A_2.\mathbb{Z}_2$. Fix an 8-space y in this orbit. Then $G_y = A_2.\mathbb{Z}_2 = A_2\langle\tau\rangle$ acts on y as on $\mathrm{Lie}(A_2)$, with τ acting on $\mathrm{Lie}(A_2)$ by transposition. When $p \neq 3$, the quadruple $(A_2.\mathbb{Z}_2, \lambda_1 + \lambda_2, p, 1)$ has generic stabilizer $T_2.\mathbb{Z}_2$, as the open set for the A_2-action is constructed like in Lemma 4.1 starting from elements in $\mathrm{Lie}(T)$ which are fixed by τ. The same is actually true also when $p = 3$. In this case, the $A_2\langle\tau\rangle$ module y is not irreducible, but we can still build an open dense subset of $\mathcal{G}_1(y)$ consisting of orbits of regular semisimple elements, such that all stabilizers are conjugate to $T_2.\mathbb{Z}_2$. Since the action on $\mathcal{G}_1(y)$ is isomorphic to the action on $\mathcal{G}_7(y)$, there is an open dense subset $\hat{X}$ of $X := \mathcal{G}_7(y)$ such that for all $x \in \hat{X}$, the stabilizer $(A_2.\mathbb{Z}_2)_x$ is $A_2.\mathbb{Z}_2$-conjugate to $T_2.\mathbb{Z}_2$. Note that $X \subset \mathcal{S}_7(V)$, and since every element of $\mathcal{S}_7(V)$ is a subspace of precisely one element of $\mathcal{S}_8'(V)$, we must have $\mathrm{Tran}_G(x, X) = G_y$ for all $x \in X$. As $\dim G - \dim \mathrm{Tran}_G(x, X) = 28 = \dim \mathcal{S}_7(V) - \dim X$, the set $\hat{X}$ is X-exact. By Lemma 2.15, we conclude that $C_{\mathcal{S}_7(V)} = T_2.\mathbb{Z}_2$. $\qquad\square$

Proposition 5.17. *Let $G = B_4$, $\lambda = \lambda_4$ with $p \neq 2$. Then $C_{\mathcal{S}_8''(V)} = A_1^3$.*

Proof. Let V_{nat} be the natural module for G and let $V_1 \oplus V_2 \oplus V_3$ be an orthogonal decomposition of V_{nat} such that $\dim V_i = 3$. Let $S = A_1^3$ be the the connected component of the stabilizer of this orthogonal decomposition. Then $N_G(S)$ is a maximal subgroup of G isomorphic to $(\mathbb{Z}_2^2 \times S).Sym(3)$. The group S acts homogeneously on V as the sum of two copies of the 8-dimensional irreducible S-module $\lambda_1 \otimes \lambda_1 \otimes \lambda_1$. Let Y be the 1-dimensional variety of nontrivial proper S-submodules of V (i.e. the set of all 8-dimensional S-submodules of V). Since $p \neq 2$, any such 8-space must be totally singular. We will now show that no element of $N_G(S)/S$ acts trivially on Y. Let τ be a pre-image under the the the canonical projection $N_G(S) \to S$ of one of the 3 nontrivial reflections in $(\mathbb{Z}_2^2 \times S)/S$. Then τ lies in $N_G(A_1D_3)$, acting as a graph automorphism on D_3. Now, A_1D_3 acts on V as $(\lambda_1 \otimes \lambda_2) \oplus (\lambda_1 \otimes \lambda_3)$, and therefore, τ swaps these two 8-spaces. Similarly, a 2-cycle $\tau \in (S.Sym(3))/S$ corresponds to an element in the D_3 component of A_1D_3 acting as a graph automorphism of $A_1^2 \leq D_3$. Here, τ does fix $(\lambda_1 \otimes \lambda_2)$ and $(\lambda_1 \otimes \lambda_3)$, although they are not isomorphic $S\langle\tau\rangle$-modules. A 3-cycle $\tau \in (S.Sym(3))/S$ also acts nontrivially on Y since it is a product of two 2-cycles that do not have the same fixed points. Finally, no product of a reflection with a transposition can act trivially, again because they do not fix the same points. Now by Lemma 2.10, there is a dense subset $\hat{Y}$ of Y on which no element of $N_G(S)/S$ has fixed points. Furthermore, there are only three proper connected subgroups of G that properly contain S, all isomorphic to A_1D_3 acting on V as $(\lambda_1 \otimes \lambda_2) \oplus (\lambda_1 \otimes \lambda_3)$. Any such A_1D_3 only fixes two 8-spaces, which are not contained in Y since they are the fixed points of the 2-cycles in $Sym(3)$. Let $y \in \hat{Y}$. We have shown that $G_y = S$, and therefore, $\mathrm{Tran}_G(y, Y) = N_G(S)$. We then get $\dim G - \dim \mathrm{Tran}_G(y, Y) = 27 = \dim \mathcal{S}_8''(V) - \dim Y$. Thus, the set $\hat{Y}$ is Y-exact, and by Lemma 2.15, we conclude that $C_{\mathcal{S}_8''(V)} = A_1^3$. $\qquad\qquad\square$

Proposition 5.18. *Let $G = B_4$, $\lambda = \lambda_4$ with $p = 2$. Then the quadruple $(G, \lambda, p, 8'')$ has no generic ts-stabilizer, but has a semi-generic ts-stabilizer A_1^3.*

Proof. Given the standard parabolic $P_1 = U_7 B_3 T_1 = U_7 L$, let $X \leq L'$ be a subgroup isomorphic to an A_1^3 acting as $2 \perp 2 \perp 2$ on $V_{B_3}(\lambda_1)$. Here, L acts on the abelian unipotent radical U_7 by fixing the longest short-root subgroup X_{1111} and as $V_{B_3}(\lambda_1)$ on U_7/X_{1111}. Then X has a 3-dimensional 1-cohomology on U_7, corresponding to the conjugacy classes of A_1^3-subgroups of U_7X. We can parametrise this by pairing the root subgroups generating X with the highest and lowest weight vectors for the action on U_7/X_{1111}. More precisely, take

$$X = \langle X_{\pm 0111}, X_{\pm 0011}, X_{\pm 0001} \rangle$$

and define

$$A_1^{(1)}(\lambda) := \langle x_{0111}(t)x_{1222}(\lambda t), x_{-0111}(t)x_{1000}(\lambda t)\rangle_{t \in K},$$
$$A_1^{(2)}(\lambda) := \langle x_{0011}(t)x_{1122}(\lambda t), x_{-0011}(t)x_{1100}(\lambda t)\rangle_{t \in K},$$
$$A_1^{(3)}(\lambda) := \langle x_{0001}(t)x_{1112}(\lambda t), x_{-0001}(t)x_{1110}(\lambda t)\rangle_{t \in K},$$
$$X_{abc} := \langle A_1^{(1)}(a), A_1^{(2)}(b), A_1^{(3)}(c)\rangle.$$

Each $A_1^{(i)}(\lambda)$ is a connected subgroup of P_1 of type A_1. Furthermore, $A_1^{(i)}(\lambda)$ and $A_1^{(j)}(\mu)$ commute if $i \neq j$, which means that X_{abc} is isomorphic to A_1^3. We can write the given generators for $A_1^{(i)}(\lambda)$ in a nice compact form in the Clifford algebra – namely,

$$A_1^{(i-1)}(\lambda) = \langle (1 + te_ie_5)(1 + te_if_5)(1 + \lambda te_ie_1), (1 + tf_ie_5)(1 + tf_if_5)(1 + \lambda tf_ie_1)\rangle_{t \in K},$$

where $i \in \{2, 3, 4\}$. Then $\mathcal{C} := \{X_{abc}\}_{a,b,c \in K}$ is a set of representatives for the conjugacy classes of A_1^3-subgroups of U_7X.

Now consider an arbitrary X_{abc}. We proceed to show that X_{abc} acts homogeneously on V as a sum of two irreducible 8-spaces. Since P_1 fixes the (totally singular) 8-space

$$V_1 := \left\langle \begin{matrix} e_1e_2, & e_1e_3, & e_1e_4, \\ e_1e_5, & e_1e_2e_3e_5, & e_1e_2e_4e_5, \\ e_1e_3e_4e_5, & e_1e_2e_3e_4 \end{matrix} \right\rangle,$$

so does X_{abc}. Secondly, let

$$W_{abc} := \left\langle \begin{matrix} e_1e_3e_4e_5 + e_3e_4, & (a+1)e_1e_2e_3e_4 + e_2e_3e_4e_5, & e_2e_3 + (a+b+1)e_1e_2e_3e_5, \\ e_3e_5 + (1+b)e_1e_3, & (1+a+c)e_1e_2e_4e_5 + e_2e_4, & (c+1)e_1e_4 + e_4e_5, \\ 1 + (b+c+1)e_1e_5, & e_2e_5 + (1+a+b+c)e_1e_2 \end{matrix} \right\rangle.$$

A simple check using the generators of X_{abc} shows that W_{abc} is fixed by X_{abc} and it is isomorphic to $\lambda_1 \otimes \lambda_1 \otimes \lambda_1$ as an A_1^3-module. Since $p = 2$, it is not guaranteed that W_{abc} is totally singular. Indeed, W_{abc} is totally singular if and only if $a + b + c = 0$, by a direct check using Lemma 5.13. We now consider the subset $\mathcal{C}^*$ of $\mathcal{C}$ given by triples (a, b, c) with $a + b + c = 0$ such that a, b, c are all distinct. Under these conditions on (a, b, c), we know that an element of $\mathcal{C}^*$ acts homogeneously on V as a sum of two totally singular 8-spaces and acts indecomposably on $V_{B_4}(\lambda_1)$ as $1/(2 \perp 2 \perp 2)/1$.

We will now show that $X_{abc} \in \mathcal{C}^*$ is the connected component of the stabilizer in G of W_{abc}. Since X acts indecomposably as $(2 \perp 2 \perp 2)/1$ on U_7, so does X_{abc}. In particular, recall that X_{1111} is fixed by X_{abc}. Therefore, if u is a nontrivial element in U_7, we must have $X_{1111} \cap \langle u, X_{abc}\rangle \neq 1$. A direct check shows that no nontrivial element of X_{1111} stabilises W_{abc}, implying $(U_7)_{W_{abc}} = 1$. Let M be a minimal connected overgroup of X_{abc}, such that $M \leq G_{W_{abc}}$. If M has a larger projection $\overline{M}$ onto L' than X_{abc}, it means that either $\overline{M} = A_1 B_2$ or $\overline{M} = L'$. In the latter case, $M = U_7 L'$, which is absurd; therefore, assume that $\overline{M} = A_1 B_2$. Without loss of generality, take $\overline{M} = \langle X_{\pm 0111}, X_{\pm 0010}, X_{\pm 0001}\rangle$. Let $u \in U_7$ and $\kappa \in K^*$ such that $u x_{0010}(1) h_{\alpha_1}(\kappa) \in M$. Since $M \cap U_7 = 1$, we must have $[u, X_{0010}] = [u, h_{\alpha_1}(\kappa)] = 1$. We cannot have $u \in X_{1222}$ since a direct check shows that $u x_{0010}(1) h_{\alpha_1}(\kappa)$ does not fix W_{abc}; so we must have $\kappa = 1$ and $u \in \langle X_{1000}, X_{1110}, X_{1111}, X_{1122}, X_{1222}\rangle$. Similarly, since

$$[X_{0010}, A_1^{(3)}(c)] = [X_{0010}, A_1^{(1)}(a)] = 1,$$

we also get $[u, X_{\pm 0011}] = [u, X_{\pm 0111}] = 1$. Thus, $u \in \langle X_{1110}, X_{1111}\rangle$. Now assume that $x := x_{1110}(t_1) x_{1111}(t_2) x_{0010}(1)$ fixes W_{abc}. Since $x.(e_1e_3e_4e_5 + e_3e_4) \in W_{abc}$, we find that $t_1 = t_2 = 0$. Therefore, $x = x_{0010}(1)$. Since $x.(e_4e_5 + (c+1)e_1e_4) \in W_{abc}$, we find that $b = c$, which is absurd by our choice of (a, b, c). This completes the proof that $(G_{W_{abc}})^0 = X_{abc}$ when a, b, c are all distinct.

The radical of the Levi $B_3 T_1$ acts by scalar multiplication on (a, b, c). Therefore, the subset $\mathcal{C}^{**}$ of $\mathcal{C}^*$ defined by the further condition $a = 1$ contains A_1^3-subgroups which are pairwise non-conjugate in $U_7 X T_1$. Let $Y = \{W_{abc} : X_{abc} \in \mathcal{C}^{**}\}$, a 1-dimensional variety of totally singular 8-spaces. By [31, Prop. 3.5.2 (D)], G-fusion of elements of $\mathcal{C}^{**}$ is controlled by $N_{L'}(X)/X \simeq Sym(3)$. We can be even more precise, and like in [31, Lemma 4.1.3] deduce that if $X_{abc}^g = X_{a'b'c'}$, then $g \in U_7 N_{B_3 T_1}(X)$. Therefore, X_{abc} is G-conjugate to $X_{a'b'c'}$ if and only if $(a', b', c') = t(\pi(a), \pi(b), \pi(c))$ for some $t \in K$ and $\pi \in Sym(\{a, b, c\})$. Also, if a, b, c are pairwise distinct and are not of the form $a, \mu a, \mu^2 a$ where μ is a root of $x^2 + x + 1$, we must have $N_G(X_{abc}) = U_1 X_{abc}$.

Therefore, there is a dense subset $\hat{Y}$ of Y such that any two distinct elements in $\hat{Y}$ have non-conjugate stabilizers in G, isomorphic to A_1^3. Let $y \in \hat{Y}$. Then by construction, $A_1^3 \leq \mathrm{Tran}_G(y, Y) \leq A_1^3.Sym(3)$, and by dimensional considerations, $\hat{Y}$ is Y-exact. By Lemma 2.16, we conclude that there is no generic stabilizer, and by Lemma 2.14, we conclude that there is a semi-generic stabilizer isomorphic to A_1^3.   □

There are now two cases left in order to complete the proof of Theorem 2. These are given by the ts-small quadruples $(C_2, 2\lambda_1, p, 5)$ $(p \neq 2)$ and $(C_3, \lambda_2, p, 7)$ $(p \neq 3)$. These two cases present considerable challenges and similarities to each other. They are the subject of the next two sections.

5.1. The case $(C_2, 2\lambda_1, p, 5)$

In this section, we handle the case of C_2 acting on maximal totally singular subspaces of its adjoint module. We shall prove that this action has a dense orbit with finite generic stabilizer. We resort to making extensive use of computational methods in Magma, with the relevant code being listed in Appendix A as well as being made available on the author's GitHub [28].

