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SIMULTANEOUS TRIANGULARIZATION OF ALGEBRAS OF POLYNOMIALLY COMPACT OPERATORS

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ABSTRACT. If A is a norm closed algebra of compact operators on a Hilbert space and if its Jacobson radical $J(A)$ consists of all quasinilpotent operators in A then $A/J(A)$ is commutative. The result is not valid for a general algebra of polynomially compact operators.

Hadwin [2] shows that an algebra A of algebraic operators on a vector space X is triangularizable if and only if $A/J(A)$ is commutative, and this in turn is true if and only if $J(A) = \{T \in A : T \text{ is quasinilpotent}\}$. Here the Jacobson radical $J(A)$ is defined as the set of all $T \in A$ such that for every $S \in A$ the operator $1 + ST$ is invertible in any unital algebra containing A. In case A is an algebra of polynomially compact operators on a Hilbert space H and triangularizability is defined in terms of closed subspaces of H (see below), he shows that A is triangularizable if $A/J(A)$ is commutative [2, Theorem 3.4]. Hadwin leaves the following questions open.

QUESTION 1. If A is a norm closed algebra of compact operators on H and $J(A)$ is the set of all quasinilpotent operators in A, then must $A/J(A)$ be commutative? What if the elements of A are all trace class operators?

QUESTION 2. Suppose that A is a triangularizable norm closed algebra of polynomially compact operators. Must $A/J(A)$ be commutative?

In the present paper we answer Question 1 in the affirmative, and Question 2 in negative.

Recall that a maximal chain of subspaces of H (resp. of invariant subspaces of a collection $A \subset B(H)$) is any chain C of subspaces of H (resp. of invariant subspaces of A) which is not a proper subchain of another chain of subspaces of H (resp. of invariant subspaces of A). (Throughout the paper H denotes a general Hilbert space and $B(H)$ is the algebra of all bounded linear operators on H; also, by a subspace of H we mean a closed subspace.) A collection $A \subset B(H)$ is called triangularizable if it has a maximal chain of invariant subspaces which is also a maximal chain of subspaces of H.

An operator $T \in B(H)$ is called polynomially compact if $p(T)$ is compact for some polynomial p. A quasinilpotent operator T is one for which $\sigma(T) = \{0\}$. The following lemmas which are partially needed for the proof of the main result, will demonstrate the

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extent to which the algebraic results of [2] can be extended. They also give a different proof of Theorem 3.4 of [2] mentioned above.

LEMMA 1. *Let A be any unital algebra over the field of complex numbers* $\mathbf{C}$. *Let K be a Banach space and assume $\phi\colon A \longrightarrow B(K)$ is an algebra homomorphism such that $\phi(T^k) = 0$ for all $T \in J(A)$, where k is a fixed positive integer. Then $\phi(A)$ has a nontrivial invariant subspace if* $\dim K > 1$ *and* $\phi(J(A)) \neq \{0\}$.

PROOF. Assume $\dim K > 1$ and $\phi(A)$ is transitive (i.e., $\phi(A)$ has no invariant subspace other than the trivial ones $\{0\}$ and H). We must show that $\phi(J(A)) = \{0\}$. By [3, Theorem 3.1], $\phi(J(A))$ has a nontrivial invariant subspace M. Fix $T \in J(A)$ and $x \in M$. Then the closure of $\{\phi(ST)x : S \in A\}$ is an invariant subspace of $\phi(A)$ contained in M. Since $\phi(A)$ is transitive, $\phi(T)x = 0$. Hence $\phi(T)|M = 0$ for all $T \in J(A)$. Next, let $0 \neq x \in M$. Since $\{\phi(S)x : S \in A\}$ is dense in H and since $\phi(T)\phi(S)x = \phi(TS)x = 0$ for all $S \in A$ and all $T \in J(A)$, it follows that $\phi(T) = 0$ for all $T \in J(A)$. Thus $\phi(J(A)) = \{0\}$. ∎

LEMMA 2. *Let A be a Banach algebra. Assume $\phi\colon A \longrightarrow B(K)$ is a continuous homomorphism such that $\phi(T)$ is a nilpotent operator on a Banach space K for all $T \in J(A)$. Then there exists a positive integer k such that $\phi(T^k) = 0$, for all $T \in J(A)$.*

The proof is an imitation of the proof of a similar result due to Grabiner [1].

