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A NOTE ON THE ANALOGUE OF THE BOGOMOLOV TYPE THEOREM ON DEFORMATIONS OF CR-STRUCTURES

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ABSTRACT. Let $(M, {}^{\circ}T'')$ be a compact strongly pseudo-convex CR-manifold with trivial canonical line bundle. Then, in [A-M2], a *weak version* of the Bogomolov type theorem for deformations of CR-structures was shown by an analogy of the Tian-Todorov method. In this paper, we show that: *in the very strict sense*, there is a counterexample.

Introduction. Let $(M, {}^{\circ}T'')$ be a compact strongly pseudo-convex CR-manifold with trivial canonical line bundle and $\dim_{\mathbb{R}} M = 2n - 1 \geq 7$. Then as in the case of compact complex manifolds, a deformation theory of CR-structures was successfully established and a versal family was constructed ([A1], [A2], [A-M1], [M1]). Recently in the case of deformations of compact Kähler manifolds with trivial canonical line bundles, Tian and Todorov ([Ti], [To]) independently established a method for proving the smoothness of the parameter space of the versal family, so called the Bogomolov theorem ([B]). Therefore it seems interesting to study an analogy of their method for proving the smoothness of the versal family of deformations of CR-structures. A weak version of the Bogomolov type theorem for deformations of CR-structures was established ([A-M2], [A-M3]). In this paper, we show that there is an obstructed compact strongly pseudo-convex CR-manifold with trivial canonial line bundle and $\dim_{\mathbb{R}} M = 2n - 1 \geq 7$, and which even has transverse symmetry. Namely, in the very strict sense, the Bogomolov type theorem does not hold in the case of deformations of CR-structures. The authors would like to thank Prof. A. Fujiki. The counterexample in §2 was suggested by him. And they also thank the referee for pointing out that "the Tian-Todorov method for proving the Bogomolov theorem" should be used instead of "the Tian-Todorov theorem" in [A-M2].

1. **Preliminaries.** Let $(M, {}^{\circ}T'')$ be a strongly pseudo-convex CR-manifold with $\dim_{\mathbb{R}} M = 2n - 1 \geq 7$. Namely, $(M, {}^{\circ}T'')$ is a subbundle of the complexfied tangent bundle $\mathbb{C} \otimes TM$ satisfying:

$$(1) \qquad {}^{\circ}T'' \cap {}^{\circ}\bar{T}'' = 0, \quad \dim_{\mathbb{C}} \mathbb{C} \otimes TM / ({}^{\circ}T'' + {}^{\circ}\bar{T}'') = 1,$$

$$(2) \qquad [\Gamma(M, {}^{\circ}T''), \Gamma(M, {}^{\circ}T'')] \subset \Gamma(M, {}^{\circ}T'').$$

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Furthermore in this paper, we assume that there is a real vector field ζ satisfying:

(3) $$\zeta_p \notin {}^\circ T'' + {}^\circ \bar{T}'' \quad \text{for every point } p \in M,$$

(4) $$[\zeta, \Gamma(M, {}^\circ T'')] \subset \Gamma(M, {}^\circ T''),$$

namely M has transverse symmetry (see [L]). We set $T' = {}^\circ\bar{T}'' + \mathbb{C}\zeta$. Then our deformation complex is

$$O \longrightarrow \Gamma(M, T') \xrightarrow{\bar\partial_b} \Gamma\big(M, T' \otimes ({}^\circ T'')^*\big) \xrightarrow{\bar\partial_b} \Gamma\big(M, T' \otimes \overset{2}{\bigwedge}({}^\circ T'')^*\big) \xrightarrow{\bar\partial_b} \cdots$$

(*cf.* [K]). On the other hand, as is well known in [A-M2], we have the canonical decomposition

$$\overset{k}{\bigwedge}(\mathbb{C} \otimes TM)^* = \sum_{r+s=k, r\geq 0, s\geq 0} \overset{r}{\bigwedge}(T')^* \wedge \overset{s}{\bigwedge}({}^\circ T'')^*,$$

and on this, we have a d''-complex,

$$0 \longrightarrow \Gamma\big(M, \overset{r}{\bigwedge}(T')^*\big) \xrightarrow{d''} \Gamma\big(M, \overset{r}{\bigwedge}(T')^* \wedge ({}^\circ T'')^*\big) \xrightarrow{d''} \Gamma\big(M, \overset{r}{\bigwedge}(T')^* \wedge \overset{2}{\bigwedge}({}^\circ T'')^*\big) \xrightarrow{d''} \cdots.$$