Suppose that $p \neq 2$. Let $G = Sp_4(K)$, with fundamental roots α_1, α_2, where α_1 is short. Let $\alpha_3 = \alpha_1 + \alpha_2$ and $\alpha_4 = 2\alpha_1 + \alpha_2$. Order the standard basis of the natural module V_{nat} as (e_1, e_2, f_2, f_1) and let $V = \mathrm{Lie}(G) \leq \mathfrak{sl}_4(K)$, on which G is acting by conjugation. Let $e_{\pm\alpha_1}, e_{\pm\alpha_2}, e_{\pm\alpha_3}, e_{\pm\alpha_4}, h_{\alpha_1}, h_{\alpha_2}$ be the corresponding Chevalley basis, where $e_{\alpha_1}, e_{\alpha_2}, e_{\alpha_3}, e_{\alpha_4}$ are respectively the matrices

$$\begin{pmatrix} & 1 & & \\ & & 0 & \\ & & & -1 \\ & & & \end{pmatrix}, \begin{pmatrix} 0 & & & \\ & & 1 & \\ & & 0 & \\ & & & \end{pmatrix}, \begin{pmatrix} & & 1 & \\ & & & 1 \\ & & & \\ & & & \end{pmatrix}, \begin{pmatrix} & & & 1 \\ & & & \\ & & & \\ & & & \end{pmatrix},$$

the elements $e_{-\alpha_1}, e_{-\alpha_2}, e_{-\alpha_3}, e_{-\alpha_4}$ are respectively their transposes, and

$$h_{\alpha_1} = \mathrm{diag}(1, -1, 1, -1), \; h_{\alpha_2} = \mathrm{diag}(0, 1, -1, 0), \; h_{\alpha_3} = \mathrm{diag}(1, 1, -1, -1), \; h_{\alpha_4} = \mathrm{diag}(1, 0, 0, -1).$$

Let T be the standard maximal torus of G. The module V is orthogonal, with quadratic form given by

$$Q(v) = \mathrm{Trace}(v^2).$$

For i a square root of -1, and ζ a square root of -2, let $W_{(i,\zeta)}$ be the totally singular 5-space of V spanned by

$$v^{(0)} = h_{\alpha_4} + i h_{\alpha_2},$$
$$v^{(1)} = e_{\alpha_1} + \zeta e_{\alpha_2},$$
$$v^{(2)} = e_{\alpha_3} + \zeta e_{-\alpha_4},$$
$$v^{(3)} = e_{-\alpha_3} + \zeta e_{\alpha_4},$$
$$v^{(4)} = e_{-\alpha_1} + \zeta e_{-\alpha_2}.$$

Let W_5 be the 5-space of V spanned by

$$u^{(0)} = h_{\alpha_4} + 2 h_{\alpha_2},$$
$$u^{(1)} = e_{\alpha_1} + 3 e_{\alpha_2},$$
$$u^{(2)} = e_{\alpha_2} + 3 e_{-\alpha_4},$$
$$u^{(3)} = e_{-\alpha_3},$$
$$u^{(4)} = e_{-\alpha_1} + e_{-\alpha_2} + 3 e_{\alpha_4},$$

a totally singular subspace if $p = 5$.

Let

$$\tau = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \text{ and } x = \begin{cases} \mathrm{diag}(\omega^{-1}, \omega^{-2}, \omega^2, \omega) \text{ with } \omega^5 = 1, \omega \neq 1, & \text{if } p \neq 5; \\ \begin{pmatrix} 1 & 2 & 1 & 1 \\ & 1 & 1 & 4 \\ & & 1 & 3 \\ & & & 1 \end{pmatrix}, & \text{if } p = 5. \end{cases}$$

Table 7. Maximal subgroups M_q of $Sp_4(q)$ (q odd).

Class	M_q	Notes	# conjugacy classes
$\mathscr{C}_1$	$q^{1+2}.((q-1) \times Sp_2(q))$		1
$\mathscr{C}_1$	$q^3.GL_2(q)$		1
$\mathscr{C}_2$	$Sp_2(q)^2.\mathbb{Z}_2$		1
$\mathscr{C}_2$	$GL_2(q).\mathbb{Z}_2$	$q \geq 5$	1
$\mathscr{C}_3$	$Sp_2(q^2).\mathbb{Z}_2$		1
$\mathscr{C}_3$	$GU_2(q).\mathbb{Z}_2$	$q \geq 5$	1
$\mathscr{C}_5$	$Sp_4(q_0).(2,r)$	$q = q_0^r$, r prime	$(2,r)$
$\mathscr{C}_6$	$\mathbb{Z}_2.\mathbb{Z}_2^4.Sym(5)$	$q = p \equiv \pm 1 \bmod 8$	2
$\mathscr{C}_6$	$\mathbb{Z}_2.\mathbb{Z}_2^4.Alt(5)$	$q = p \equiv \pm 3 \bmod 8$	1
$\mathscr{S}$	$\mathbb{Z}_2.Alt(6)$	$q = p \equiv \pm 5 \bmod 12$, $q \neq 7$	1
$\mathscr{S}$	$\mathbb{Z}_2.Sym(6)$	$q = p \equiv \pm 1 \bmod 12$	2
$\mathscr{S}$	$\mathbb{Z}_2.Alt(7)$	$q = 7$	1
$\mathscr{S}$	$SL_2(q)$	$p \geq 5$, $q \geq 7$	1

Furthermore, let

$$\tau^* = \begin{cases} \tau, & \text{if } p \neq 5; \\ \text{diag}(\alpha, 2\alpha, 4\alpha, 3\alpha), \text{ where } \alpha^2 = 2, & \text{if } p = 5. \end{cases}$$

Finally, let

$$S^* = \langle x, \tau^* \rangle, \text{ and } W^* = \begin{cases} W_{(i,\zeta)}, & \text{if } p \neq 5; \\ W_5, & \text{if } p = 5. \end{cases}$$

With this setup, it is easy to check that $S^* \leq G_{W^*}$.

Proposition 5.19. *Let $G = C_2$, $\lambda = 2\lambda_1$ with $p = 5$. Then $C_{\mathcal{S}_5'(V)} = \mathbb{Z}_4$.*

Proof. We use the setup of [10, Lemma 4.8(i)] and its proof. Let $h_0 = \text{diag}(-1, -2, 2, 1)$, a regular semisimple element of $\text{Lie}(T)$, and set $\mathfrak{G} = \langle h_0 \rangle$. For a subspace U of $\text{Lie}(G)$, write $\text{Ann}_{\text{Lie}(G)}(U)$ for the subspace $\{v \in \text{Lie}(G) : [v, U] \leq U\}$. A straightforward calculation shows that $\text{Ann}_{\text{Lie}(G)}(W_{(3,\zeta)}) = \mathfrak{G}$. Let $S = G_{W_{(3,\zeta)}}$ and take $g \in S$. We have $\mathfrak{G} = \text{Ann}_{\text{Lie}(G)}(g.W_{(3,\zeta)}) = g.\mathfrak{G}$. Therefore, $g.\mathfrak{G} = \mathfrak{G}$, which is easily seen to imply $g \in T.\langle \tau \rangle$. A direct calculation shows that $T \cap S = \pm 1$, which implies that $S = \langle \tau \rangle = Z(G).\mathbb{Z}_4$. Since $\dim G - \dim S = \dim G = \dim \mathcal{S}_5'(V)$, we conclude that $C_{\mathcal{S}_5'(V)} = \mathbb{Z}_4$. $\square$

Remark 5.20. Note that the subspace $W_{2,\zeta}$ does not belong to the same D_5-orbit as $W_{3,\zeta}$ by Lemma 2.6; however, it also does not have a finite stabilizer. Indeed, it is not difficult to see that it has a stabilizer isomorphic to U_3T_2.

The following lemma describes the subgroup structure of $Sp_4(q)$.

Lemma 5.21. [5, §8.2]. *Assume that $p < \infty$ and let $q = p^e$ for some $e \in \mathbb{Z}_{\geq 1}$. Then the maximal subgroups of $Sp_4(q)$ (q odd) are as in Table 7, and the maximal subgroups of $SL_2(q)$ are as in Table 8. In both cases, see [5] for more details, including the precise notation.*

Lemma 5.22. *Assume $p = 5$. Let $H = G_{\langle u^{(1)} \rangle}$. Then $H_{W^*} = S^* \simeq Z(G).\mathbb{Z}_5.\mathbb{Z}_4$.*

Proof. The element $u^{(1)}$ is a regular nilpotent element, and a simple calculation shows that $H = U_2T_1$, where

$$U_2 = \left\{ \begin{pmatrix} 1 & -b & -b^2 & a \\ & 1 & 2b & b^2 \\ & & 1 & b \\ & & & 1 \end{pmatrix} : a, b \in K \right\}, \quad T_1 = \{\text{diag}(\kappa^3, \kappa, \kappa^{-1}, \kappa^{-3}) : \kappa \in K^*\}.$$

Table 8. *Maximal subgroups M_q of $SL_2(q)$.*

Class	$\mathbf{M_q}$	Notes	# conjugacy classes
$\mathscr{C}_1$	$q.(q-1)$		1
$\mathscr{C}_2$	$Q_{2(q-1)}$	$q \neq 5$; q odd	1
$\mathscr{C}_2$	$D_{2(q-1)}$	q even	1
$\mathscr{C}_3$	$Q_{2(q+1)}$	q odd	1
$\mathscr{C}_3$	$D_{2(q+1)}$	q even	1
$\mathscr{C}_5$	$SL_2(q_0).(2,r)$	$q = q_0^r$, r prime, q odd	$(2,r)$
$\mathscr{C}_5$	$PSL_2(q_0)$	$q = q_0^r$, r prime, $q_0 \neq 2$, q even	1
$\mathscr{C}_6$	$\mathbb{Z}_2.\mathbb{Z}_2^2.Sym(3)$	$q = p \equiv \pm 1 \bmod 8$	2
$\mathscr{C}_6$	$\mathbb{Z}_2.\mathbb{Z}_2^2.\mathbb{Z}_3$	$q = p \equiv \pm 3, 5, \pm 11, \pm 13, \pm 19 \bmod 40$	1
$\mathscr{S}$	$SL_2(5)$	$q = p \equiv \pm 1 \bmod 10$	2
$\mathscr{S}$	$SL_2(5)$	$q = p^2$, $p \equiv \pm 3 \bmod 10$	2

Let $g = \begin{pmatrix} 1 & -b & -b^2 & a \\ & 1 & 2b & b^2 \\ & & 1 & b \\ & & & 1 \end{pmatrix} \begin{pmatrix} \kappa^3 & & & \\ & \kappa & & \\ & & \kappa^{-1} & \\ & & & \kappa^{-3} \end{pmatrix} \in H_{W^*}$. We have $gu^{(0)}g^{-1} = u^{(0)} - bu^{(1)} + (b^3 + 3a)e_{\alpha_4}$. This forces $a = 3b^3$. Also, $gu^{(3)}g^{-1} = \frac{1}{\kappa^4}(u^{(3)} - b^2 u^{(0)} + 2b^3 u^{(1)} + 2bu^{(4)} + (b^5 - b)e_{\alpha_4})$, forcing $b = b^5$. Finally, $gu^{(2)}g^{-1} = \frac{1}{\kappa^6}(-b^3 u^{(0)} + b(1 - \kappa^8)e_{\alpha_3} + u^{(2)} + 3bu^{(3)} + 3b^2 u^{(4)} - b^4 u^{(1)} + (\kappa^8 - 1)e_{\alpha_2} + b^2(\kappa^8 - 1)e_{\alpha_4})$, implying $\kappa^8 = 1$. This allows us to conclude that $H_{W^*} = \langle x, \tau^* \rangle = S^*$, as claimed. □

Lemma 5.23. *Let $H = N_G(\langle x \rangle)$. Then $H_{W^*} = S^* \simeq Z(G).\mathbb{Z}_5.\mathbb{Z}_4$.*

Proof. Assume $p \neq 5$. Since x is a regular semisimple element, it is easy to see that $H = T.\langle \tau \rangle$, and one quickly finds that $T_{W^*} = \pm\langle x \rangle$. Since $\tau \in G_{W^*}$, we conclude that $H_{W^*} = \langle x, \tau \rangle = Z(G).\langle x \rangle.\mathbb{Z}_4$. If $p = 5$, we have $H \leq \langle C_G(x), T \rangle$. It is easy to see that $C_G(x)$ is the unipotent radical of $G_{\langle u^{(1)} \rangle}$ and that

$$N_T(\langle x \rangle) = \langle \tau^* \rangle \leq \{\mathrm{diag}(\kappa^3, \kappa, \kappa^{-1}, \kappa^{-3}) : \kappa \in K^*\}.$$

Therefore, $H \leq G_{\langle u^{(1)} \rangle}$, and we can conclude by Lemma 5.22. □

Lemma 5.24. *Suppose that $H \leq G$ is a reducible subgroup of G containing $\langle x \rangle$. Then $H_{W^*} \leq S^*$.*

Proof. If $p \neq 5$, the only 1-spaces of V_{nat} stabilised by the semisimple element x are spanned by a standard basis vector; therefore, $H \leq G_U$, where $U \leq V_{nat}$ is a 1-space, a totally singular 2-space or a nondegenerate 2-space, in each case spanned by standard basis vectors. However, if $p = 5$, then $H \leq G_U$, where $U = \langle e_1 \rangle$ or $U = \langle e_1, e_2 \rangle$, as x is a regular unipotent element contained in the standard Borel subgroup. In all the cases where U is totally singular (i.e., $G_U \simeq P_1$ or $G_U \simeq P_2$), it is easily seen that G_U stabilises a unique 6-space of V. We intersect this 6-space with W^*, identifying a 1-space spanned by a regular nilpotent element v that must be stabilised by H_{W^*}. This then reduces the problem to computing the stabilizer of W^* within a $U_2 T_1$. If $p = 5$, we find that $\langle v \rangle = \langle u^{(1)} \rangle$, concluding by Lemma 5.22.

Therefore, from now on, assume that $p \neq 5$. If $H = G_{\langle e_1 \rangle}$, then H stabilises $\langle e_{\pm\alpha_2}, e_{\alpha_1}, e_{\alpha_3}, e_{\alpha_4}, h_{\alpha_2} \rangle$, which intersects W^* in $\langle v^{(1)} \rangle$. We find that $G_{\langle v^{(1)} \rangle} = U_2 T_1$, where

$$U_2 = \left\{ \begin{pmatrix} 1 & b & \frac{b^2\zeta}{2} & a \\ & 1 & \zeta b & -\frac{b^2\zeta}{2} \\ & & 1 & -b \\ & & & 1 \end{pmatrix} : a, b \in K \right\}, \quad T_1 = \{\mathrm{diag}(\kappa^3, \kappa, \kappa^{-1}, \kappa^{-3}) : \kappa \in K^*\}.$$

A direct calculation like in the proof of Lemma 5.22 then shows that $(U_2 T_1)_{W^*} = \pm\langle x \rangle$. The other cases with $G_U \simeq P_1$ or $G_U \simeq P_2$ are similarly dealt with. Here, we report just the intersection $\langle v \rangle$. If $U = \langle e_2 \rangle$, then $\langle v \rangle = \langle v^{(2)} \rangle$; if $U = \langle f_1 \rangle$, then $\langle v \rangle = \langle v^{(4)} \rangle$; if $U = \langle f_2 \rangle$, then $\langle v \rangle = \langle v^{(3)} \rangle$; if $U = \langle e_1, e_2 \rangle$, then $\langle v \rangle = \langle v^{(1)} \rangle$; if $U = \langle e_1, f_2 \rangle$, then $\langle v \rangle = \langle v^{(3)} \rangle$; if $U = \langle e_2, f_1 \rangle$, then $\langle v \rangle = \langle v^{(4)} \rangle$.

