Canad. Math. Bull. Vol. **42** (3), 1999 pp. 298–306

Semigroup Algebras and Maximal Orders

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Abstract. We describe contracted semigroup algebras of Malcev nilpotent semigroups that are prime Noetherian maximal orders.

Maximal orders in simple Artinian rings of quotients have attracted considerable interest. In particular, it has been shown that various algebraic ring constructions yield examples of Noetherian maximal orders. In [2], Brown described when a group algebra $K[G]$ of a polycyclic-by-finite group G is a prime maximal order. It is always the case if G is torsion-free. In this paper we investigate when a contracted semigroup algebra is a Noetherian prime maximal order. For standard terminology and notation on semigroups and semigroup algebras we refer to [4], [16].

For notation and terminology on (maximal) orders we refer to [15]. However, we recall some of this in the semigroup context (see for example [6] and [17]). A cancellative monoid S which has a left and right group of fractions G is called an *order*. Such a monoid S is called a *maximal order* if there does not exist a submonoid S' of G properly containing S and such that $aS'b \subseteq S$ for some $a, b \in G$. For subsets A, B of G we denote by $(A :_l B) = \{g \in G \mid gB \subseteq A\}$ and by $(A :_r B) = \{g \in G \mid Bg \subseteq A\}$. Note that S is a maximal order if and only if $(I :_l I) = (I :_r I) = S$ for every *fractional ideal* I of S. The latter means that $SIS \subseteq I$ and $cI, Id \subseteq S$ for some $c, d \in S$. If S is a maximal order, then note that $(S :_l I) = (S :_r I)$ for any fractional ideal I; we simply denote this fractional ideal by $(S : I)$ or by I^{-1}. Recall that I is said to be *divisorial* if $I = I^*$, where $I^* = \big(S : (S : I)\big)$. The divisorial product $I * J$ of two divisorial ideals I and J is defined as $(IJ)^*$.

A cancellative monoid S is said to be a Krull order if and only if S is a maximal order satisfying the ascending chain condition on integral divisorial ideals, that is fractional ideals contained in S.

Chouinard [3] showed that commutative monoid algebras $K[S]$ are Krull domains if and only S is a Krull order. Later, Wauters [17] characterised monoids S that are Krull orders in case every element n of S is *normal*, that is $Sn = nS$. Hence, because of a result of Brown [2] on group algebras of polycylic-by-finite groups, using standard arguments, one can extend Chouinard's result as follows. The group of invertible elements of a monoid is denoted by $\mathcal{U}(S)$. If S is a subsemigroup of a group G, then $gr(S)$ denotes the subgroup of G generated by S.

Theorem 1 *Let S be a cancellative monoid of normal elements, G its group of fractions and K a field. If G is torsion-free polycyclic-by-finite, then the following conditions are equivalent:*

Received by the editors March 25, 1998.

Work supported in part by NSERC of Canada, grant number OGP0036631. The second author would like to thank Memorial University of Newfoundland for the warm hospitality while being a visiting scientist during the academic year 1997-1998.

AMS subject classification: Primary: 16S36, 16H05; secondary: 20M25.

1. $K[S]$ is a Krull domain.
2. S is a Krull order.
3. $S/\mathcal{U}(S)$ is an abelian Krull order.

Moreover, an abelian monoid A with trivial unit group is a Krull order if and only if $A = gr(A) \cap F_+$, where F_+ is the positive cone of a free abelian monoid F.

We note that, in the situation described in the theorem above, $K[S]$ is a Noetherian maximal order if and only if S is a finitely generated Krull order.

It remains an unsolved problem to characterize when an arbitrary semigroup algebra is a prime Noetherian maximal order. Since any semigroup algebra can be considered as a contracted semigroup algebra, we investigate the question in this more general framework. This way we handle a much wider class of prime algebras. Apart from the semigroups listed in Theorem 1, an answer to the question has only been obtained for some special classes of semigroups, such as for example *binomial semigroups* [5], [9], or for more restrictive orders, such as principal ideal rings [8].