LEMMA 3. *Let A be a unital algebra. Assume $\phi\colon A \longrightarrow B(K)$ is a homomorphism such that $\phi(A)$ is a transitive algebra of polynomially compact operators on a Banach space K. Then every element of $\phi(J(A))$ is nilpotent.*

PROOF. If $\dim K < \infty$, then $\phi(A) = B(K)$ and hence $\phi(J(A)) = J(\phi(A)) = \{0\}$ (Burnside's theorem). Therefore, we assume without loss of generality that $\dim K = \infty$. Let $T \in J(A)$ and assume, if possible, that $q(\phi(T)) = a_0 + a_1\phi(T) + \cdots + a_n\phi(T^n)$ is a nonzero compact operator for some complex numbers $a_0, a_1, \ldots, a_n$. Since $a_0 + T(a_1 + \cdots + a_nT^{n-1})$ is not invertible, $a_0 = 0$ and hence $q(T) \in J(A)$. By Lomonosov's lemma [4] (or [7, Lemma 8.22]), there exists $S \in A$ and $y \in K$ such that $\phi(S)\phi(q(T))y = y$ and thus $\phi(1 - Sq(T))$ is not invertible, a contradiction. Hence $J(A)$ contains no nonzero compact operator and for every nonzero $T \in J(A)$ there exists a minimal monomial p such that $p(\phi(T)) = 0$. Since $z - T$ is invertible for all $z \neq 0$, it follows that $p(z) = z^k$. ∎

LEMMA 4. *Let A be a triangularizable norm closed algebra of compact operators on H, and let C be a maximal chain of subspaces of H which are invariant subspaces of A. For each $M \in C$, let M_- be the closed span of all proper subspaces of M belonging to C. Then $T \in A$ is quasinilpotent if and only if $T^M = 0$ for all $M \in C$, where T^M is the operator induced by $T|M$ on M/M_-.*

PROOF. If $T \in A$ is quasinilpotent, so are $T|M$ and T^M. Hence $T^M = 0$ because $\dim M/M_- \leqq 1$ for all $M \in C$. Conversely, assume $T^M = 0$ for all $M \in C$. Then, by a result of Ringrose [8; 9], T is quasinilpotent. ∎

LEMMA 5. *Let A be a unital algebra over* **C**. *Assume $xy - yx = 1 + q$ for some $q \in J(A)$. Then $\sigma(yx)$ is unbounded if it is nonempty. (As usual $\sigma(u)$ denotes the set of all complex numbers λ such that $\lambda - u$ is not invertible.)*

PROOF. It is well-known that $\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}$, and $\sigma(u+v) = \sigma(u)$ for all $u \in A$ and all $v \in J(A)$. Thus $\sigma(xy) = \sigma(yx + 1 + q) = \sigma(1 + yx) = 1 + \sigma(yx)$, and hence $\sigma(yx) \cup \{0\} = (1 + \sigma(yx)) \cup \{0\}$. It is now easy to observe that a nonempty set Δ of complex numbers satisfying $\Delta \cup \{0\} = (1 + \Delta) \cup \{0\}$ is necessarily unbounded.■

LEMMA 6. *Let A be a norm closed algebra of polynomially compact operators. Assume $J(A) = \{T \in A : T$ is quasinilpotent$\}$. Then $A/J(A)$ is commutative.*

PROOF. Assume without loss of generality that $I \in A$. Let $S, T \in A$ and $U = ST - TS$. We must show that U is quasinilpotent. Assume, if possible, that $\sigma(U) \neq \{0\}$. Then $\sigma(U)$ contains an isolated nonzero point z. Let f be an analytic function defined on a neighbourhood of $\sigma(U)$ which is identically 1 in a neighbourhood of z and is identically 0 in a neighbourhood of $\sigma(U) \setminus \{z\}$. By the Riesz-Dunford functional calculus, the operator $P = f(U)$ is a nonzero idempotent in A such that $PU = UP = zP + Q$, where $Q \in J(A)$. Since the operators $PS(I-P)$, $(I-P)SP$, $PT(I-P)$, and $(I-P)TP$ are nilpotent, it follows that $zP+Q = PUP = P(ST-TS)P = (PSP)(PTP)-(PTP)(PSP)+Q'$ for some $Q' \in J(A)$. Letting $B = PAP$, $x = PSP$, and $y = z^{-1}PTP$, we observe that B is a unital Banach algebra with unit P, $J(B) \supset PJ(A)P$, and $xy - yx = 1 + q$ for some $q \in J(B)$. Thus $\sigma(yx) \neq \emptyset$ and hence, in view of Lemma 5, it is unbounded; a contradiction. ■

Now, we prove the main result of the paper. The equivalence of (1) and (3) is known; a rather different proof is given in [5; 6].