It remains to consider the case $H \leq G_{\langle e_1, f_1 \rangle}$. Here, H fixes the subspace $\langle h_{\alpha_1}, h_{\alpha_2}, e_{\pm \alpha_2}, e_{\pm \alpha_4} \rangle$, which intersects W^* in $\langle v^{(0)} \rangle$. Therefore, $H_{W^*} \leq N_G(T)$, and we conclude by Lemma 5.23. $\qquad \square$

Lemma 5.25. *Suppose that $\langle x \rangle \leq H.\mathbb{Z}_2 < G$, where H is an arbitrary subgroup of G. Then $(H.\mathbb{Z}_2)_{W^*} \leq S^*$ if and only if $H_{W^*} \leq S^*$.*

Proof. The forward direction is trivial. Suppose that $H_{W^*} \leq \langle x, \tau^* \rangle = S^*$. Since x has order 5, we must have $\langle x \rangle \leq H$. Since $H_{W^*} \leq S^*$, the subgroup $\langle x \rangle$ is the unique subgroup of order 5 in H_{W^*}. Therefore, since $H_{W^*} \lhd (H.\mathbb{Z}_2)_{W^*}$, the subgroup $\langle x \rangle$ is normal in $(H.\mathbb{Z}_2)_{W^*}$. By Lemma 5.23, we know that the stabilizer of W^* in $N_G(\langle x \rangle)$ is S^*, concluding. $\qquad \square$

Proposition 5.26. *Let $G = C_2$, $\lambda = 2\lambda_1$ with $p \neq 2$. Then $C_{\mathcal{S}'_5(V)} = \mathbb{Z}_5.\mathbb{Z}_4$ and $C_{\mathcal{S}''_5(V)} = \mathbb{Z}_{5/(p,5)}.\mathbb{Z}_4$.*

Proof. Let $S = G_{W^*}$. We shall prove that $S \leq N_G(\langle x \rangle)$. This will conclude the proof of the proposition as follows. By Lemma 5.23, we have $(N_G(\langle x \rangle))_{W^*} = S^*$, which then implies $S = S^*$ and that W^* is in a dense G-orbit on one of the two D_5-orbits on $\mathcal{S}_5(V)$. If $p \neq 5$, then $W_{(i,\zeta)}$ and $W_{(-i,\zeta)}$ intersect in a 4-dimensional subspace, and therefore, by Lemma 2.6, they belong to distinct D_5-orbits on $\mathcal{S}_5(V)$. They each have stabilizer S^*, concluding the $p \neq 5$ case. If $p = 5$, the subspace W^* intersects $W_{(3,\zeta)}$ trivially, which by Lemma 2.6 implies that W^* and $W_{(3,\zeta)}$ belong to distinct D_5-orbits on $\mathcal{S}_5(V)$. Again, $G_{W^*} = S^*$, concluding.

In order to prove that $S \leq N_G(\langle x \rangle)$, we show that for all $p < \infty$ and $e \in \mathbb{Z}_{\geq 1}$, if

$$\langle x \rangle \leq R \leq Sp_4(p^e) = Sp_4(q) < G, \text{ with } R \not\leq N_G(\langle x \rangle),$$

then R does not stabilise W^*. Note that this is indeed sufficient, since if $g \in S \setminus N_G(\langle x \rangle)$, then there must exist $e \in \mathbb{Z}_{\geq 1}$ such that $g \in Sp_4(p^e)$, with $R = \langle x, g \rangle$ satisfying the condition above. We shall make extensive use of maximal subgroups of $Sp_4(q)$, often combined with exhaustive computations in Magma. The commented code is made available both in the Appendix as well as on the author's Github [28]. The $p = \infty$ case then follows from the $p < \infty$ case since they are equivalent once p is sufficiently large.

Suppose that

$$\langle x \rangle \leq R \leq M_q < Sp_4(q) < G, \text{ with } R \not\leq N_G(\langle x \rangle),$$

where M_q is a maximal subgroup of $Sp_4(q)$, as listed in Table 7. The goal is to prove that R does not stabilize W.

If M_q is as in one of the first 6 rows of Table 7, then by Lemma 5.25, we can assume that R is reducible, and Lemma 5.24 implies that R does not stabilize W. If $M_q = Sp_4(q_0).(2, r)$ where r is prime and $q = q_0^r$, then Lemma 5.25 allows us to reduce to one of the other cases.

Suppose that M_q is the double cover of $Alt(6)$, $Sym(6)$ or $Alt(7)$, in which case $q = p$. An exhaustive search using Magma shows that $Z(G).\langle x \rangle \leq R^* \leq R$ where $R^* = Z(G).Alt(5)$, the double cover of $Alt(5)$, isomorphic to $SL_2(5)$. The general strategy adopted for this exhaustive search is the following. We set up R as an abstract group. Then for all conjugacy classes of elements of order 5 of R, we take a representative g_5 and go through all subgroups of R that contain g_5, determining which ones do not normalise $\langle g_5 \rangle$. By Lemma 5.24, we can assume that R^* is irreducible in G. If $p \neq 5$, since $p \neq 2$ by assumption and $p \neq 3$ by choice of M_q, we can use ordinary character theory to show that R^* does not fix any 5-space of V. The subgroup R^* must be embedded in G via its unique irreducible symplectic character χ of degree 4. We then verify that $S^2(\chi) = \chi_4 + \chi_5 + \chi_7$, where χ_4, χ_5, χ_7 are irreducible characters of degrees 3, 3, 4. Therefore, R^* fixes no 5-space of V. See Listing 3 for the corresponding Magma code. If $p = 5$, we can use a direct construction of $2.Alt(6) \leq Sp_4(5)$ to check that R^* acts on V with composition factors of dimensions 3, 3, 3, 1. Again, this means that R fixes no 5-space of V. See Listing 3 for the Magma code proving this.

Suppose that $M_q = Z(G).\mathbb{Z}_2^4.Sym(5)$, the normalizer of an extraspecial subgroup of G of minus type. Similarly to the previous case, an exhaustive search shows that R must contain $R^* = Z(G).\mathbb{Z}_2^4.\langle x \rangle$

or $R^* = Z(G).Alt(5)$. By Lemma 5.24, we can assume that R^* is irreducible in G. In the second case, we have already seen that R^* does not fix a 5-space of V when $p \neq 3$. If $p = 3$, the same holds, which can be checked directly in $Sp_4(3)$ by taking an explicit construction of $Z(G).Alt(5)$. Therefore, assume that $R^* = Z(G).\mathbb{Z}_2^4.\langle x \rangle$, a group with GAP Id $(160, 199)$. If $p \neq 5$, we can use the ordinary characters of R^* to check that R^* does not fix any totally singular 5-spaces of V. The subgroup R^* must be embedded in G via its unique irreducible symplectic character χ of degree 4. One then finds that $S^2(\chi) = \psi_1 + \psi_2$, where ψ_1 and ψ_2 are distinct self-dual irreducible characters of degree 5. Therefore, R^* stabilises exactly two nondegenerate 5-spaces of V. See Listing 2 for the Magma code. If $p = 5$, we can use a direct construction of $\mathbb{Z}_2.\mathbb{Z}_2^4.Alt(5) \leq Sp_4(5)$ to show that the KR^*-module $V \downarrow R^*$ has two self-dual non-isomorphic composition factors, implying that R^* does not stabilise a totally singular 5-space. See details of the computations in Listing 4.

It remains to consider the case $M_q = SL_2(q)$ with $p \geq 5$ and $q \geq 7$, as in the last row of Table 7. In order to handle this case, we consider the subgroup structure of the maximal subgroups of $SL_2(q)$, as classified in Table 8. First note that $SL_2(q)$ does not fix a 5-space of V, as if $p = 5$ it acts on V with composition factors of dimensions $4, 3, 3$, while if $p > 5$, it acts on V with composition factors of dimension 7 and 3. Therefore, assume that $R \leq M_q^* < SL_2(q)$, where M_q^* is a maximal subgroup of $SL_2(q)$, as described by Table 8. If M_q^* is as in one of the first five rows of Table 8, then by Lemma 5.25, we can assume that R is reducible, and Lemma 5.24 implies that R does not fix W^*. If $M_q^* = SL_2(q_0).(2, r)$ where r is prime and $q = q_0^r$, then Lemma 5.25 allows us to reduce to one of the other cases. Since $p \neq 2$, the case $M_q^* = PSL_2(q_0)$ is excluded, while since x has order 5, the cases $M_q^* = \mathbb{Z}_2.\mathbb{Z}_2^2.Sym(3)$ and $M_q^* = \mathbb{Z}_2.\mathbb{Z}_2^2.\mathbb{Z}_3$ are not possible. The only other possibility is $M_q^* = SL_2(5)$ with $p \neq 5$, as per the last two rows of Table 8. In this case, we must have $R = M_q^* = SL_2(5)$, which we have already dealt with. This completes the case-by-case analysis. □

5.2. The case $(C_3, \lambda_2, p, 7)$

In this section, we handle the last remaining case needed to complete the proof of Theorem 2. In particular, we shall prove that the ts-small quadruples $(C_3, \lambda_2, p, 7')$ and $(C_3, \lambda_2, p, 7'')$ have a finite generic stabilizer. The strategy is entirely similar to the one used for the $(C_2, 2\lambda_1, p, 5)$ case. We shall, however, make even greater use of computational methods, sometimes resorting to solving large systems of equations using Magma. Again, the code can be found in Appendix A, as well as on the author's GitHub [28].

Suppose that $p \neq 3$. Let $G = Sp_6(K)$ and order the standard basis of the natural module V_{nat} as $(e_1, e_2, e_3, f_3, f_2, f_1)$. Let V be the submodule of $\bigwedge^2 V_{nat}$ defined by

$$V = \langle e_i \wedge e_j, f_i \wedge f_j, e_i \wedge f_j, \sum \alpha_i e_i \wedge f_i : i \neq j, \sum \alpha_i = 0 \rangle.$$

Then $V = V_G(\lambda_2)$. Let ω be a primitive cube root of unity and let $(v_1, \ldots, v_{14})$ be the ordered basis of V given by

$v_1 = e_1 \wedge e_2,$	$v_5 = e_2 \wedge f_3,$	$v_{10} = e_3 \wedge f_2,$
$v_2 = e_1 \wedge e_3,$	$v_6 = e_1 \wedge f_2,$	$v_{11} = e_3 \wedge f_1,$
$v_3 = e_2 \wedge e_3,$	$v_7 = e_1 \wedge f_1 + \omega e_2 \wedge f_2 + \omega^2 e_3 \wedge f_3,$	$v_{12} = f_2 \wedge f_3,$
$v_4 = e_1 \wedge f_3,$	$v_8 = e_1 \wedge f_1 + \omega^2 e_2 \wedge f_2 + \omega e_3 \wedge f_3,$	$v_{13} = f_1 \wedge f_3,$
	$v_9 = e_2 \wedge f_1,$	$v_{14} = f_1 \wedge f_2.$

Then it is easy to check that G fixes a nondegenerate quadratic form on V, given by

$$Q\left(\sum_1^{14} \alpha_i v_i\right) = \sum_1^7 \alpha_i \alpha_{15-i}.$$

Let T be the standard maximal torus of G. For i a primitive fourth root of unity, let $W_{(\omega,i)}$ be the totally singular 7-space of V spanned by

$$
\begin{aligned}
v^{(0)} &= e_1 \wedge f_1 + \omega e_2 \wedge f_2 + \omega^2 e_3 \wedge f_3, \\
v^{(1)} &= e_2 \wedge f_3 + i e_1 \wedge f_2, \\
v^{(2)} &= e_1 \wedge f_3 - i e_2 \wedge e_3, \\
v^{(3)} &= f_1 \wedge f_2 + i e_1 \wedge e_3, \\
v^{(4)} &= e_1 \wedge e_2 + i f_1 \wedge f_3, \\
v^{(5)} &= e_3 \wedge f_1 - i f_2 \wedge f_3, \\
v^{(6)} &= e_3 \wedge f_2 + i e_2 \wedge f_1.
\end{aligned}
$$

Let W_7 be the 7-space of V spanned by

$$
\begin{aligned}
u^{(0)} &= e_1 \wedge f_1 + 4 e_2 \wedge f_2 + 2 e_3 \wedge f_3, \\
u^{(1)} &= e_1 \wedge e_2 + 3 e_3 \wedge f_1 + 3 f_2 \wedge f_3, \\
u^{(2)} &= e_2 \wedge e_3 + 4 f_1 \wedge f_2, \\
u^{(3)} &= e_1 \wedge f_3 + 4 f_1 \wedge f_2, \\
u^{(4)} &= e_2 \wedge f_3 + 2 e_1 \wedge f_2, \\
u^{(5)} &= e_2 \wedge f_1 + 5 e_3 \wedge f_2, \\
u^{(6)} &= f_1 \wedge f_3,
\end{aligned}
$$

a totally singular subspace if $p = 7$. Let

$$
\tau = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{ and } x = \begin{cases} \mathrm{diag}(\zeta^{-1}, \zeta^{-2}, \zeta^{-3}, \zeta^3, \zeta^2, \zeta) \text{ with } \zeta^7 = 1, \zeta \neq 1, & \text{if } p \neq 7; \\[4pt] \begin{pmatrix} 1 & 1 & 4 & 6 & 2 & 1 \\ & 1 & 1 & 4 & 1 & 5 \\ & & 1 & 1 & 3 & 6 \\ & & & 1 & 6 & 4 \\ & & & & 1 & 6 \\ & & & & & 1 \end{pmatrix}, & \text{if } p = 7. \end{cases}
$$

Furthermore, let

$$
\tau^* = \begin{cases} \tau, & \text{if } p \neq 7; \\ \mathrm{diag}(\alpha, 5\alpha, 4\alpha, 6\alpha, 2\alpha, 3\alpha), \text{ where } \alpha^2 = 5, & \text{if } p = 7. \end{cases}
$$

Finally, let

$$
S^{\dagger} = \langle x, (\tau^*)^4 \rangle, \ S^* = \langle x, \tau^* \rangle, \text{ and } W^* = \begin{cases} W_{(\omega,i)}, & \text{if } p \neq 7; \\ W_7, & \text{if } p = 7. \end{cases}
$$

With this setup, it is easy to check that $S^{\dagger} \leq S^* \leq G_{W^*}$.