In this paper we solve the problem for semigroup algebras of finitely generated semigroups that are nilpotent in the sense of Malcev (see for example [7] for the definition). First we deal with cancellative nilpotent monoids that have a finitely generated group of fractions. Note that for such monoids S it is known that $K[S]$ is right Noetherian if and only if S satisfies the ascending chain condition on right ideals. Furthermore, such monoids have been fully characterized in [10], and it follows that $K[S]$ is right Noetherian if and only if it is left Noetherian. We simply say that $K[S]$ is Noetherian in this case.

Theorem 2 *Let K be a field and S a submonoid of a finitely generated torsion-free nilpotent group G. Then,*

1. $K[S]$ is a maximal order if and only if S is a maximal order.
2. If, moreover, S satisfies the ascending chain condition on right ideals and S is a maximal order, then all elements of S are normal.

Proof It is well known that the group algebra of a finitely generated torsion-free nilpotent group is a Noetherian maximal order (see for example [2]). Since G is an ordered group, the first assertion then follows using standard techniques.

For the second statement, assume S is a maximal order satisfying the ascending chain condition on right ideals. Without loss of generality we may assume that G is the group of fractions of S. As mentioned above, since S satisfies the ascending chain condition on right ideals, $K[S]$ is left and right Noetherian. In particular, it follows from the characterization in [10] that S and G are finitely generated and S contains a subgroup F such that F is normal in G and G/F is abelian-by-finite. The n-th centre of G is denoted by $Z_n(G)$.

If $\mathcal{U}(S)$ is trivial, then G is nilpotent and abelian-by-finite. Since it is also torsion-free, we obtain that (see for example [13]) G is abelian. Hence the result is clear.

So in the remainder of the proof we assume that $\mathcal{U}(S)$ is not trivial. We claim that then also $\mathcal{U}(S) \cap Z_1(G)$ is not trivial. Let n be the minimal positive integer such that there exists $1 \neq u \in \mathcal{U}(S) \cap Z_n(G)$. Then, from [10] we know that for every $a \in S$ there exists $m = m(a) \geq 1$ so that $au^m a^{-1} \in \mathcal{U}(S)$, and thus $au^m a^{-1} u^{-m} \in \mathcal{U}(S) \cap Z_{n-1}(G)$. So

$au^m a^{-1} u^{-m} = 1$. Hence, since G is torsion-free and nilpotent, a and u commute. Because a is arbitrary it follows that $u \in \mathcal{U}(S) \cap Z_1(G)$, as desired.

We now show that the group $G/\big(\mathcal{U}(S) \cap Z_1(G)\big)$ is torsion-free. For this let $\overline{g} = g\big(\mathcal{U}(S) \cap Z_1(G)\big) \in \overline{G} = G/\big(\mathcal{U}(S) \cap Z_1(G)\big)$ so that $\overline{g}^n = 1$ for some $n \geq 1$. As $G/Z_1(G)$ is torsion-free, it follows that $g \in Z_1(G)$. Let $I = S \cup gS \cup \cdots \cup g^{n-1}S$. Then I is a fractional ideal of S and $gI \subseteq I$. Because of the maximal order assumption, this yields $g \in S$. Hence, as $g^n \in \mathcal{U}(S)$, we get that $g \in \mathcal{U}(S) \cap Z_1(G)$. So $\overline{g} = \overline{1}$, as desired.

Let $\overline{S}$ be the natural image of S in $\overline{G}$. It is readily verified that for any fractional ideal $\overline{I}$ in $\overline{S}$ the following property holds

$$(\overline{I} :_l \overline{I}) = \overline{(I : I)} = (\overline{I} :_r \overline{I}),$$

where the former and the latter are defined in $\overline{G}$ and I is the inverse image of $\overline{I}$ in S. Since S is a maximal order it follows that $\overline{S}$ also is a maximal order in its torsion-free nilpotent group of fractions $\overline{G}$.

Next we claim that for any $s, t \in S$, $sts^{-1}t^{-1} \in \mathcal{U}(S)$. We prove this by induction on the Hirsch number $h(G)$ of G. If $h(G) = 1$, this is obvious, as in this case G is cyclic. Since $h(\overline{G}) < h(G)$, the induction hypothesis applied to $\overline{S} \subseteq \overline{G}$ implies that $\overline{sts^{-1}t^{-1}} \in \mathcal{U}(\overline{S})$. Hence $sts^{-1}t^{-1} \in \mathcal{U}(S)$.