Now suppose that the canonical line bundle $K_M = \bigwedge^n (T')^*$ is trivial in CR-sense, then we have a quasi-isomorphism between these complexes

$$\begin{array}{ccccc}
0 & \longrightarrow & \Gamma(M, T') & \xrightarrow{\bar\partial_b} & \Gamma\big(M, T' \otimes ({}^\circ T'')^*\big) \\
 & & i_0 \big\downarrow & & i_1 \big\downarrow \\
0 & \longrightarrow & \Gamma\big(M, \bigwedge^{n-1}(T')^*\big) & \xrightarrow{d''} & \Gamma\big(M, \bigwedge^{n-1}(T')^* \wedge ({}^\circ T'')^*\big)
\end{array}$$

$$\begin{array}{ccc}
\xrightarrow{\bar\partial_b} & \Gamma\big(M, T' \otimes \bigwedge^2({}^\circ T'')^*\big) & \xrightarrow{\bar\partial_b} \cdots \\
 & i_2 \big\downarrow & \\
\xrightarrow{d''} & \Gamma\big(M, \bigwedge^{n-1}(T')^* \wedge \bigwedge^2({}^\circ T'')^*\big) & \xrightarrow{d''} \cdots
\end{array}$$

On the above decomposition of $\bigwedge^k(\mathbb{C} \otimes TM)^*$, we also have operator d'. However, contrary to the complex manifold case, $d'd'' + d''d'$ does not necessarily vanish and some Kähler identities fail in our case. So in order to avoid this difficulty, we introduce

$$F^{p,q} = \Big\{u \in \Gamma\big(M, (\mathbb{C}\zeta)^* \wedge \overset{p}{\bigwedge}({}^\circ\bar{T}'')^* \wedge \overset{q}{\bigwedge}({}^\circ T'')^*\big), Lu = 0\Big\},$$

where L means the canonical (1,1) form defined by the Levi metric (see §3 in [A-M2]). And we introduce

$$Z^q = \{u \in F^{n-2,q}, d'u = 0\}.$$

We set

$$J^{n-2,q} = (F^{n-2,q} \cap \operatorname{Ker} d'' \cap d'F^{n-3,q}) / (d''F^{n-2,q-1} \cap d'F^{n-3,q}).$$

Note that $J^{n-2,q}$ is canonically embedded in $H^q_{d''}\big(M, \bigwedge^{n-1}(T')^*\big) \simeq H^q_{\bar\partial_b}(M, T')$ for $q \geq 2$ (*cf.* [A-M2, §5]). Then we have a weak version of the Bogomolov type theorem.

A WEAK VERSION OF THE BOGOMOLOV TYPE THEOREM ([A-M2, MAIN THEOREM]). *Let* $(M, {}^\circ T'')$ *be a strongly pseudo-convex* CR-*manifold having transverse symmetry and with* $\dim_{\mathbb{R}} M = 2n-1 \geq 7$. *And we assume that its canonical line bundle* $K_M = \bigwedge^n (T')^*$ *is trivial in* CR-*sense. Then the obstructions of deformations in* $i_1^{-1}(Z^1)$ *appear in* $J^{n-2,2}$. *That is, if* $J^{n-2,2} = 0$, *then any deformation of* CR-*structures in* $i_1^{-1}(Z^1)$ *is unobstructed.*

2. **An example of obstructed CR-manifolds.** In this section, we give an obstructed strongly pseudo-convex CR-manifold with trivial canonical line bundle, which was suggested by Prof. A. Fujiki. Let V be a projective algebraic manifold with ample canonical line bundle and satisfying: there is an element σ in $H^1(V, \Theta_V)$ satisfying $[\sigma, \sigma] \neq 0$ in $H^2(V, \Theta_V)$, where Θ_V denotes the sheaf of germs of holomorphic tangent vector fields on V. In fact, in [H] it is proved that some quintic surface satisfies the above. So setting $X = V \times Y$, where Y is an arbitrary projective algebraic manifold with ample canonical line bundle, then X is a projective algebraic manifold with ample canonical line bundle and satisfying: there is an element σ' in $H^1(X, \Theta_X)$ satisfying $[\sigma', \sigma'] \neq 0$ in $H^2(X, \Theta_X)$. We consider

$$\begin{array}{ccc} K_X^{-1} & \supset & U = K_X^{-1} \setminus 0 \\ \downarrow{\scriptstyle \pi} & & \\ X & & \end{array}$$

Then $K_U = \pi^*(K_X^{-1})^{-1} \otimes \pi^* K_X$, and K_U is trivial since $\pi^* K_X$ is trivial on U. Now we show the following theorem.