Proposition 5.27. *Let $G = C_3$, $\lambda = \lambda_2$ with $p = 7$. Then $C_{\mathcal{S}_7'(V)} = \mathbb{Z}_6$.*

Proof. We use the setup of [10, Lemma 4.8(i)] and its proof. Let $h_0 = \mathrm{diag}(-1, -2, -3, 3, 2, 1)$, a regular semisimple element of $\mathrm{Lie}(T)$, and set $\mathfrak{G} = \langle h_0 \rangle$. For a subspace U of $\mathrm{Lie}(G)$, write $\mathrm{Ann}_{\mathrm{Lie}(G)}(U)$ for the

subspace $\{v \in \mathrm{Lie}(G) : [v, U] \le U\}$. A straightforward calculation shows that $\mathrm{Ann}_{\mathrm{Lie}(G)}(W_{(2,i)}) = \mathfrak{G}$. Let $S = G_{W_{(2,i)}}$ and take $g \in S$. We have $\mathfrak{G} = \mathrm{Ann}_{\mathrm{Lie}(G)}(g.W_{(2,i)}) = g.\mathfrak{G}$. Therefore, $g.\mathfrak{G} = \mathfrak{G}$, which is easily seen to imply $g \in T.\langle \tau \rangle$. A direct calculation shows that $T \cap S = \pm 1$, which implies that $S = \langle \tau \rangle = Z(G).\mathbb{Z}_6$. Since $\dim G - \dim S = \dim G = \dim \mathcal{S}_7'(V)$, we conclude that $C_{\mathcal{S}_7'(V)} = \mathbb{Z}_6$. $\qquad\square$

Lemma 5.28. *Suppose that $H \le G$ is a reducible subgroup of G containing $S^\dagger$. Then $H_{W^*} \le S^*$.*

Proof. Suppose that $p = 7$. Since $x \in S^\dagger$ is a regular unipotent element contained in the standard Borel subgroup B, we have that H is contained in $G_{\langle e_1 \rangle}$, $G_{\langle e_1, e_2 \rangle}$ or $G_{\langle e_1, e_2, e_3 \rangle}$. We consider each of these cases and deduce that $H_{W^*} \le B$. We then use Magma to directly show that $B_{W^*} = S^*$, concluding as required. Suppose that $H \le G_{\langle e_1, e_2 \rangle}$. The group $G_{\langle e_1, e_2 \rangle}$ stabilises $U_6 = \langle e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3, e_1 \wedge f_3, e_2 \wedge f_3, e_1 \wedge f_1 + e_2 \wedge f_2 - 2e_3 \wedge f_3 \rangle$, and therefore, H_{W^*} must stabilise $W^* \cap U_6 = \langle e_1 \wedge f_3 - e_2 \wedge e_3 \rangle$ as well as $W^* \cap (U_6)^\perp = \langle e_1 \wedge f_3 - e_2 \wedge e_3, e_2 \wedge f_3 + 2e_1 \wedge f_2 \rangle$. The latter implies that H_{W^*} stabilises $\langle e_1, e_2, e_3, f_3, f_2 \rangle$ and therefore also its radical $\langle e_1 \rangle$. Let $g \in H_{W^*}$. Since g stabilises $\langle e_1 \rangle$, $\langle e_1, e_2 \rangle$, $\langle e_1, e_2 \rangle^\perp$ and $\langle e_1 \wedge f_3 - e_2 \wedge e_3 \rangle$, it is easy to see that $g.e_3 \in \langle e_1, e_2, e_3 \rangle$. Therefore, $g \in B$ and $H_{W^*} \le B$. Now consider the case $H \le G_{\langle e_1 \rangle}$. Any element of H must also stabilise $\langle e_1 \rangle^\perp = \langle e_1, e_2, e_3, f_3, f_2 \rangle$, and similarly to the previous case, we find that H_{W^*} stabilises $\langle e_1 \wedge f_3 - e_2 \wedge e_3, e_2 \wedge f_3 + 2e_1 \wedge f_2 \rangle$. Let $g = (a_{ij})_{ij} \in H_{W^*}$. We have $g.(e_1 \wedge f_3 - e_2 \wedge e_3) = a_{11}e_1 \wedge (a_{14}e_1 + a_{24}e_2 + a_{34}e_3 + a_{44}f_3 + a_{54}f_2) - (a_{12}e_1 + a_{22}e_2 + a_{32}e_3 + a_{42}f_3 + a_{52}f_2) \wedge (a_{13}e_1 + a_{23}e_2 + a_{33}e_3 + a_{43}f_3 + a_{53}f_2)$. Since $g.(e_1 \wedge f_3 - e_2 \wedge e_3) = \gamma_1(e_1 \wedge f_3 - e_2 \wedge e_3) + \gamma_2(e_2 \wedge f_3 + 2e_1 \wedge f_2)$ for some $\gamma_1, \gamma_2 \in K$, we must have

$$\det\begin{pmatrix} a_{22} & a_{23} \\ a_{52} & a_{53} \end{pmatrix} = \det\begin{pmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{pmatrix} = \det\begin{pmatrix} a_{32} & a_{33} \\ a_{52} & a_{53} \end{pmatrix} = \det\begin{pmatrix} a_{42} & a_{43} \\ a_{52} & a_{53} \end{pmatrix} = 0.$$

If $(a_{52}, a_{53}) \ne (0, 0)$, we get

$$\gamma_1 = \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} = 0 = \det\begin{pmatrix} a_{22} & a_{23} \\ a_{42} & a_{43} \end{pmatrix} = \gamma_2,$$

a contradiction. Therefore, $a_{52} = a_{53} = 0$. Considering the image of the second basis vector $e_2 \wedge f_3 + 2e_1 \wedge f_2$, we similarly find that $a_{52} = a_{54} = 0$. Therefore, g stabilises $\langle e_1, e_2, e_3, f_3 \rangle$, and therefore also its radical $\langle e_1, e_2 \rangle$, reducing to the case $H \le G_{\langle e_1, e_2 \rangle}$. The case $H \le G_{\langle e_1, e_2, e_3 \rangle}$ follows similarly. This proves that $H_{W^*} \le B$. It remains to show that $B_{W^*} = S^*$. Given $g \in B$, we can write it as

$$g = h_{\alpha_1}(t_1)h_{\alpha_2}(t_2)h_{\alpha_3}(t_3)x_{100}(a_1)x_{110}(a_2)x_{010}(a_3)x_{221}(a_4)x_{121}(a_5)$$
$$x_{111}(a_6)x_{021}(a_7)x_{011}(a_8)x_{001}(a_9),$$

where $t_1, t_2, t_3 \in K^*$ and $a_i \in K$ for $1 \le i \le 9$. Let U be the subspace of V with basis given by

$$u_1, \ldots, u_7 = e_1 \wedge e_3, \; e_1 \wedge f_2, \; e_1 \wedge f_1 - e_2 \wedge f_2, \; e_3 \wedge f_2, \; e_3 \wedge f_1, \; f_2 \wedge f_3, \; f_1 \wedge f_2.$$

Then $V = W^* \oplus U$. For each basis vector $u^{(i)}$ of W^*, write $g.u^{(i)}$ as $w^* + \sum_{j=1}^{7} f_{ij} u_j$, where $w^* \in W^*$ and we view f_{ij} as an element of $\mathbb{F}_7[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, a_1, \ldots, a_9]$. Then to determine B_{W^*} it suffices to determine the zero locus of the ideal $I \otimes K \le K[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, a_1, \ldots, a_9]$, where $I \le \mathbb{F}_7[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, a_1, \ldots, a_9]$ is the ideal generated by $\{f_{ij} : 1 \le i, j \le 7\}$. We do this by first determining a Groebner basis for I using Magma. See Listing 5 for the code that does this. We find that I is generated by

$$a_1 + 6a_9, \; a_2 + 3a_9^2, \; a_3 + 6a_9, \; a_4 + a_9^5, \; a_5 + 6a_9^4, \; a_6 + a_9^3, \; a_7 + 2a_9^3, \; a_8 + 3a_9^2, \; a_9^7 + 6a_9,$$
$$t_1 + 6t_2^{-2}t_3^{-3}, \; t_2 + 6t_2^{-2}, \; t_3 + 6t_3^{-3}, \; t_1^{-1} + 6(t_2 t_3)^{-1}, \; t_2^{-3} + 6, \; t_3^{-4} + 6.$$

It is now easy to find the zero locus of $I \otimes K$, to determine that

$$h_{\alpha_1}(t_1)h_{\alpha_2}(t_2)h_{\alpha_3}(t_3)x_{100}(a_1)x_{110}(a_2)x_{010}(a_3)x_{221}(a_4)$$

$$x_{121}(a_5)x_{111}(a_6)x_{021}(a_7)x_{011}(a_8)x_{001}(a_9) \in B_{W^*}$$

$$\iff a_1 \in \mathbb{F}_7, \, a_9 = a_1, \, a_2 = 4a_1^2, \, a_3 = a_1, \, a_4 = 6a_1^5, \, a_5 = a_1^4, \, a_6 = -a_1^3, \, a_7 = 5a_1^3, \, a_8 = 4a_1^2,$$

$$t_2^3 = t_3^4 = 1, t_1 = t_2 t_3.$$

Taking $a_1 = t_1 = t_2 = t_3 = 1$, we get the element x, while taking $a_1 = 0$ we get the subgroup generated by τ^*. This completes the proof that $B_{W^*} = S^*$.

We now turn to the case $p \neq 7$. The only proper nonzero subspaces of V stabilised by $S^\dagger$ are $\langle e_1, e_2, f_3 \rangle$ and $\langle e_3, f_1, f_2 \rangle$. Suppose that $H \leq G_{\langle e_1, e_2, f_3 \rangle}$. Then it is easy to see that H_{W^*} stabilises the subspace $W^\ddagger = \langle v^{(0)}, v^{(1)}, v^{(2)}, v^{(4)} \rangle$. Let

$$g = u^- n t u,$$

where

$$u = x_{100}(a_1)x_{110}(a_2)x_{010}(a_3)x_{221}(a_4)x_{121}(a_5)x_{111}(a_6)x_{021}(a_7)x_{011}(a_8)x_{001}(a_9),$$
$$t = h_{\alpha_1}(t_1)h_{\alpha_2}(t_2)h_{\alpha_3}(t_3),$$
$$n \in \{1, n_1, n_2, n_1 n_2, n_2 n_1, n_1 n_2 n_1\},$$
$$u^- = x_{100}(b_1)x_{110}(b_2)x_{010}(b_3)x_{221}(b_4)x_{121}(b_5)x_{111}(b_6)x_{021}(b_7)x_{011}(b_8)x_{001}(b_9),$$
$$u^- \in \langle X_\alpha : n^{-1}.\alpha \in \Phi^-, \alpha \in \Phi^+ \rangle,$$

for $a_1, \ldots, a_9, b_1, \ldots, b_9 \in K$ and $t_1, t_2, t_3 \in K^*$. Then g is an arbitrary element of $G_{\langle e_1, e_2, e_3 \rangle}$, written in terms of its Bruhat decomposition, and therefore, g^{n_3} is an arbitrary element of $G_{\langle e_1, e_2, f_3 \rangle}$.

For each possible n, we use Magma to solve the system of equations corresponding to $g^{n_3} \in G_{W^\ddagger}$, similarly to how we did for $p = 7$. More care is now required for the setup of the computations in Magma since the characteristic is arbitrary. What we do is find a Groebner basis over $\mathbb{Q}$ and also output the list of primes that the algorithm divided by in its various steps. If p is not in such list, we can use the Groebner basis to easily solve the system; otherwise, we simply run the Groebner basis algorithm again over $\mathbb{F}_p$. This is done as per Listing 6 and Listing 7. We find that g^{n_3} does not stabilise $W^\ddagger$ when $n \in \{n_1, n_2, n_1 n_2 n_1\}$. When $n \in \{1, n_1 n_2, n_2 n_1\}$, we find that g^{n_3} stabilises $W^\ddagger$ if and only if $u = u^- = 1$ and t_1, t_2, t_3 satisfy the following:

$t_1 = t_2^{-2} t_3^{-1},$	$t_2^7 = 1,$	$t_3^2 = 1,$	when $n = 1$;
$t_1 = -t_2^{-2} t_3^{-1},$	$t_2^7 = 1,$	$t_3^2 = 1,$	when $n = n_1 n_2$;
$t_1 = t_2^{-2} t_3^{-1},$	$t_2^7 = -1,$	$t_3^2 = 1,$	when $n = n_2 n_1$.

This is easily seen to be equivalent to $g^{n_3} \in S^*$. The case $H \leq G_{\langle e_3, f_1, f_2 \rangle}$ follows similarly, with an arbitrary element of $G_{\langle e_3, f_1, f_2 \rangle}$ being given by $g^{n_1 n_2 n_3 n_2 n_1 n_2 n_3 n_2}$. □

Lemma 5.29. *Let $H = N_G(S^\dagger)$. Then $H_{W^*} = S^*$.*

Proof. Assume $p \neq 7$. Since x is a regular semisimple element, we have $H \leq N_G(T)$, from which it is easy to check that $H \leq T.\langle \tau^* \rangle$. A quick calculation shows that $T_{W^*} = \pm \langle x \rangle$, concluding as required.

If $p = 7$, we have $H \leq B$, where B is the standard Borel subgroup of G. Then we conclude by Lemma 5.28. □

Lemma 5.30. *Suppose that $S^\dagger \leq H.\mathbb{Z}_2 < G$, where H is an arbitrary subgroup of G. Then $(H.\mathbb{Z}_2)_{W^*} \leq S^*$ if and only if $H_{W^*} \leq S^*$.*

Proof. The forward direction is trivial. Suppose that $H_{W^*} \leq S^*$. Since $S^\dagger$ has order 21, we must have $S^\dagger \leq H$. Since $H_{W^*} \leq S^*$, the subgroup $S^\dagger$ is the unique subgroup of order 21 in H_{W^*}. Therefore, $H_{W^*} \lhd (H.\mathbb{Z}_2)_{W^*}$ implies that $S^\dagger$ is normal in $(H.\mathbb{Z}_2)_{W^*}$. By Lemma 5.29, we know that the stabilizer of W^* in $N_G(S^\dagger)$ is S^*, concluding. □

Proposition 5.31. *Let $G = C_3$, $\lambda = \lambda_2$ with $p \neq 3$. Then $C_{\mathcal{S}'_7(V)} = \mathbb{Z}_7.\mathbb{Z}_6$ and $C_{\mathcal{S}''_7(V)} = \mathbb{Z}_{7/(p,7)}.\mathbb{Z}_6$.*

Proof. Let $S = G_{W^*}$. We shall prove that $S \leq N_G(S^\dagger)$. This will conclude the proof of the proposition as follows. By Lemma 5.29, we have $(N_G(S^\dagger))_{W^*} = S^*$, which then implies $S = S^*$ and that W^* is in a dense G-orbit on one of the two D_7-orbits on $\mathcal{S}_7(V)$. If $p \neq 7$, then $W_{(\omega,i)}$ and $W_{(\omega^2,i)}$ intersect in a 6-dimensional subspace, and therefore, by Lemma 2.6, they belong to distinct D_7-orbits on $\mathcal{S}_7(V)$. They each have stabilizer S^*, concluding the $p \neq 7$ case. If $p = 7$, the subspace W^* intersects $W_{(2,i)}$ trivially, which by Lemma 2.6 implies that W^* and $W_{(4,\zeta)}$ belong to distinct D_7-orbits on $\mathcal{S}_7(V)$. Again $G_{W^*} = S^*$, concluding.

In order to prove that $S \leq N_G(\langle x \rangle)$, we show that for all $p < \infty$ and $e \in \mathbb{Z}_{\geq 1}$, if

$$S^\dagger \leq R \leq Sp_6(p^e) = Sp_6(q) < G, \text{ with } R \not\leq N_G(S^\dagger),$$

then R does not stabilise W^*. Note that this is indeed sufficient since if $g \in S \setminus N_G(S^\dagger)$, then there must exist $e \in \mathbb{Z}_{\geq 1}$ such that $g \in Sp_6(p^e)$, with $R = \langle S, g \rangle$ satisfying the condition above. We shall make extensive use of maximal subgroups of $Sp_6(q)$, combined with computations in Magma. The $p = \infty$ case then follows from the $p < \infty$ case since they are equivalent once p is sufficiently large.