THEOREM 1. *Let A be a norm closed algebra of compact operators on H. Then the following are equivalent. (1) A is triangularizable. (2) Every maximal chain of invariant subspaces of A is a maximal chain of subspaces of H. (3) The algebra $A/J(A)$ is commutative. (4) $J(A) = \{T \in A : T$ is quasinilpotent$\}$.*

PROOF. The proof of (2) $\Rightarrow$ (1) is trivial, and the proof of (4) $\Rightarrow$ (3) is given in Lemma 6. It remains to show that (1) $\Rightarrow$ (4) and (3) $\Rightarrow$ (2).

Assume (1) is true and C is a maximal chain making A triangularizable. Let $T \in A$ be quasinilpotent. Since $T^M = 0$, it follows that $(ST)^M = 0$ for all $S \in A$, where $M \in C$ is arbitrary (see Lemma 4). Thus ST is quasinilpotent and hence $T \in J(A)$. This proves (4).

Finally, assume (3) is true and let C be a maximal chain of invariant subspaces of A. We claim $\dim M/M_- \leqq 1$ for all $M \in C$. Assume, if possible, that $\dim M/M_- > 1$. Let $\phi : A \longrightarrow B(M/M_-)$ be the algebra homomorphism sending $T \in A$ to the operator T^M induced by $T|M$ on M/M_-. Since C is a maximal chain of invariant subspaces of A, it follows from Lemmas 1, 2, and 3 that $\phi(J(A)) = \{0\}$. Thus $\phi(A)$ is a commutative algebra of compact operators. Since $\phi(A)$ is transitive, every element of $\phi(A)$ is a scalar

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multiple of identity and dim $M/M_- = 1$. This shows that C is a maximal chain of subspaces of H. ∎

A re-examination of the proof of Theorem 1 suggests the proof of the following corollary.

COROLLARY 1. *The implications* (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) *remain true if in the hypothesis of Theorem 1, A is assumed only to be a norm closed algebra of polynomially compact operators.*

As we acknowledged before, a different proof of the implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) in Corollary 1 is given in [2].

The following example shows that the triangularizability of a norm closed algebra of polynomially compact operators does not necessarily imply that $A/J(A)$ is commutative or $J(A) = \{T \in A : T \text{ is quasinilpotent}\}$.

EXAMPLE. Let A be the algebra of all operators T on $H = L^2(0,1) \oplus L^2(0,1)$ defined by $T(f \oplus g) = (af + hg) \oplus (cf + dg)$, where a, b, c, and d are arbitrary complex numbers. For each $t \in [0, 1]$ the subspaces

$$M = \{f \oplus g : f(x) = g(x) \ a.e. \text{ on } [t, \ 1]\}$$

is an invariant subspace of A. Since $\{M_t : 0 \leqq t \leqq 1\}$ is a maximal chain of subspaces of H, it follows that A is triangularizable. If $T \in A$, then

$$(\star) \qquad T = \begin{bmatrix} aI & bI \\ cI & dI \end{bmatrix} \quad \text{on } L^2(0,1) \oplus L^2(0,1),$$

and $(a-T)(d-T)-bc = 0$. (Here I denotes the identity on $L^2(0,1)$.) Thus every element of A is algebraic, and hence A is a triangularizable algebra of polynomially compact operators. We claim $J(A) = \{0\}$. Let T be as in ($\star$). It is easy to see that the map sending T to the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an algebra isomorphism between A and the algebra of all 2×2 complex matrices. Thus $J(A) = \{0\}$ and hence neither $A/J(A)$ is commutative nor $J(A)$ is equal to $\{T \in A : T \text{ is quasinilpotent}\}$.

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