THEOREM 1. *Let* U *be as above. Then* U *is obstructed.*

PROOF. It suffices to show that there is an element σ in $H^1(U, \Theta_U)$ satisfying $[\sigma, \sigma] \neq 0$ in $H^2(U, \Theta_U)$. Here $[\ ,\ ]$ means; for $\theta = \{\theta_{ij}\}$, $\phi = \{\phi_{ij}\}$ in $H^1(U, \Theta_U)$, $[\theta, \phi]_{ijk} = \frac{1}{2}\{[\theta_{ij}, \phi_{jk}] + [\phi_{ij}, \theta_{jk}]\}$. Consider the diagram of cohomology groups

$$\begin{array}{ccc} H^q(U, \Theta_U) & \xrightarrow{\beta_q} & H^q(U, \pi^*\Theta_X) \\ & & \uparrow{\scriptstyle \gamma_q} \\ & & H^q(X, \Theta_X) \end{array}$$

where β_q is a homomorphism induced from a sheaf-homomorphism $\Theta_U \xrightarrow{d\pi} \pi^*\Theta_X$ and γ_q is an embedding as the component of degree 0 with respect to the grading $H^q(U, \pi^*\Theta_X) = \oplus_{\nu \in \mathbb{Z}} H^q(X, \Theta_X \otimes K_X^\nu)$ (*cf.* [S, Lemma 1]). Note that β_q and γ_q commute with $[\ ,\ ]$-operation, in particular $[\beta_1(\sigma), \beta_1(\tau)] = \beta_2([\sigma, \tau])$ and $[\gamma_1(\sigma'), \gamma_1(\tau')] = \gamma_2([\sigma', \tau'])$ hold. By the assumption, there is an element σ' in $H^1(X, \Theta_X)$ satisfying $[\sigma', \sigma'] \neq 0$ in $H^2(X, \Theta_X)$. This σ' defines a family of deformations of X over $\mathrm{Spec}\big(\mathbb{C}[t]/(t^2)\big)$. Namely, take a system of local charts of X $\{V_i, (z_i^1, \ldots, z_i^n)\}_{i \in I}$ with transition functions

$$z_i^\alpha = f_{ij}^\alpha(z_j)\ (\alpha = 1, \ldots, n) \text{ on } V_i \cap V_j.$$

With these, the above family can be represented by local charts $\{V_i \times \mathrm{Spec}(\mathbb{C}[t]/(t^2)), (z_i^1, \ldots, z_i^n)\}_{i \in I}$ with transition functions

$$z_i^\alpha = f_{ij}^\alpha(z_j) + \sigma_{ij}^\alpha(z_j)t \ (\alpha = 1, \ldots, n) \text{ on } V_i \times \mathrm{Spec}(\mathbb{C}[t]/(t^2)) \cap V_j \times \mathrm{Spec}(\mathbb{C}[t]/(t^2)),$$

where $\sigma' = \{\sum_{\alpha=1}^n \sigma_{ij}^\alpha \frac{\partial}{\partial z_i^\alpha}\}$. Then this family naturally induces a family of deformations of U, over $\mathrm{Spec}(\mathbb{C}[t]/(t^2))$, by

$$\begin{cases} z_i^\alpha = f_{ij}^\alpha(z_j) + \sigma_{ij}^\alpha(z_j)t & (\alpha = 1, \ldots, n) \\ \zeta_i = (k_{ij}(z_j) + k_{ij|1}(z_j)t)\zeta_j \end{cases}$$

where $\{k_{ij}(z_j)\}$ is the transition function of K_X^{-1} and $\{k_{ij|1}(z_j)\}$ is given by

$$\det\Big(\frac{\partial f_{ij}^\alpha}{\partial z_j^\beta} + \frac{\partial \sigma_{ij}^\alpha}{\partial z_j^\beta}t\Big) = k_{ij}(z_j) + k_{ij|1}(z_j)t + o(t^2).$$

So we have an element $\sigma = \{\sum_{\alpha=1}^n \sigma_{ij}^\alpha(z_j)\frac{\partial}{\partial z_i^\alpha} + k_{ij|1}(z_j)\zeta_j \frac{\partial}{\partial \zeta_i}\} \in H^1(U, \Theta_U)$ satisfying $\beta_1(\sigma) = \gamma_1(\sigma')$. Then σ satisfies $[\sigma, \sigma] \neq 0$ in $H^2(U, \Theta_U)$. In fact, if $[\sigma, \sigma] = 0$, then $\beta_2([\sigma, \sigma])$ must be zero because $\beta_2([\sigma, \sigma]) = \gamma_2([\sigma', \sigma'])$. But, by the assumption, $[\sigma', \sigma'] \neq 0$. Hence, we have $[\sigma, \sigma] \neq 0$ since γ_2 is injective. ∎

Now let M be the unit tangent sphere bundle in U and ${}^\circ T''$ the induced CR-structure from the complex structure of U. Then $(M, {}^\circ T'')$ is strongly pseudo-convex since K_X is ample. Since the natural restriction map $H^q(U, \Theta_U) \longrightarrow \varinjlim_{W \supset M} H^q(W, \Theta_W)$ is an isomorphism for $q = 1, 2$, the obstructed deformation family of U defined by $\sigma \in H^1(U, \Theta_U)$ induces an obstructed deformation family of U as germs along M. Hence, by [B-M] or [M2], we have an obstructed deformation of CR-structures on M.

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