Suppose that

$$S^\dagger \leq R \leq M_q < Sp_6(q) < G, \text{ with } R \not\leq N_G(S^\dagger),$$

where M_q is a maximal subgroup of $Sp_6(q)$, as listed in [5, Table 8.28, Table 8.29]. By Lemma 5.28, we can assume that M_q is an irreducible subgroup, with order divisible by $|S^\dagger| = 21$, and by Lemma 5.30, we can assume that M_q does not contain a reducible subgroup of index 2. The goal is to prove that R does not stabilize W^*. Going through (5, Table 8.28,Table 8.29) for the maximal subgroups of $Sp_6(q)$, and (5, Table 8.30,Table 8.31,Table 8.32,Table 8.33,Table 8.34) for the maximal subgroups of $SO_6^+(q)$, $SO_6^-(q)$ and $G_2(q)$ in even characteristic, we reduce to having to consider the cases in the following table:

Class	M_q	Notes
$\mathscr{C}_2$	$(Sp_2(q)^3).Sym(3)$	
$\mathscr{C}_4$	$Sp_2(q) \otimes GO_3(q)$	$p \neq 2$
$\mathscr{C}_5$	$Sp_6(q_0).\mathbb{Z}_2$	$q = q_0^2$
$\mathscr{S}$	$Z(G).PSL_2(7).\mathbb{Z}_2$	
$\mathscr{S}$	$Z(G).PSL_2(13)$	
$\mathscr{S}$	$(Z(G) \times U_3(3)).\mathbb{Z}_2$	
$\mathscr{S}$	$Z(G).J_2$	
$\mathscr{S}$	$SL_2(q)$	$p \geq 7$

If M_q is $Sp_6(q_0).\mathbb{Z}_2$, then by Lemma 5.30, we reduce to one of the other cases.

Suppose that $M_q = (Sp_2(q)^3).Sym(3)$. By Lemma 5.30, we can assume that $R \leq (Sp_2(q)^3).\mathbb{Z}_3$. Since x has order 7, it must be contained in $Sp_2(q)^3$. Therefore, $p \neq 7$ since $Sp_2(q)^3$ does not contain a regular unipotent element. Then $Sp_2(q)^3$ must be the stabilizer of the orthogonal sum $\langle e_1, f_1 \rangle \perp \langle e_2, f_2 \rangle \perp \langle e_3, f_3 \rangle$ in $Sp_6(q)$. This implies that $(Sp_2(q)^3).\mathbb{Z}_3 = (Sp_2(q)^3).\langle \tau^4 \rangle$. Since $\tau^4 \in S^\dagger$, it remains to determine the stabilizer of W^* in $(Sp_2(q)^3)$, which is easily seen to be $\langle x, \tau^3 \rangle \leq S^*$, concluding.

Suppose that $M_q = Sp_2(q) \otimes GO_3(q)$ with $p \neq 2$. Then M_q does not contain a regular unipotent element, and therefore, $p \neq 7$. Comparing the actions of M_q and $S^\dagger$ on V, we also see that M_q does not contain $S^\dagger$.

Suppose that $M_q = Z(G).PSL_2(7).\mathbb{Z}_2$. By Lemma 5.30, we can assume that $R \leq Z(G).PSL_2(7)$. Here, we use Magma to determine that the only possibility for R is $Z(G).PSL_2(7)$ itself, which acts on V as a sum of a 6-dimensional irreducible and an 8-dimensional irreducible when $p \neq 7$, and with composition factors of dimension $5, 1, 3, 5$ when $p = 7$, therefore not stabilising W^*. The Magma code used here and for the next cases can be found in Listing 8.

Suppose that $M_q = Z(G).PSL_2(13)$. Then M_q does not contain a subgroup of order 21, a contradiction. Suppose that $M_q = (Z(G) \times U_3(3)).\mathbb{Z}_2$. Clearly, we can reduce to the case $R \leq U_3(3)$. Here, we find that the only possibility for R is $PSL_2(7)$, concluding like for the $M_q = Z(G).PSL_2(7).\mathbb{Z}_2$ case.

Suppose that $M_q = Z(G).J_2$. Then we find that all the possibilities for R contain one of $PSL_2(7)$, $SL_2(7)$, $U_3(3)$, concluding via the previous analysis. Finally, if $M_q = SL_2(q)$, a similar treatment using the maximal subgroups of $SL_2(q)$ allows us to conclude. □

6. Proof of Theorem 4

In this section, we shall prove Theorem 4. Unlike with previous work, we shall not be interested in determining the exact structure of the (semi-)generic stabilizers, if they exist. Instead, we will often resort to finding a dense open subset of the variety where the stabilizers have a certain minimal dimension in order to exclude the possibility of a dense orbit, in a very similar fashion to the work we have done to determine (semi-)generic stabilizers. The following lemma is a crucial tool for narrowing down the cases we will have to consider.

Lemma 6.1. *Let $H = Cl(V_1) \otimes Cl(V_2) \leq Cl(V_1 \otimes V_2) = Cl(V)$. Assume that $\dim V_2 = k \dim V_1$ for some $k \geq 1$. Let $G = Cl(V_1) \otimes Cl(V_2') \leq Cl(V_1 \otimes V_2') = Cl(V')$ with $\dim V_2' \geq \dim V_2$ and $Cl(V), Cl(V') = SO(V), SO(V')$ or $Cl(V), Cl(V') = Sp(V), Sp(V')$. Then H has a dense orbit on $\mathcal{S}_k(V)$ if and only if G has a dense orbit on $\mathcal{S}_k(V')$.*

Proof. Write $\dim V_1 = d_1$, $\dim V_2 = d_2 = kd_1$, $\dim V_2' = d_2'$; then $\dim V = d_1 d_2$, $\dim V' = d_1 d_2'$. We may assume $V_2 \leq V_2'$; let V_2'' be the orthogonal complement to V_2 in V_2', so that $\dim V_2'' = d_2' - d_2$. Let ϵ_V be 1 or -1 according as V and V' are both orthogonal or both symplectic, and similarly ϵ_{V_2} be 1 or -1 according as V_2, V_2' and V_2'' are all orthogonal or all symplectic. Then

$$\dim G - \dim \mathcal{S}_k(V') = \dim Cl(V_1) + \dim Cl(V_2') - kd_1 d_2' + \frac{3k^2 + \epsilon_V k}{2},$$

$$\dim H - \dim \mathcal{S}_k(V) = \dim Cl(V_1) + \dim Cl(V_2) - kd_1 d_2 + \frac{3k^2 + \epsilon_V k}{2};$$

thus,

$$
\begin{aligned}
&(\dim G - \dim \mathcal{S}_k(V')) - (\dim H - \dim \mathcal{S}_k(V)) \\
&= \dim Cl(V_2') - \dim Cl(V_2) - kd_1(d_2' - d_2) \\
&= \frac{1}{2}d_2'(d_2' - \epsilon_{V_2}) - \frac{1}{2}d_2(d_2 - \epsilon_{V_2}) - d_2(d_2' - d_2) \\
&= \frac{1}{2}(d_2' - d_2)(d_2' - d_2 - \epsilon_{V_2}) \\
&= \dim Cl(V_2'').
\end{aligned}
$$

Now let $v_1, \ldots, v_{d_1}$ be a fixed basis of V_1. Given $y \in \mathcal{S}_k(V')$, choose a basis $x_1, \ldots, x_k$ of y and write each x_j uniquely as $\sum_{i=1}^{d_1} v_i \otimes u_{ij}$ with each $u_{ij} \in V_2'$; set $\mathrm{supp}_2(y) = \langle u_{ij} : 1 \leq i \leq d_1, 1 \leq j \leq k \rangle \leq V_2'$. Define

$$Y = \{y \in \mathcal{S}_k(V') : \mathrm{supp}_2(y) \text{ is nondegenerate of dimension } d_2\}.$$

The set Y is dense in $\mathcal{S}_k(V')$ because the set $\{y \in \mathcal{S}_k(V') : \dim \mathrm{supp}_2(y) < d_2\}$ is a proper closed subvariety of $\mathcal{S}_k(V')$, and nondegenerate d_2-spaces are dense in the variety of all d_2-spaces in V_2'; likewise, $Y \cap \mathcal{S}_k(V)$ is dense in $\mathcal{S}_k(V)$. Moreover, all nondegenerate d_2-spaces in V_2' lie in a single $Cl(V_2')$-orbit; thus, given $y \in Y$, by applying an element of G, we may assume that $\mathrm{supp}_2(y) = V_2$,

and then the stabilizer of y in G must fix V_2 and hence V_2'', whence $G_y = H_y \times Cl(V_2'')$, so that $\dim G_y - \dim H_y = \dim Cl(V_2'')$, and hence,

$$\dim \mathcal{S}_k(V') - (\dim G - \dim G_y) = \dim \mathcal{S}_k(V) - (\dim H - \dim H_y).$$

Now if H has a dense orbit on $\mathcal{S}_k(V)$, it must meet $Y \cap \mathcal{S}_k(V)$; thus, it contains some $y \in Y \cap \mathcal{S}_k(V)$ for which the right side of the above equation is zero, whence the left side is also zero and so G has a dense orbit on $\mathcal{S}_k(V')$. Conversely, if G has a dense orbit on $\mathcal{S}_k(V')$, it must meet Y, and therefore, $Y \cap \mathcal{S}_k(V)$; thus, it contains some $y \in Y \cap \mathcal{S}_k(V)$ for which the left side of the above equation is zero, whence the right side is also zero and H has a dense orbit on $\mathcal{S}_k(V)$. □

Lemma 6.2. *Let* $G = SO(V_1) \otimes SO(V_2)$ *with* $3 \leq \dim V_1 \leq \dim V_2$. *Then* G *does not have a dense orbit on* $\mathcal{S}_k(V_1 \otimes V_2)$ *for all* $1 \leq k \leq \frac{1}{2} \dim V_1 \dim V_2$.

Proof. Let $m = \dim V_1$, $n = \dim V_2$ and assume that G does have a dense orbit on $\mathcal{S}_k(V_1 \otimes V_2)$. Then

$$\dim G = \frac{1}{2}(m^2 + n^2 - m - n) \geq mnk - \frac{3k^2 + k}{2} = \dim \mathcal{S}_k(V_1 \otimes V_2).$$

We first show that $mk \leq n$. If $k = 1$ this is immediate, so assume $k > 1$. Write $n = am$ for some $a \geq 1$. Then if we define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = m^2 + m^2 x^2 - m - mx - 2m^2 xk + 3k^2 + k,$$

we have $g(a) = 2(\dim G - \dim \mathcal{S}_k(V_1 \otimes V_2)) \geq 0$. The discriminant of $g(x)$ is $(-m - 2m^2 k)^2 - 4m^2(m^2 - m + 3k^2 + k) = m^2 h(m)$, where we define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = 1 - 4k - 12k^2 + 4(k+1)x + 4(k^2 - 1)x^2.$$

In turn, the discriminant of $h(x)$ is $16(k+1)^2 - 16(k^2 - 1)(1 - 4k - 12k^2) = 16(k+1)(12k^3 - 8k^2 - 4k + 2) \geq 0$, so that the equation $y = h(x)$ has real roots; the positive root is

$$m = \frac{-(k+1) + \sqrt{(k+1)(12k^3 - 8k^2 - 4k + 2)}}{2(k^2 - 1)},$$

which is easily seen to be always less than 2. Thus, as $m \geq 3$, we have $h(m) > 0$, so that $g(x)$ has positive discriminant, and therefore, the equation $y = g(x)$ has real roots

$$r_1, r_2 = \frac{1 + 2km \pm \sqrt{h(m)}}{2m},$$

where $r_1 < r_2$. We claim that $r_1 < a$. Since $a \geq 1$, the claim is certainly true if $r_1 < 1$; we have $r_1 < 1 \Leftrightarrow h(m) > (2km - 2m + 1)^2 \Leftrightarrow 2km^2 - 2m^2 + 2m - 3k^2 - k > 0$. If $k \leq 4$, the last inequality holds as $m \geq 3$, so we may assume $k \geq 5$. If we had $k \leq \frac{1}{2}m^2$, this would force $m \geq 4$, and then

$$2km^2 - 2m^2 + 2m - 3k^2 - k = \frac{2}{5}(k-5)m^2 + 2m + k\left(\frac{8}{5}m^2 - 3k - 1\right)$$

$$\geq \frac{2}{5}(k-5)m^2 + 2m + k\left(\frac{8}{5}m^2 - \frac{3}{2}m^2 - 1\right)$$

$$= \frac{2}{5}(k-5)m^2 + 2m + k\left(\frac{1}{10}m^2 - 1\right)$$

$$> 0,$$

so we may assume $k \geq \frac{1}{2}m^2$. Write $k = \theta m^2$; as $k \leq \frac{1}{2}mn = \frac{1}{2}am^2$, we have $\frac{1}{2} < \theta \leq \frac{1}{2}a$. Then $r_1 < 2\theta$ if and only if

$$1 + 2\theta m^3 - \sqrt{4\theta^2 m^6 - 12\theta^2 m^4 + 4\theta m^3 - 4(1+\theta)m^2 + 4m + 1} < 4\theta m,$$

which reduces to

$$m(\theta^2(m^2 - 4) - \theta - 1) + (1 + 2\theta) > 0,$$

which is true for all $m \geq 3$ and $\theta > \frac{1}{2}$. Thus, we do have $r_1 < a$, as claimed; so as $g(a) \geq 0$, we must have $r_2 < a$, whence $a > \frac{r_1 + r_2}{2} = \frac{1 + 2km}{2m} > k$ as required.

Let $U \leq V_2$ be a nondegenerate subspace of dimension $k \dim V_1$. By Lemma 6.1, we know that $SO(V_1) \otimes SO(U)$ has a dense orbit on $\mathcal{S}_k(V \otimes U)$. The dimension requirement is that $m^2 - m + m^2k^2 - mk \geq 2m^2k^2 - 3k^2 - k$, which is absurd when $m \geq 3$ and $k \geq 1$. This contradicts G having a dense orbit on $\mathcal{S}_k(V_1 \otimes V_2)$. □

Lemma 6.3. *Let* $G = Sp(V_1) \otimes SO(V_2)$ *with* $4 \leq \dim V_1 \leq \dim V_2$ *or* $3 \leq \dim V_2 \leq \dim V_1$. *Then* G *does not have a dense orbit on* $\mathcal{S}_k(V_1 \otimes V_2)$ *for all* $2 \leq k \leq \frac{1}{2} \dim V_1 \dim V_2$.

Proof. This is similar to Lemma 6.2. Note that here $k \geq 2$. □

Lemma 6.4. *Let* $G = Sp(V_1) \otimes Sp(V_2)$ *with* $4 \leq \dim V_1 \leq \dim V_2$. *Then if* G *has a dense orbit on* $\mathcal{S}_k(V_1 \otimes V_2)$, *we must have* $k = 1$.

Proof. Let $m = \frac{\dim V_1}{2}$, $n = \frac{\dim V_2}{2}$ and assume that G does have a dense orbit on $\mathcal{S}_k(V_1 \otimes V_2)$. Then

$$\dim G = 2m^2 + 2n^2 + m + n \geq 4mnk - \frac{3k^2 + k}{2} = \dim \mathcal{S}_k(V_1 \otimes V_2), .$$

Similar calculations as in Lemma 6.2 show that $k \dim V_1 \leq \dim V_2$. By Lemma 6.1, we then deduce that $Sp(V_1) \otimes Sp(U_2)$ has a dense orbit on $\mathcal{S}_k(V_1 \otimes U_2)$, where $\dim U_2 = k \dim V_1$. Then dimensional considerations rule out $k \geq 2$. □

Proposition 6.5. *Let* $G = Sp(V_1) \otimes Sp(V_1)$ *with* $\dim V_1 \geq 6$. *Then* G *does not have a dense orbit on* $\mathcal{S}_1(V_1 \otimes V_1)$.