From the previous we obtain that $sts^{-1} \in S$ for any $s, t \in S$. Hence $S \subseteq s^{-1}Ss$. The maximal order assumption therefore implies $S = s^{-1}Ss$ and thus $Ss = sS$ for any $s \in S$. This proves the result. ■

Corollary 3 *Let S and T be maximal orders in a torsion-free nilpotent group G. If there exist $x, y \in G$ such that $xTy \subseteq S$, then $S = T$.*

Proof Because of Theorem 2, $gS = Sg$ for any $g \in G$. Hence $xTy \subseteq S$ implies $T \subseteq hS$, where $h = x^{-1}y^{-1}$. It follows that $ST = TS \subseteq hS$ and thus ST is a fractional S-ideal. Hence, since $T \subseteq (ST : ST)$ and S is a maximal order, it follows that $T \subseteq S$. The reverse inclusion follows by symmetry. So the result follows. ■

Now we handle arbitrary contracted semigroup algebras of nilpotent semigroups that are prime Noetherian maximal orders. For this we first recall a recent result on Morita contexts. For simplicity we formulate the result for the situations needed for our applications.

Proposition 4 **(Marubayashi, Zhang and Yang [14])**

1. *If R is a prime maximal order and $e = e^2 \in R$, then also eRe is a prime maximal order.*
2. *Let $M = \left[\begin{smallmatrix} R & V \\ W & S \end{smallmatrix}\right]$ be a Morita context with R and S prime Goldie rings with the same classical ring of quotients Q, and $V, W \subseteq Q$. Then M is a prime maximal order (in its left and right ring of quotients) if and only if the following conditions are satisfied:*
   - *(a) R and S are prime maximal orders;*
   - *(b) V and W are nonzero;*
   - *(c) $(R :_l W) = (S :_r W) = V$ and $(R :_r V) = W = (S :_l V)$.*

https://doi.org/10.4153/CMB-1999-036-2 Published online by Cambridge University Press

For a subsemigroup T of a completely 0-simple semigroup $M = \mathcal{M}(G, n, n; P)$; we write $T_{ij} = \{(g, i, j) \mid g \in G, (g, i, j) \in M\}$. Clearly, each T_{ij} is closed under left multiplication by T_{ii} and closed under right multiplication by T_{jj}. If M is (Malcev) nilpotent, then we know [7] that, up to isomorphism, P is the identity matrix $I = \Delta$. We simply denote T as $[T_{ij}]$. Recall that T is said to be *uniform* in M if each T_{ij} is nonzero and the group of fractions of each T_{ii} is a maximal subgroup of M.

By e_{ij} we denote the elementary matrix in a matrix ring $M_n(R)$ with 1 in the (i, j)-entry and zeroes elsewhere.

Theorem 5 *Let S be a finitely generated nilpotent semigroup and K a field. The contracted semigroup algebra $K_0[S]$ is a prime maximal order if and only if $K_0[S] = K_0[T]$, where $T = [T_{ij}]$ (with $T_{ij} = e_{ii}S^1e_{jj} \setminus \{0\}$) is a uniform subsemigroup in $\hat{T} = \mathcal{M}\big(gr(T_{11}), n, n; \Delta\big)$ such that*

(M1) $T_{11} = T_{22} = \cdots = T_{nn}$ is a monoid;
(M2) $gr(T_{11})$ is a torsion-free nilpotent group;
(M3) T_{11} is a maximal order;
(M4) each T_{ij} is a divisorial T_{11}-ideal;
(M5) $T_{ij} * T_{jk} = T_{ik}$.

Moreover, for $K_0[S]$ to be a right Noetherian prime maximal order, one has to add the conditions:

(N1) T_{11} satisfies the ascending chain condition on right ideals;
(N2) each T_{ij} is a finitely generated right T_{11} module.

Proof Suppose $R = K_0[S]$ is a prime maximal order in its classical ring of quotients $Q = M_n(D)$, where D is a division algebra. Without loss of generality we may assume that S has a zero element. From [7] and [12, Lemma 1.6] we know that the elements of s (together with the zero element) of minimal nonzero rank form an ideal I that is uniform in $\hat{I} = \mathcal{M}(G, n, n; \Delta)$, the completely 0-simple closure of I in the multiplicative semigroup $M_n(D)$. Here G is the group of fractions of the diagonal components of I. Furthermore

$$K_0[I] \subseteq R \subseteq K_0[\hat{I}] = M_n\big(K[G]\big) \subseteq M_n(D),$$

and $M_n(D)$ is also the classical ring of quotients of $K_0[\hat{I}]$. Also $M_n(K[G])$ is a localization of R. So this algebra is also a prime ring, and thus the group G is finitely generated torsion-free nilpotent. We define $I_{ij} = \{(g, i, j) \mid g \in G^0, (g, i, j) \in I\}$. So $I = \bigcup_{i,j} I_{ij}$. Since $K_0[S]$ is a maximal order and $e_{ij}K_0[I] \subseteq K_0[I]$, we get that $e_{ij} \in R$. Hence,

$$R = \sum_i e_{ii}Re_{jj}.$$

Since S is a nilpotent semigroup, it follows from the proof of Theorem 3.5 in [7] that the elements s of S are represented as monomial matrices in $M_n(K[G])$ with entries in G^0; that is, at most one nonzero entry occurs in each row and column of s. It follows that

$$T_{ij}T_{jk} \subseteq T_{ik},$$

https://doi.org/10.4153/CMB-1999-036-2 Published online by Cambridge University Press

where $T_{ij} = e_{ii}S^1 e_{jj} \setminus \{0\}$. So each T_{ii} is a submonoid of G and $T = \bigcup_{1\le i,j\le n} T_{ij} \cup \{0\}$ is a uniform subsemigroup of $\hat{I}$. Furthermore,

$$R = \oplus_{ij} K_0[T_{ij}] = K_0[T],$$

where the contracted semigroup algebra notation $K_0[A]$ is also used to denote the K-subspace of $K[G]$ spanned by a subset A of G. From Proposition 4 we get that each $K_0[T_{ii}]$ is a maximal order, and by Corollary 3, $T_{ii} = T_{11}$ for all i.

Proposition 4 also implies that, for any distinct $1 \le i, j \le n$,

$$(e_{ii} + e_{jj})K_0[T](e_{ii} + e_{jj}) = \begin{bmatrix} K_0[T_{ii}] & K_0[T_{ij}] \\ K_0[T_{ji}] & K_0[T_{jj}] \end{bmatrix}$$

is a prime maximal order and $K_0[T_{ji}] = (K_0[T_{11}] : K_0[T_{ij}]) = K_0[(T_{11} : T_{ij})]$. Hence each T_{ij} is a divisorial T_{11}-ideal and $T_{ji} = T_{ij}^{-1}$.

Furthermore,

$$K_0[T_{ij}]K_0[T_{jk}] \subseteq K_0[T_{ik}]$$

and thus, because each $K_0[T_{ij}]$ is divisorial,

$$K_0[T_{ij}] * K_0[T_{jk}] \subseteq K_0[T_{ik}].$$

Hence, also

$$K_0[T_{ik}] * K_0[T_{jk}]^{-1} = K_0[T_{ik}] * K_0[T_{kj}] \subseteq K_0[T_{ij}].$$

Consequently $K_0[T_{ik}] \subseteq K_0[T_{ij}] * K_0[T_{jk}]$ and it follows that

$$K_0[T_{ij}] * K_0[T_{jk}] = K_0[T_{ik}]$$

and thus

$$T_{ij} * T_{jk} = T_{ik}.$$

This proves the necessity of the conditions (M1)–(M5). Conditions (N1)–(N2) are easily shown to be equivalent with $K_0[T] = \big[K_0[T_{ij}]\big]$ being right Noetherian (see for example Lemma 2.1 in [18]).