Proof. Let $2n = \dim V_1$ and $e_1, \ldots, e_n, f_n, \ldots, f_1$ be the standard basis of V_1. Let

$$Y = \left\{ \left\langle \sum_{i=1}^{n} a_i e_i \otimes f_i \right\rangle : \sum a_i^2 = 0 \right\},$$

a subvariety of $\mathcal{S}_1(V_1 \otimes V_1)$. Let

$$\hat{Y} = \left\{ \left\langle \sum_{i=1}^{n} a_i e_i \otimes f_i \right\rangle : \sum a_i^2 = 0, a_i \neq a_j \text{ if } i \neq j \right\},$$

a dense subset of Y. Let $y \in \hat{Y}$. Then $\mathrm{Tran}_G(y, Y)$ contains an A_1^n stabilising all elements of y, which projects onto each $Sp(V_1)$ as $\bigcap Sp(V_1)_{\langle e_i, f_i \rangle}$. By assumption on the a_i's, it is easy to see that this is the connected component of $\mathrm{Tran}_G(y, Y)$. Since $\dim G - \dim \mathrm{Tran}_G(y, Y) = 4n^2 + 2n - 3n = 4n^2 - 2 - (n-2) = \dim \mathcal{S}_1(V_1 \otimes V_1) - \dim Y$, we find that y is Y-exact. Therefore, by Lemma 2.14, we conclude that there is an open dense subset of the variety of singular 1-spaces of $V_1 \otimes V_1$ such that all stabilizers are $3n$-dimensional. Therefore, $3n$ is the lower bound for the dimension of the stabilizer of any singular 1-space. Since $\dim G - 3n = 4n^2 - 2 - (n-2) > \dim \mathcal{S}_1(V_1 \otimes V_1)$ when $n \geq 3$, we conclude that G does not have a dense orbit on $\mathcal{S}_1(V_1 \otimes V_1)$. □

Lemma 6.6. *Let $G = Sp_2 \otimes Sp(V_2) \leq SO(V)$ with $\dim V = 2 \dim V_2 \geq 20$. Suppose that G has a dense orbit on $\mathcal{S}_k(V)$ for some $2 \leq k \leq \dim V_2$. Then k is either $2, 3, 4, \dim V_2 - 1, \dim V_2'$ or $\dim V_2''$.*

Proof. Calculations similar to the ones in Lemma 6.2 show that if $k < \dim V_2 - 1$, we have $2k \leq \dim V_2$. We can then apply Lemma 6.1 to determine that $Sp_2 \otimes Sp_{2k} = Sp(V_1) \otimes Sp(U)$ must have a dense orbit on $\mathcal{S}_k(V_1 \otimes U)$. Dimensional considerations then give $3 + 2k^2 + k \geq 4k^2 - \frac{3}{2}k^2 - \frac{k}{2}$, which implies $k \leq 4$. $\qquad\square$

Lemma 6.7. *Let $G = Sp_2 \otimes Sp(V_2) \leq SO(V)$ with $\dim V = 2 \dim V_2 \leq 16$. Suppose that G has a dense orbit on $\mathcal{S}_k(V)$ for some $2 \leq k \leq \dim V_2 - 2$. Then either $k = 2$ or $(k, \dim V_2)$ is one of $(3,6), (3,8), (4,6), (4,8), (6,8)$.*

Proof. This is simply a matter of checking the dimension of $\mathcal{S}_k(V)$ in all finitely many cases. $\qquad\square$

Proposition 6.8. *Let $G = Sp_2 \otimes Sp_{2n} \leq SO_{4n} = SO(V)$. Then G has a dense orbit on $\mathcal{S}'_{2n}(V)$ (and $\mathcal{S}''_{2n}(V)$) if and only if $n = 1, 2, 3$.*

Proof. When $n = 1, 2$ the group G is spherical in $SO(V)$. Therefore, assume that $n \geq 3$. Let $V_1 = \langle e, f \rangle$ and $V_2 = \langle e_1, \ldots, e_n, f_n, \ldots, f_1 \rangle$ so that $G = Sp(V_1) \otimes Sp(V_2)$ and the given bases are the standard bases for V_1 and V_2. Given $\mathbf{a} = (a_1, \ldots, a_n) \in K^n$, define $W_{\mathbf{a}} = \langle (e + a_i f) \otimes e_i, (e + a_i f) \otimes f_i : 1 \leq i \leq n \rangle$, a $2n$-dimensional totally singular subspace of V. Define

$$Y = \{W_{\mathbf{a}} : \mathbf{a} \in K^n\},$$

an n-dimensional subvariety of $\mathcal{S}'_{2n}(V)$. Let

$$\hat{Y} = \{W_{\mathbf{a}} : \mathbf{a} \in K^n, a_1 \ldots a_n \neq 0, a_i \neq a_j \text{ for all } i \neq j\},$$

a dense subset of Y. Let $y \in \hat{Y}$. It is easily seen that $\mathrm{Tran}_G(y, Y)$ has connected component $A_1^{n+1} = Sp(V_1) \otimes \bigcap Sp(V_2)_{\langle e_i + f_i \rangle}$, while $G_y^0 = 1 \otimes \bigcap Sp(V_2)_{\langle e_i + f_i \rangle} \simeq A_1^n$. Since

$$\dim G - \dim \mathrm{Tran}_G(y, Y) = \dim \mathcal{S}'_{2n}(V) - \dim Y,$$

we have that $\hat{Y}$ is Y-exact. Therefore, by Lemma 2.14 and Corollary 2.12, we know that $\dim A_1^n$ is the minimum dimension for the stabilizer of any $y \in \mathcal{S}'_{2n}(V)$. Dimensional considerations rule out $n \geq 4$, while for $n = 3$, we have $\dim G - \dim A_1^3 = 15 = \dim \mathcal{S}'_{2n}(V)$. By Lemma 2.6, changing the definition of $W_{\mathbf{a}}$ by swapping the first two generators $(e + a_1 f) \otimes e_1, (e + a_1 f) \otimes f_1$ with $e \otimes (e_1 + a_1 f_1), f \otimes (e_1 + a_1 f_1)$ leads to the same result for the action on $\mathcal{S}''_{2n}(V)$. $\qquad\square$

Proposition 6.9. *Let $G = Sp_2 \otimes Sp_{2n} \leq SO(4n) = SO(V)$. Then G has a dense orbit on $\mathcal{S}_{2n-1}(V)$ if and only if $n = 1, 2, 3$.*

Proof. Suppose that G has a dense orbit on $\mathcal{S}_{2n-1}(V)$. Every $y \in \mathcal{S}_{2n-1}(V)$ is contained in precisely one element of $\mathcal{S}'_{2n}(V)$ and one of $\mathcal{S}''_{2n}(V)$. Let $\mathcal{O}$ be the dense orbit of G on $\mathcal{S}_{2n-1}(V)$; then its complement $\mathcal{S}_{2n-1}(V) \setminus \mathcal{O}$ is contained in a proper closed subvariety X of $\mathcal{S}_{2n-1}(V)$. Let Z be the set of elements of $\mathcal{S}'_{2n}(V)$ all of whose hyperplanes lie in X; then Z is a proper closed subvariety of $\mathcal{S}'_{2n}(V)$, so its complement $\mathcal{S}'_{2n}(V) \setminus Z$ is a dense subset of $\mathcal{S}'_{2n}(V)$ with the property that any of its elements has a hyperplane lying in $\mathcal{O}$. Thus, given two elements of $\mathcal{S}'_{2n}(V) \setminus Z$, we can choose hyperplanes within them and an element of G which sends one hyperplane to the other and therefore one element of $\mathcal{S}'_{2n}(V) \setminus Z$ to the other; so $\mathcal{S}'_{2n}(V) \setminus Z$ lies in a single G-orbit, and therefore, G has a dense orbit on $\mathcal{S}'_{2n}(V)$. Replacing $\mathcal{S}'_{2n}(V)$ by $\mathcal{S}''_{2n}(V)$ shows that G also has a dense orbit on $\mathcal{S}''_{2n}(V)$. By Proposition 6.8, we therefore have $n = 1, 2, 3$. When $n = 1, 2$ the group G is spherical in $SO(V)$. Therefore, assume that $n = 3$. Given y in the dense G-orbit on $\mathcal{S}'_6(V)$, the group induced by G_y on y is A_1^3 acting as a sum of three natural modules for A_1. Therefore, G_y has a dense orbit on 5-spaces of y, concluding that G has a dense orbit on $\mathcal{S}_5(V)$. $\qquad\square$

Proposition 6.10. *Let $G = Sp_2 \otimes Sp_6 \leq SO(12) = SO(V)$. Then G has a dense orbit on $\mathcal{S}_3(V)$.*

Proof. Let $V_1 = \langle e, f \rangle$ and $V_2 = \langle e_1, e_2, e_3, f_3, f_2, f_1 \rangle$ so that $G = Sp(V_1) \otimes Sp(V_2)$ and the given bases are the standard bases for V_1 and V_2. Let

$$W = \langle e \otimes e_1 + f \otimes e_2, \quad e \otimes f_2 + f \otimes f_1, \quad e \otimes (e_2 + e_3) + f \otimes (f_3 - f_2) \rangle,$$

a totally singular 3-space of V. Let $S = G_W$. Let $g \in S$ such that $g = 1 \otimes g_1$. Then g_1 fixes $\langle e_1, f_2, e_2 + e_3 \rangle$ and $\langle e_2, f_1, f_3 - f_2 \rangle$, and consequently their radicals $\langle e_1 \rangle$ and $\langle f_1 \rangle$. One then quickly reaches the conclusion that g_1 must also fix $\langle e_2 \rangle$, $\langle f_2 \rangle$, $\langle e_2 + e_3 \rangle$ and $\langle f_3 - f_2 \rangle$. However, this would mean that g acts on W by sending $e \otimes (e_2 + e_3) + f \otimes (f_3 - f_2) \mapsto \lambda e \otimes (e_2 + e_3) + \lambda^{-1} f \otimes (f_3 - f_2)$, implying that $g_1 = \pm 1$. To conclude, we observe that given $g = g_1 \otimes 1$, by a simple application of Witt's Lemma, we can find $g' = 1 \otimes g_2$ such that $gg' \in S$. Therefore, $S^0 = Sp_2$. Since $\dim G - \dim S = 21 = \dim \mathcal{S}_3(V)$, we conclude that G has a dense orbit on $\mathcal{S}_3(V)$. □

Proposition 6.11. *Let $G = Sp_2 \otimes Sp_6 \leq SO_{12} = SO(V)$. Then G has no dense orbit on $\mathcal{S}_4(V)$.*

Proof. Let $V_1 = \langle e, f \rangle$ and $V_2 = \langle e_1, e_2, e_3, f_3, f_2, f_1 \rangle$ so that $G = Sp(V_1) \otimes Sp(V_2)$ and the given bases are the standard bases for V_{nat}. The stabilizer of an element in a dense orbit would have dimension 1. We will show that already the group $G_2 := 1 \otimes Sp_6 < G$ acts on $\mathcal{S}_4(V)$ with stabilizers that are at least 3-dimensional. Let W_{abcd} be the totally singular 4-space spanned by

$$e \otimes e_1 + f \otimes (ae_1 + be_2 + ce : 3 + de_4), \ e \otimes e_2 + f \otimes (be_1 + de_2),$$
$$e \otimes f_1 + f \otimes (af_1 + bf_2 + cf_3 + df_4), \quad e \otimes f_2 + f \otimes (bf_1 + df_2).$$

Let $Y = \{W_{abcd} : a, b, c, d \in K\}$, a 4-dimensional subvariety of $\mathcal{S}_4(V)$. Let $\hat{Y} = \{W_{abcd} : a, b, c, d \in K^*\}$, a dense subset of Y. Take $y = W_{abcd} \in \hat{Y}$. Since $y \subseteq e \otimes \langle e_1, f_2, e_2, f_2 \rangle + f \otimes V_6$, the stabilizer $(G_2)_y$ must preserve $\langle e_1, f_1, e_2, f_2 \rangle$, and therefore its orthogonal complement $\langle e_3, f_3 \rangle$. Since for all $g \in (G_2)_y$ we must have $g.(e \otimes e_2 + f \otimes (be_1 + de_2)) \in y$ and $g.(e \otimes f_2 + f \otimes (bf_1 + df_2)) \in y$, we get that $\langle e_2, f_2 \rangle$ must also be preserved by $(G_2)_y$. Therefore, $(G_2)_y \leq A_1^3$. It is now immediate to see that the image of any e_i or f_i in $\langle e_i, f_i \rangle$ completely determines the element $g \in (G_2)_y$. Since the standard diagonal subgroup $A := A_1 \leq A_1^3$ fixes y, we must then have $(G_2)_y = A$. Now assume that $g \in \mathrm{Tran}_{G_2}(y, Y)$. Again, we have $g \in A_1^3$, and since A clearly fixes any element of Y, it is the stabilizer of $g.y$. Therefore, $g \in N_{A_1^3}(A)$, which is a finite extension of A. This shows that $\dim \mathrm{Tran}_{G_2}(y, Y) = 3$, and therefore, $\mathrm{codim}\, \mathrm{Tran}_{G_2}(y, Y) = 18 = 22 - 4 = \mathrm{codim}\, Y$. This shows that all points in $\hat{Y}$ are Y-exact, and Lemma 2.14 allows us to conclude that $3 = \dim A_1$ is the lower bound for the dimension of any stabilizer for the G_2-action on $\mathcal{S}_4(V)$. In particular, this proves that G has no dense orbit on $\mathcal{S}_4(V)$. □

Proposition 6.12. *Let $G = Sp_2 \otimes Sp_8 \leq SO_{16} = SO(V)$. Then G has no dense orbit on $\mathcal{S}_4(V)$ and on $\mathcal{S}_6(V)$.*

Proof. Let $V_1 = \langle e, f \rangle$ and $V_2 = \langle e_1, e_2, e_3, e_4, f_4, f_3, f_2, f_1 \rangle$ so that $G = Sp(V_1) \otimes Sp(V_2)$ and the given bases are the standard bases for V_1 and V_2. The stabilizer of an element in a dense orbit would have dimension 2. We will show that already the group $G_2 := 1 \otimes Sp_8 < G$ acts on $\mathcal{S}_4(V)$ with stabilizers that are at least 3-dimensional. Given $\mathbf{a} \in K^7$, let $W_{\mathbf{a}}$ be the totally singular 4-space spanned by vectors

$$e \otimes e_1 + f \otimes (a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4), \ e \otimes e_2 + f \otimes (a_2 e_1 + a_5 e_2 + a_6 e_3 + a_7 e_4),$$
$$e \otimes f_1 + f \otimes (a_1 f_1 + a_2 f_2 + a_3 f_3 + a_4 f_4), \ e \otimes f_2 + f \otimes (a_2 f_1 + a_5 f_2 + a_6 f_3 + a_7 f_4).$$

Let $Y = \{W_{\mathbf{a}} : \mathbf{a} \in K^7\}$, a 7-dimensional subvariety of $\mathcal{S}_4(V)$. Let $\hat{Y}_1 = \{W_{\mathbf{a}} : \mathbf{a} \in (K^*)^7\}$, a dense subset of Y. The standard diagonal $A_1 \leq A_1^4 = \bigcap (G_2)_{\langle e_i, f_i \rangle}$ fixes any $y \in Y$. Call this A_1-subgroup A. Let $y \in \hat{Y}$. We will now prove that the connected component of $(G_2)_y$ is A. We begin by observing that $(G_2)_y$ fixes $\langle e_1, f_1, e_2, f_2 \rangle$, and therefore, $(G_2)_y \leq C_2 C_2$. Let $\pi_i(A)$ ($i = 1, 2$) denote the projection of A onto each C_2. Assume $p \neq 2$. Then $\pi_i(A)$ is a diagonal A_1 in C_2, which lies in two opposite parabolic subgroups

of C_2, acting irreducibly on their unipotent radicals, and in precisely one Levi subgroup L_i. Also, $\pi_i(A)$ is maximal in infinitely many A_1^2's stabilising a sum of two nondegenerate 2-spaces. Let M be a minimal connected overgroup of A in C_2C_2. Then $\pi_i(M)$ is one of A_1, A_1^2, L_i, U_3A_1. If $\pi_1(M) \simeq \pi_2(M)$, then M is diagonal in C_2C_2 by minimality. Suppose that $\pi_1(M) \not\simeq \pi_2(M)$. If $\pi_i(M) \not\leq L_i$ for $i = 1$ or $i = 2$, then M contains $\pi_1(A) \times \pi_2(A)$. Summarising, the minimal connected ovegroups of A in C_2C_2 are as follows:

(i) A_1^2 diagonal in C_2C_2;
(ii) $A_1^2 = \pi_1(A) \times \pi_2(A)$;
(iii) AT_1, where $T_1 < Z(L_1) \times Z(L_2)$ is a 1-dimensional torus;
(iv) U_3A diagonal in C_2C_2.

Assume we are in the first case and $M = A_1^2$ is diagonal in C_2C_2. Then $\pi_1(A)$ is maximal in an A_1^2 fixing $\langle e_1 + \lambda_1 e_2, f_1 + \lambda_1 f_2 \rangle \perp \langle e_1 - \lambda_1^{-1} e_2, f_1 - \lambda_1^{-1} f_2 \rangle$, while $\pi_2(A)$ is maximal in an A_1^2 fixing $\langle e_3 + \lambda_2 e_4, f_3 + \lambda_2 f_4 \rangle \perp \langle e_3 - \lambda_2^{-1} e_4, f_3 - \lambda_2^{-1} f_4 \rangle$, for some $\lambda_1, \lambda_2 \in K^*$. Let $y \in \hat{Y}_1$ and assume that M fixes y. Considering a 1-dimensional torus in M but not in A, one finds four independent equations in terms of the entries of $\mathbf{a}$ and λ_1, λ_2 that all need to be satisfied since M fixes y. As the variety of diagonal A_1^2's from case (i) is 2-dimensional, the elements y of Y such that there is some such diagonal A_1^2 fixing y lie in a subvariety of Y whose codimension is at least 2. Therefore, there is a dense subset $\hat{Y}_2$ of Y with the property that no minimal connected overgroup of A of type (i) fixes some $y \in \hat{Y}_2$. The same can be quickly deduced for the other cases. This shows that there exists a dense subset $\hat{Y}$ of Y such that A is the connected component of the stabilizer of any $y \in \hat{Y}$. Now take $y \in \hat{Y}$ and $g \in \mathrm{Tran}_{G_2}(y, Y)$. Again, we find $g \in C_2C_2$, and since A clearly fixes all elements of Y, it must be the connected component of the stabilizer of $g.y$. Therefore, $g \in N_{C_2C_2}(A)$. The connected component of $N_{C_2C_2}(A)$ is AT_2, where T_2 is a 2-dimensional torus, and therefore, codim $\mathrm{Tran}_{G_2}(y, Y) \geq 31$. However, codim $Y = 31$, which means that codim $\mathrm{Tran}_{G_2}(y, Y) = 31$ and that all points of $\hat{Y}$ are Y-exact. By 2.14, this proves that the minimal dimension for the stabilizer of any totally singular 4-space of V is 3.

If instead $p = 2$, the reasoning is similar, with the difference that $\pi_i(A)$ is contained in a single parabolic subgroup of C_2 and in no Levi subgroup, and has connected centralizer U_1. Therefore, the minimal connected ovegroups of A in C_2C_2 are as follows:

(i) A_1^2 diagonal in C_2C_2;
(ii) $A_1^2 = \pi_1(A) \times \pi_2(A)$;
(iii) U_1A, where $U_1 \leq C_{C_2}(\pi_1(A)) \times C_{C_2}(\pi_2(A))$ is a 1-dimensional unipotent subgroup.

The same analysis then concludes. The case $\mathcal{S}_6(V)$ is entirely similar. $\qquad\square$

Proof of Theorem 4. Recall that $V = V_1 \otimes V_2$. If $G = SO(V_1) \otimes SO(V_2) \leq SO(V)$, Lemma 6.2 shows that G has no dense orbit on $\mathcal{S}_k(V)$. If $G = Sp(V_1) \otimes SO(V_2)$, Lemma 6.3 likewise shows that G has no dense orbit on $\mathcal{S}_k(V)$ if $k \geq 2$; if instead $k = 1$, then $\mathcal{S}_k(V) = \mathcal{G}_k(V)$, and so G has a dense orbit on $\mathcal{S}_k(V)$ if and only if K^*G has a dense orbit on V, that is, if and only if (K^*G, V) is a prehomogeneous vector space, giving case (i) in the statement of Theorem 4.

For the remainder of the argument, assume $G = Sp(V_1) \otimes Sp(V_2) \leq SO(V)$ with $\dim V_1 \leq \dim V_2$, and G has a dense orbit on $\mathcal{S}_k(V)$. If $k = 1$, Proposition 6.5 and Lemma 6.1 between them show that we must have $\dim V_1 = 2$ or 4; in both possibilities, [26, Thm. 3] shows that G does indeed have a dense orbit on $\mathcal{S}_k(V)$, giving cases (ii) and (vii) respectively in the statement of Theorem 4. Now suppose $k \geq 2$. Lemma 6.4 shows that we must have $\dim V_1 = 2$; write $\dim V_2 = 2n$. If $n \geq 5$, Lemma 6.6 shows that $k = 2, 3, 4, 2n - 1, (2n)'$ or $(2n)''$; if instead $n \leq 4$, Lemma 6.7 shows that either $k = 2$ or $(k, 2n) = (3, 6), (3, 8), (4, 6), (4, 8)$ or $(6, 8)$. If $k = (2n)'$ or $(2n)''$, Proposition 6.8 shows that we must have $n = 1, 2, 3$ (and that then G does indeed have a dense orbit), giving case (v); if $k = 2n - 1$, Proposition 6.9 shows the same thing, giving case (vi). Propositions 6.11 and 6.12 show that $(k, 2n) \neq (4, 6), (4, 8)$ or $(6, 8)$, and then Lemma 6.1 shows that $k \neq 4$. If $k = 3$, Proposition 6.10 shows that G does have a dense orbit if $n = 3$, and then Lemma 6.1 shows that the same is true if $n > 3$,

giving case (iv). Finally, if $k = 2$, the sphericality of $Sp_2 \otimes Sp_4$ shows that G does have a dense orbit if $n = 2$, and then Lemma 6.1 shows that the same is true if $n > 2$, giving case (iii).   □

7. Proof of Theorem 5

We conclude with the proof of Theorem 5.

Proof of Theorem 5. By Theorem 2.1, either both G and H are parabolic subgroups, and by the Bruhat decomposition we have finitely many (G, H)-double cosets in Γ; or they are both reductive, and by [6, Thm. A], there is a dense double coset if and only if there is a single double coset (hence a factorization $\Gamma = GH$); or one of the two subgroups, say H, is a maximal parabolic, and G is reductive. It remains to deal with this last case, so assume that $H = P_k$ is a maximal parabolic and G is reductive. If $\Gamma = SL(V)$ or if $\Gamma = Sp(V)$ with $k = 1$, there is a dense (G, H)-double coset in Γ if and only if G acts on the Grassmannian $\mathcal{G}_k(V)$ with a dense orbit. This is equivalent to $GL_k \otimes G$ acting on $K^k \otimes V$ with a dense orbit – which is equivalent to the pair $(GL_k \otimes G, K^k \otimes V)$ being a prehomogeneous vector space, as classified by [29][32][33].

Now assume that $\Gamma \neq SL(V)$ and $k \neq 1$ if $\Gamma = Sp(V)$. There are 4 options for G. The first option is for G to be simple and irreducible on V, in which case there is a dense (G, H)-double coset if and only if G has a dense orbit on the variety of totally singular subspaces corresponding to Γ/H, as classified by Theorem 3.

The second option is for G to be the connected component of the stabilizer of an orthogonal sum (i.e., $G = Sp(V_1) \times Sp(V_2)$ and $\Gamma = Sp(V_1 \perp V_2)$ or $G = SO(V_1) \times SO(V_2)$ and $\Gamma = SO(V_1 \perp V_2)$), and in both cases, G is a spherical subgroup of Γ. Thirdly, we can have G being the stabilizer of a degenerate but nonsingular 1-space of V, where $p = 2$ and V is orthogonal. In this case, G is again spherical in Γ.

Lastly, G can be semisimple but not simple, acting irreducibly and tensor decomposably on V. The possibilities for (G, k) are then given by Theorem 4.   □

A. Magma code

Listing 1. Double covers of Sym(6) and Alt(7)

```
//Construct the double cover of Sym(6).
G := Sym(6);
F := FPGroup(G);
F2 := pCover(G, F, 2);
G2 := PermutationGroup(F2);

//There is a single conjugacy class of elements of order 5. Such an element
    generates <x>
X2 := sub<G2|ConjugacyClasses(G2)[8][3]>;

/*
List the chief factors of overgroups of X2 in G2 that do not normalise X2.
Note how they all contain a double cover of Alt(5).
To run the check for Alt(7), change G to Alt(7) and set X2 :=
    sub<G2|ConjugacyClasses(G2)[6][3]>;
*/

im:=IntermediateSubgroups(G2,X2);
imNN:=[H:H in im|IsNormal(H,X2) eq false];
for H in imNN do
        ChiefFactors(H);
```

```
end for;

// We now determine how 2.Alt(5) = SL(2,5) acts on V

G := SL(2,5);
C := CharacterTable(G);

chi := C[6];
IsSymplecticCharacter(chi);
sym2chi := Symmetrization(chi,[2,0]);
[InnerProduct(C[i],sym2chi) : i in [1..#C]];

// The output [ 0, 0, 0, 1, 1, 0, 1, 0, 0 ] indicates that 2.Alt(5) acts on V
    as 3+3+4.
```

Listing 2. *The case* $2.2^4.Sym(5)$

```
// Get 2.2^4.Sym(5) directly from Sp(4,7)

cms := ClassicalMaximals("S",4,7);
G:= cms[#cms-1];
X := sub<G|ConjugacyClasses(G)[22][3]>;
im:=IntermediateSubgroups(G,X);
imNN:=[H:H in im|IsNormal(H,X) eq false];
for H in imNN do
        ChiefFactors(H);
end for;

// The first element of imNN is the subgroup 2.2^4.5

G:=imNN[1];
C := CharacterTable(G);

chi:= C[6];
sym2chi := Symmetrization(chi,[2,0]);
for i in [1..#C] do
        if InnerProduct(C[i],sym2chi) eq 1 then
                C[i];
        end if;
end for;

// The output consists of the two consitutents of S^2(chi), two distinct self
    dual characters of degree 5.
```

Listing 3. *The case* $2.Alt(6)$ *in characteristic 5*

```
// Get the subgroup M = 2.Alt(6) in Sp(4,5)
cms:=ClassicalMaximals("S",4,5);
M:=cms[#cms];

// There are 2 conjugacy classes of elements of order 5, leading to the same
    result
U := sub<M|ConjugacyClasses(M)[6][3]>;
```

```
/*
List the chief factors of overgroups of U in M that do not normalise U.
Note how they all contain a double cover of Alt(5) = SL(2,5).
In each case determine the composition factors for the action on V, by taking
    the symmetric square of their module.
Also check that all composition factors are indeed absolutely irreducible.
*/

im:=IntermediateSubgroups(M,U);
imNN:=[H:H in im|IsNormal(H,U) eq false];
for H in imNN do
        ChiefFactors(H);
        V_H := GModule(H);
        CompositionFactors(V_H);
        CompositionFactors(SymmetricSquare(V_H));
        // Output "true" as all composition factors of the symmetric square
            are absolutely irreducible.
        &and[IsAbsolutelyIrreducible(comp) : comp in
            CompositionFactors(SymmetricSquare(V_H))];
end for;
```

Listing 4. *The case* $2.2^4.5$ *in characteristic 5*

```
// Get the subgroup M = 2.2^4.Alt(5) in Sp(4,5)
cms:=ClassicalMaximals("S",4,5);
M:=cms[#cms-1];

// There are 2 conjugacy classes of elements of order 10, leading to the same
    result
U := sub<M|ConjugacyClasses(M)[15][3]>;

/*
List the chief factors of overgroups of U in M that do not normalise U.
In each case determine the composition factors for the action on V, by taking
    the symmetric square of their module.
Also check that all composition factors are indeed absolutely irreducible.
*/

im:=IntermediateSubgroups(M,U);
imNN:=[H:H in im|IsNormal(H,U) eq false];

for H in imNN do
        ChiefFactors(H);
        V_H := GModule(H);
        CompositionFactors(V_H);
        CompositionFactors(SymmetricSquare(V_H));
        // Output "true" as all composition factors of the symmetric square
            are absolutely irreducible.
        &and[IsAbsolutelyIrreducible(comp) : comp in
            CompositionFactors(SymmetricSquare(V_H))];
end for;
```

```
// Check that 2.2^4.5 has two self-dual non-isomorphic composition factors on
    V
H := imNN[1];
V_H := GModule(H);
comps := CompositionFactors(SymmetricSquare(V_H));
&and[IsSelfDual(comp) : comp in comps];
IsIsomorphic(comps[1],comps[2]);
```