The sufficiency of conditions (M1)–(M5) follows easily, by induction on n, from Proposition 4 and Theorem 2. ■

Note that one can now easily deduce a characterization of semigroup algebras of nilpotent semigroups that are prime Noetherian Dedekind rings, hence obtaining one of the main results in [12]. Indeed, since the rings are of dimension 0 or 1, one obtains easily that the group G has to be cyclic and thus $K_0[I_{11}]$ is a principal ideal domain. So $K_0[I_{ij}]K_0[I_{ij}]^{-1} = K_0[I_{11}]$. Hence one verifies that $K_0[S]$ is isomorphic with a full matrix ring over a semigroup algebra $K_0[T]$, where T is either $\mathbf{N}$ or $\mathbf{Z}$ or trivial. Note that arbitrary semigroup algebras that are left and right principal ideal rings have been classified in [8].

The above result shows that our maximal orders are generalised matrix rings $R = [R_{ij}]$ with $R_{11} = \cdots = R_{nn}$ and each R_{ij} a divisorial R_{ii}-ideal. Furthermore $R_{ij} * R_{jk} = R_{ik}$. We could call such rings *divisorially graded* rings. The following example shows that in general these semigroup algebras are not full matrix rings. Indeed, let $R = K[S] = K[xz, xw, yz, yw] \subseteq K[x, y, z, w]$, a polynomial ring in four commuting variables. It is shown in [1] that $K[S]$ is a Noetherian maximal order that has $P = (xz, xw)$ as a height one prime ideal. It is easily seen that $x^{-1}y \in P^{-1} = (R : P) \subseteq \{x^a y^b z^c w^d \mid a \geq -1, b, c, d \geq 0\}$. Hence, $yz, yw \in x^{-1}yP \subseteq P^{-1}P$, and thus $M \subseteq P^{-1}P$, where M is the ideal of $K[S]$ consisting of the elements with zero constant term.

We claim that $P^{-1}P = M$, and thus $K[S]/P^{-1}P \cong K$. For this it is sufficient to show that $1 \notin P^{-1}P$. Suppose the contrary, then, since $P^{-1} \subseteq \{x^a y^b z^c w^d \mid a \geq -1, b, c, d \geq 0\}$, we get $1 \in x^{-1}y^\alpha z^\beta w^\delta P$ for some $\alpha, \beta, \delta \geq 0$. Now note that, $x^{-1}y^\alpha z^\beta w^\delta(xz) = y^\alpha z^{\beta+1}w^\delta \in K[S]$ implies that $y^\alpha z^{\beta+1} w^\delta \in \langle yz, yw\rangle$. Since the latter is a free semigroup, we obtain $\beta + \delta + 1 = \alpha$. Consequently $x^{-1}y^\alpha z^\beta w^\delta = (x^{-1}y)y^\beta y^\delta z^\beta w^\delta = (x^{-1}y)(yz)^\beta(yw)^\delta \in x^{-1}yK[S]$. So,

$$1 \in x^{-1}y^\alpha z^\beta w^\delta P \subseteq x^{-1}yP \subseteq yzK[S] + ywK[S],$$

which is a contradiction.

Let $T = \left[\begin{smallmatrix} S & P \\ P^{-1} & S \end{smallmatrix}\right]$, a uniform semigroup in $\mathcal{M}\big(gr(x, y, z, w), 2, 2; \Delta\big)$. From the theorem we know that

$$K_0[T] = \begin{bmatrix} K[S] & K[P] \\ K[P^{-1}] & K[S] \end{bmatrix}$$

is a prime Noetherian maximal order. However it is not a full matrix ring, since

$$I = \begin{bmatrix} K[S] & K[P] \\ K[P^{-1}] & K[M] \end{bmatrix}$$

is an ideal of $K[S]$ and $K[S]/I \cong K$ is not a 2×2-matrix ring. Note that $K_0[T] = K_0[T']$, with

$$T' = \begin{bmatrix} S & P \\ P^{-1} & P^{-1}P \end{bmatrix} \cup \{\Delta\}.$$

So T' only contains one of the elementary diagonal idempotents.