***Listing 5.** Groebner basis for I in characteristic 7*

```
//Construct C_3 and its lambda_2 representation over the field of fractions
    of a polynomial ring over GF(7).
R<a1,a2,a3,a4,a5,a6,a7,a8,a9,t1,t2,t3,t1inv,t2inv,t3inv> :=
    PolynomialRing(GF(7),15);
F<b1,b2,b3,b4,b5,b6,b7,b8,b9,x1,x2,x3,x1inv,x2inv,x3inv> :=
    FieldOfFractions(R);
C := GroupOfLieType("C3",F:Isogeny:="SC");
f:=HighestWeightRepresentation(C,[0,1,0]);
V3:=VectorSpace(F,3);

//Define an arbitrary g in the standard Borel.
g:=elt<C|<3,a9>,<5,a8>,<7,a7>,<6,a6>,<8,a5>,<9,a4>,<2,a3>,
    <4,a2>,<1,a1>,V3![t1,t2,t3]>;

// Define a basis for W7, which is easily seen to correspond to W^*.
V:=VectorSpace(F,14);
v1:=V.1+4*V.11+3*V.12;
v2:=V.3+3*V.14;
v3:=V.4+4*V.14;
v4:=V.5+5*V.6;
v5:=V.7+3*V.8;
v6:=V.9+5*V.10;
v7:=V.13;
W7:=sub<V|v1,v2,v3,v4,v5,v6,v7>;
//Extend the basis
B:=[v1,v2,v3,v4,v5,v6,v7,V.2,V.6,V.8,V.10,V.11,V.12,V.14];
VB:=VectorSpaceWithBasis(B);

// Build set of generators of the ideal I
polys := {x1*x1inv-1,x2*x2inv-1,x3*x3inv-1};
for v in Basis(W7) do
        for poly in Coordinates(VB,v*f(g))[8..14] do
                Include(~polys,poly);
        end for;
end for;

// Build ideal I and determine its Groebner basis. It takes under 1 second.
I := ideal<R|[Numerator(p):p in polys]>;
time GroebnerBasis(I);
```

```
/*
Let P_3 = G_<e1,e2,e3> and n = n3 or n1n2n3n2n1n2n3n2, so that
P_3^n is the stabilizer of <e1,e2,f3> or <e3,f1,f2>.
The following function is a wrapper for the routine of finding the Groebner
    basis
of the system of polynomials that determine the stabilizer in P_3^n of W^*.
The variable field can be Rationals() or a finite field, while j is an integer
between 1 and 6, corresponding to the 6 possibilities for the Bruhat
decomposition of an element in P_3. If field == Rationals(), the function
    also returns
a list of primes that need to be checked individually.
*/


findGroebnerBasis := function(field, j)

        // Define the group and the representation.
    R<a1, a2, a3, a4, a5, a6, a7, a8, a9, b1, b2, b3, b4, b5, b6, b7, b8, b9,
        t1, t2, t3, t1inv, t2inv, t3inv, om, i> := PolynomialRing(field, 26);
    F<a1_f, a2_f, a3_f, a4_f, a5_f, a6_f, a7_f, a8_f, a9_f, b1_f, b2_f, b3_f,
        b4_f,
        b5_f, b6_f, b7_f, b8_f, b9_f, t1_f, t2_f, t3_f, t1inv_f, t2inv_f,
            t3inv_f,
        om_f, i_f> := FieldOfFractions(R);
    C := GroupOfLieType("C3", F : Isogeny := "SC");
    f := HighestWeightRepresentation(C, [0, 1, 0]);

    V3 := VectorSpace(F, 3);
    V := VectorSpace(F, 14);

    v1 := V.8 - om^2 * V.7;
    v2 := V.5 - i * V.6;
    v3 := V.4 + i * V.3;
    v4 := V.14 - i * V.2;
    v5 := V.1 - i * V.13;
    v6 := V.11 + i * V.12;
    v7 := V.10 + i * V.9;

    // Here Wdd is the subspace W^\ddag, while the span of all 7 vectors
        v1,...,v7 is W^*.
    Wdd := sub<V | v1, v2, v3, v5>;

    // Extend the basis for Wdd to a basis for the whole module
    B := [v1, v2, v3, v5, V.8, V.6, V.4, V.13, V.2, V.9, V.10, V.11, V.12,
        V.14];
    VB := VectorSpaceWithBasis(B);

    // Define the list of (preimages) of Weyl group elements that belong to
        P_3
    ns := [Identity(C), elt<C | 1>, elt<C | 2>, elt<C | 1, 2>,
        elt<C | 2, 1>, elt<C | 1, 2, 1>];
```

```
        // Each such Weyl group element has a corresponding u^-, generated by
            positive root elements
        // that are sent to negative root elements by the Weyl group element.
        u_minuss := [Identity(C), elt<C | <1, b9>>, elt<C | <2, b7>>,
                        elt<C | <2, a7>, <4, a8>>,
                    elt<C | <4, b8>, <1, b9>>,
                    elt<C | <2, a7>, <4, a8>, <1, a9>>];
        n := ns[j];
        u_minus := u_minuss[j];

        // Write an arbitrary element g belonging to the double coset B n B.
        g := elt<C | <3, a1>, <5, a2>, <7, a3>, <6, a4>, <8, a5>, <9, a6>,
                <2, a7>, <4, a8>, <1, a9>, V3![t1, t2, t3]> * n * u_minus;

        // Conjugate by either n3 or n2n3n2n1n2n3n2n1 to get an arbitrary element
            of G_<e1,e2,f3> or G_<e3,f1,f2>
        g := g^elt<C | 3>;

        // Initialise the list of polynomials, encoding the fact that the ti's
            are non-zero,
        // and that om and i are primitive fourth and third roots of unity
            respectively.
        polys := {t1_f * t1inv_f - 1, t2_f * t2inv_f - 1, t3_f * t3inv_f - 1,
            om_f + om_f^2 + 1, i_f^2 + 1};

        // Complete set of polynomials by adding the conditions required for g to
            fix Wdd
        for v in Basis(Wdd) do
            for poly in Coordinates(VB, v * f(g))[5..14] do
                Include(~polys, poly);
            end for;
        end for;

        // If field is finite, output the Groebner basis
        if IsFinite(field) then
            A1, A2 := GroebnerBasis([Numerator(p) : p in polys]);
            return A1;
        end if;

        // Otherwise also output the list of primes the F4 algorithm divided by.
            These
        // need to be checked individually by running the function again.
        if field eq Rationals() then
            SetGBGlobalModular(false);
            A1, A2, A3 := GroebnerBasis([Numerator(p) : p in polys] :
                ReturnDenominators := true);
            return A1, A3;
        end if;

    end function;
```

Listing 7. Executing the Groebner basis search using the function findGroebnerBasis

```
for i in [1..6] do

        time B, badPrimes := findGroebnerBasis(Rationals(), i);
        "The case i = ", i;
        B;

        for p in [x : x in badPrimes | x in [3,7] eq false] do
                "Checking the prime p = ", p;
                findGroebnerBasis(GF(p), i);
        end for;

end for;
```

Listing 8. *Code for subgroups of* $Sp_6(q)$

```
// SL(2,7) case (p not 2).

G := SL(2,7);

// Range over all conjugacy classes of subgroups isomorphic to S^\dag

for rec in Subgroups(G) do
        S := rec'subgroup;
        if IdentifyGroup(S) eq <21,1> then
                // Find all intermediate subgroups between S and G
                im:=IntermediateSubgroups(G,S);
                // Filter out the ones that normalise S
                imNN:=[H:H in im|IsNormal(H,S) eq false];
                // Print the ChiefFactors of such subgroups
                for H in imNN do
                        ChiefFactors(H);
                end for;
        end if;
end for;
// There are no non-normalising overgroups.


// SL(2,13) case (p not 2).

G := SL(2,13);

// Range over all conjugacy classes of subgroups isomorphic to S^\dag

for rec in Subgroups(G) do
        S := rec'subgroup;
        if Order(S) eq 21 then
                S;
        end if;
end for;
// There are no subgroups of order 21.
```

```
// U(3,3) case.

G := SU(3,3);

// Range over all conjugacy classes of subgroups isomorphic to S^\dag

for rec in Subgroups(G) do
      S := rec'subgroup;
      if Order(S) eq 21 and IdentifyGroup(S) eq <21,1> then
             // Find all intermediate subgroups between S and G
             im:=IntermediateSubgroups(G,S);
             // Filter out the ones that normalise S

             imNN:=[H:H in im|IsNormal(H,S) eq false];
             // Print the ChiefFactors of such subgroups
             for H in imNN do
                    ChiefFactors(H);
             end for;
      end if;
end for;
// The only possibility is PSL(2,7).


// J_2 case.

// Construct 2.J2 by taking the appropriate maximal subgroup of Sp_6(5).
G := ClassicalMaximals("S",6,5)[10];

// Range over all conjugacy classes of subgroups isomorphic to S^\dag

for rec in Subgroups(G) do
      S := rec'subgroup;
      if Order(S) eq 21 and IdentifyGroup(S) eq <21,1> then
             // Find all intermediate subgroups between S and G
             im:=IntermediateSubgroups(G,S);
             // Filter out the ones that normalise S
             imNN:=[H:H in im|IsNormal(H,S) eq false];
             // Print the ChiefFactors of such subgroups
             for H in imNN do
                    ChiefFactors(H);
             end for;
      end if;
end for;

// The only possibilities are PSL(2,7), SL(2,7), U(3,3).
```

Competing interest. The author have no competing interest to declare.

References

[1] E. M. Andreev, É. B. Vinberg and A. G. Élashvili, 'Orbits of greatest dimension in semi-simple linear Lie groups', *Funct. Anal. Appl.* **1** (1967), 257–261.

[2] M. Aschbacher, *Finite Group Theory* (Cambridge Studies in Advanced Mathematics) second edn. (Cambridge University Press, Cambridge, 2000).

[3] A. Babic and V. Chernousov, 'Lower bounds for essential dimensions in characteristic 2 via orthogonal representations', *Pacific J. Math.* **279** (2015), 37–63. issn: 0030-8730. doi: 10.2140/pjm.2015.279.37.

[4] N. Bourbaki, *Lie Groups and Lie Algebras: Chapters 4-6* (Elements de mathematique [series]) pts. 4–6 (Springer Berlin Heidelberg, 2008). isbn: 9783540691716.

[5] J. N. Bray, D. F. Holt and C. M. Roney-Dougal, *The Maximal Subgroups of the Low-Dimensional Finite Classical Groups* (London Mathematical Society Lecture Note Series) Vol. 407 (Cambridge University Press, Cambridge, 2013). isbn: 978-0-521-13860-4. doi: 10.1017/CBO9781139192576.

[6] J. Brundan, 'Double coset density in classical algebraic groups', *Trans. Amer. Math. Soc.* **352** (2000), 1405–1436. issn: 0002-9947. doi: 10.1090/S0002-9947-99-02258-8.

[7] R. W. Carter, *Simple Groups of Lie Type* (Wiley, Chichester, 1989). isbn: 0471137359.

[8] W. E. Duckworth, 'Infiniteness of double coset collections in algebraic groups', *J. Algebra* **273**(2) (2004), 718–733. issn: 0021-8693. doi: https://doi.org/10.1016/j.jalgebra.2003.08.011.

[9] S. Garibaldi and D. K. Nakano, 'Bilinear and quadratic forms on rational modules of split reductive groups', *Canad. J. Math.* **68**(2) (2016), 395–421. doi: 10.4153/CJM-2015-042-5.

[10] R. Guralnick and R. Lawther, *Generic Stabilizers in Actions of Simple Algebraic Groups* (Memoirs of the American Mathematical Society) (American Mathematical Society, 2024).

[11] R. M. Guralnick, M. W. Liebeck, D. Macpherson and G. M. Seitz, 'Modules for algebraic groups with finitely many orbits on subspaces', *J. Algebra* **196** (1997), 211–250. issn: 0021-8693. doi: 10.1006/jabr.1997.7068.

[12] J. Igusa, 'A classification of spinors up to dimension twelve', *Amer. J. Math.* **92** (1970), 997–1028. issn: 0002-9327. doi: 10.2307/2373406.

[13] P. Kleidman and M. Liebeck, *The Subgroup Structure of the Finite Classical Groups* (London Mathematical Society Lecture Note Series) vol. 129 (Cambridge University Press, Cambridge, 1990). isbn: 0-521-35949-X. doi: 10.1017/CBO9780511629235.

[14] F. Knop and G. Röhrle, 'Spherical subgroups in simple algebraic groups', *Compos. Math.* **151** (2015), 1288–1308. doi: 10.1112/S0010437X1400791X.

[15] M. Korhonen, 'Invariant forms on irreducible modules of simple algebraic groups', *J. Algebra* **480** (2017), 385–422. issn: 0021-8693. doi: 10.1016/j.jalgebra.2017.02.031.

[16] M. W. Liebeck and J. Saxl, 'On the orders of maximal subgroups of the finite exceptional groups of Lie type', *Proc. Lond. Math. Soc.* s3-55.2 (1987), 299–330. issn: 0024-6115. doi: 10.1093/plms/s3-55˙2.299. eprint: https://academic.oup.com/plms/article-pdf/s3-55/2/299/4522254/s3-552-299.pdf.

[17] M. W. Liebeck, J. Saxl and G. M. Seitz, 'Factorizations of simple algebraic groups', *Trans. Amer. Math. Soc.* **348** (1996), 799–822. issn: 0002-9947. doi: 10.1090/S0002-9947-96-01447-X.

[18] M. W. Liebeck and G. M. Seitz, 'On the subgroup structure of classical groups', *Invent. Math.* **134** (1998), 427–453. issn: 0020-9910. doi: 10.1007/s002220050270.

[19] M. W. Liebeck and G. M. Seitz, 'On the subgroup structure of exceptional groups of Lie type', *Trans. Amer. Math. Soc.* **350** (1998), 3409–3482.

[20] F. Lübeck, 'Small degree representations of finite Chevalley groups in defining characteristic', *LMS J. Comput. Math.* **4** (2001), 135–169. issn: 1461-1570. doi: 10.1112/S1461157000000838.

[21] B. Martin, 'Generic stabilisers for actions of reductive groups', *Pacific J. Math.* **279**(1) (2015), 397–422.

[22] P. E. Newstead, *Introduction to Moduli Problems and Orbit Spaces* (Tata Institute of Fundamental Research Lectures on Mathematics and Physics) vol. 51 (Tata Institute of Fundamental Research, Bombay; Narosa Publishing House, New Delhi, 1978). isbn: 0-387-08851-2.

[23] V. L. Popov, 'Classification of spinors of dimension fourteen', *Trans. Moscow Math. Soc.* **1** (1980), 181–232.

[24] P. Revoy, 'Trivecteurs de rang 6', in *Colloque sur les formes quadratiques (Montpellier, 1977)* (Mémoires de la Société Mathématique de France 59) (Société mathématique de France, 1979), 141–155. doi: 10.24033/msmf.256.

[25] R. W. Richardson, 'Deformations of Lie subgroups and the variation of isotropy subgroups', *Acta Math.* **129**(1) (1972), 35–73.

[26] A. Rizzoli, 'Finite singular orbit modules for algebraic groups', *J. Algebra* **570** (2021), 75–118. issn: 0021-8693. doi: https://doi.org/10.1016/j.jalgebra.2020.11.006.

[27] A. Rizzoli, 'Modules for algebraic groups with finitely many orbits on totally singular 2-spaces', *J. Algebra* **606** (2022), 104–169. issn: 0021-8693. doi: https://doi.org/10.1016/j.jalgebra.2022.05.006.

[28] A. Rizzoli, *Magma code used for computations*. Version 0.1. 2024. url: https://github.com/alunik/generic-stabs.

[29] M. Sato and T. Kimura, 'A classification of irreducible prehomogeneous vector spaces and their relative invariants', *Nagoya Math. J.* **65** (1977), 1–155. doi: 10.1017/S0027763000017633.

[30] R. Steinberg, *Lectures on Chevalley Groups (University Lecture Series)* vol. 66 (American Mathematical Society, Providence, RI, 2016). isbn: 978-1-4704-3105-1. doi: 10.1090/ulect/066.

[31] D. I. Stewart, *The Reductive Subgroups of F* (Memoirs of the American Mathematical Society) (American Mathematical Society, 2013). isbn: 9780821898734.

[32] C. Zhijie, 'A classification of irreducible prehomogeneous vector spaces over an algebraically closed field of characteristic 2 (I)', *Acta Math. Sin.* **2**(2) (1986), 168–177. doi: 10.1007/BF02564878.

[33] C. Zhijie, 'A classification of irreducible prehomogeneous vector spaces over an algebraically closed field of characteristic p II', *Chinese Ann. Math. Ser. A* **9**(1) (1988), 10–22. issn: 1000-8314.