If in the theorem, all e_{ii} belong to S, then $S = T$. However, the previous example shows that this is not necessarily the case. We now show that S contains at least one of the idempotents e_{ii}. For a group G we denote by $\mathcal{M}_n(G)$ the semigroup of *monomial matrices* over G; that is, the submonoid of the multiplicative semigroup $M_n(K[G])$ that consists of all matrices having at most one nonzero entry in each row and column, and, moreover, each nonzero entry belongs to G. It follows from the proof of the previous theorem that if $K_0[S]$ is a prime Noetherian maximal order and S is nilpotent, then $S \subseteq \mathcal{M}_n(G)$ and $\mathcal{M}_n(G)$ is naturally embedded in $M_n(K[G])$. For $s \in S$, we denote by $\operatorname{rank}(s)$ the *rank* of s; that is, the number of nonzero rows in s.

Proposition 6 *Let S be a subsemigroup of $\mathcal{M}_n(G)$ (with G a group) and assume $e_{11}, e_{22}, \ldots, e_{nn} \in K_0[S] \subseteq M_n(K[G])$ and $\mathcal{M}_n(G)$ is naturally embedded in $M_n(K[G])$. Then S contains at least one idempotent e_{ii}.*

Proof Clearly we may assume $0 \in S$. We prove this by induction on n. If $n = 1$, then $K_0[S] \subseteq K[G]$. So $1 = e_{11} \in K_0[S]$ and thus $1 \in S$. By induction we may now assume that

https://doi.org/10.4153/CMB-1999-036-2 Published online by Cambridge University Press

every nonzero diagonal idempotent e of $\mathcal{M}_n(G)$ of rank less than n is not in S (otherwise replace S be eSe).

Suppose $s \in S$ is of rank n. Let

$$1 \neq e = \sum_{i=1}^{r} \lambda_i s_i, \quad \lambda_i \in K, \quad s_i \in S,$$

for a nonzero diagonal idempotent $e \in M_n(K[G])$. Then $es = \sum_{i=1}^{r} \lambda_i s_i s$ and $\{s_1 s, \dots, s_r s\}$ is linearly independent. So, if $es \in S$, then $r = 1$ and we must have $e = \lambda_1 s_1$. Since $e^2 = e$, it follows that $e = s_1 \in S$. As S does not contain idempotents of rank less than n, we obtain $1 = e \in S$, which is a contradiction. So we may assume that, for every $s \in S$ of rank n and every nonzero diagonal idempotent $e \neq 1$, we have $es \notin S$. We call this the (NS) condition.

Write $e_{11} = \sum \lambda_i s_i$. Clearly, $(s_i)_{11} = 1$ for some i, say $i = 1$. So $e_{11} = e_{11} s_1^k = s_1^k e_{11}$ and thus $e_{11} = \sum \lambda_i s_i s_1^k = \sum \lambda_i s_1^k s_i$ for every $k \geq 1$. Let k be such that s_1^k is diagonal. So, from the previous equalities, it follows that all $s_i \in eSe$, where e is the diagonal idempotent with the same pattern of nonzero entries as s_1^k. Now $\sum \lambda_i s_i = \sum \lambda_i s_i s_1^{km}$, for any $m \geq 1$, implies that $s_1^k = e$ (as s_1^k permutes the support of e_{11}). Since S does not contain idempotents of rank less than n, we obtain $s^k = e = 1$.

Let $j < n$ be maximal such that S has matrices of rank j. Suppose $j \neq 0$. Denote by M_j all the matrices in $\mathcal{M}_n(G)$ of rank at most j. It is easily verified [11] that each M_j is an ideal in $\mathcal{M}_n(G)$ and $M_j/M_{j-1} \cong \mathcal{M}\big(G_j, \binom{n}{j}, \binom{n}{j}; \Delta\big)$, where G_j is a group extension of G^j by the symmetric group S_j. It follows that

$$K_0\big[\mathcal{M}_n(G)\big] \cong K_0[M_n/M_{n-1}] \oplus \cdots \oplus K_0[M_2/M_1] \oplus K_0[M_1].$$

Hence we obtain a natural homomorphism

$$\varphi\colon K_0\big[\mathcal{M}_n(G)\big] \to K_0[M_j/M_{j-1}] \cong M_{\binom{n}{j}}\big(K[G_j]\big).$$

We claim that $\ker(\varphi) \cap K_0[S] = K_0[M_{j-1}] \cap K_0[S]$; whence $\varphi(K_0[S]) \cong K_0[S/(S \cap M_{j-1})]$. For this let $a = a_1 + a_2 \in \ker(\varphi) \cap K_0[S]$, where $\operatorname{supp}(a_1) \subseteq (M_n \setminus M_{n-1})$ and $\operatorname{supp}(a_2) \subseteq M_j$. The fact that $\varphi(a) = 0$ means that $f_k a \in K_0[M_{j-1}]$ for every $k = 1, 2, \dots, \binom{n}{j}$, where $f_1, \dots, f_{\binom{n}{j}}$ are the nonzero idempotents of $M_j \setminus M_{j-1}$. If $a_2 \neq 0$, then

$$f_k a = \sum_i \alpha_i f_k x_i + \sum_l \beta_l f_k y_l,$$

where $a_1 = \sum \alpha_i x_i$, $a_2 = \sum \beta_l y_l$ with $\alpha_i, \beta_l \in K$ and $x_i \in (M_n \setminus M_{n-1})$, $y_l \in M_j$. Each $f_k y_l$ is either y_l or it is in M_{j-1}. Because of condition (NS), $y_l \neq f_k x_i$ for any l, k, i. Therefore it follows that $f_k a_1 \in K_0[M_{j-1}]$ and $f_k a_2 \in K_0[M_{j-1}]$ (for every k). Since $\operatorname{rank}(f_k x_i) = j$, it follows that $f_k a_1 = 0$ for every k. In particular, $K_0[M_j]a_1 = 0$ and consequently $e_{ii} a_1 = 0$ for every $1 \leq i \leq n$. Since $1 = e_{11} + \cdots + e_{nn}$, we get $a_1 = 0$, which proves the claim.

So, replacing S by $\varphi(S)$, we may assume that all nonzero matrices in S have rank n or rank 1. Now, $e_{11} = \sum_{i=1}^{r} \lambda_i s_i$ with $1 \in \{s_1, \dots, s_r\}$. We claim that all elements s_i are of rank n. Indeed, otherwise (considering $M_n(K[G])$ as a K-vector space with basis $\mathcal{M}(G, n, n; \Delta)$)

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we see that there exists an element, say s_n, of rank one in the support of e_{11} with the only nonzero entry also a matrix entry of an element, say s_1, with $\operatorname{rank}(s_1) = n$; (note that $s_n \neq e_{11}$ as $e_{11} \notin S$). However then $s_n = es_1$ for some nonzero diagonal idempotent $e \neq 1$, in contradiction with condition (NS).

Now looking at the first row of e_{11}, since all s_i have rank n, we get

$$\sum_{i,(s_i)_{11}=1} \lambda_i = 1, \qquad \sum_{i,(s_i)_{11}=g\neq 1} \lambda_i = 0, \qquad \sum_{i,(s_i)_{1t}=g} \lambda_i = 0,$$

for $g \in G$, $t = 2, \ldots, n$. So $\sum_{i=1}^{r} \lambda_i = 1$. A similar argument applied to the second row of e_{11} yields $\sum_{i=1}^{r} \lambda_i = 0$, which is a contradiction. This completes the proof of the result. ∎

Remark Let S be a semigroup as in the statement of Theorem 5. The construction of the semigroup S and its algebra can now be explained by the following process. From Proposition 6 we obtain that S contains one of the diagonal idempotents of rank 1, say e_{nn}. Hence, by the Pierce decomposition,

$$K_0[S] = K_0[e_{nn}S \cup Se_{nn}] + (1 - e_{nn})K_0[S](1 - e_{nn}).$$

Clearly $Se_{nn} \cup e_{nn}S$ are matrices of rank 1 and

$$(1 - e_{nn})K_0[S](1 - e_{nn}) \cong K_0\big[(1 - e_{nn})S'(1 - e_{nn})\big],$$

where S' is the semigroup $\{s \in S \mid e_{nn}s = e_{nn}se_{nn} = se_{nn}\}$. The contracted semigroup algebra $K_0[(1 - e_{nn})S'(1 - e_{nn})]$ again is a prime maximal order, but the semigroup is embedded in $\mathcal{M}_{n-1}(G)$. So now this ring (and the respective semigroup) is decomposed similarly.

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https://doi.org/10.4153/CMB-1999-036-2 Published online by Cambridge University Press