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FIXED POINT THEOREMS AND EQUILIBRIUM POINTS IN ABSTRACT ECONOMIES

DONAL O'REGAN

New fixed point theorems are given which have applications in the theory of abstract economies.

1. INTRODUCTION

The purpose of this paper is twofold. First we present new fixed point theorems for DKT maps. These will then be used to establish new criteria for the existence of an equilibrium point for an abstract economy. The results in this paper improve and complement those in [**2**, **8**, **9**].

Let Z be a subset of a Hausdorff topological space E_1 and W a subset of a topological space E_2. We say $F \in DKT(Z,W)$ if W is convex, and there exists a map $B : Z \to W$ with $co\,(B(x)) \subseteq F(x)$ for all $x \in Z$, $B(x) \neq \emptyset$ for each $x \in Z$ and the fibres $B^{-1}(y)$ are open (in Z) for each $y \in W$. The following selection theorem will be used in Section 2 (see [**2**, p.206] for a proof).

THEOREM 1.1. *Let Z be a nonempty, paracompact Hausdorff topological space and W a nonempty, convex subset of a topological vector space. Suppose $F \in DKT(Z,W)$. Then F has a continuous selection (that is, there exists a continuous single valued map $f : Z \to W$ of F).*

2. FIXED POINT THEORY AND APPLICATIONS

We begin by establishing some new fixed point results for DKT maps. Our analysis will rely on Theorem 1.1 and on well known fixed point results in the literature (see [**3**]). The first result concerns compact maps (see [**2**]).

THEOREM 2.1. *Let I be an index set and $\{Q_i\}_{i\in I}$ a family of nonempty, convex sets each in a locally convex Hausdorff linear topological space E_i. For each $i \in I$, let $G_i \in DKT(Q,Q_i)$ be a compact map; here $Q = \prod_{i\in I} Q_i$. Then there exists $x^\star \in Q$ with $x^\star \in G(x^\star) \equiv \prod_{i\in I} G_i(x^\star)$ that is, $x_i^\star \in G_i(x^\star)$ for all $i \in I$ (here $x_i^\star$ is the projection of $x^\star$ on E_i).*

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PROOF: For each $i \in I$, choose a compact set $X_i \subseteq E_i$ such that $G_i(Q) \subseteq X_i \subseteq Q_i$. Let $X = \prod_{i \in I} X_i$. Note X is compact. Let $Y = co(X)$ and notice [**2**, p.206] that Y is paracompact. Also it is easy to see that $G_i \in DKT(Y, co(X_i))$ for each $i \in I$ (note $G_i(Y) \subseteq G_i(Q) \subseteq X_i \subseteq co(X_i)$). By Theorem 1.1, G_i has a continuous selection $g_i : Y \to co(X_i)$ for each $i \in I$ (in fact $g_i : Y \to X_i$ since $g_i(Y) \subseteq G_i(Y) \subseteq X_i$). Let $g : Y \to Y$ be defined by

$$g(x) = \prod_{i \in I} g_i(x);$$

note $g(Y) \subseteq \prod_{i \in I} X_i = X \subseteq Y$. Also g is continuous and compact since $g(Y) \subseteq X$. Now the Schauder–Tychonoff Theorem [**3**] implies that there exists $x^\star \in Y \subseteq Q$ such that

$$x^\star = g(x^\star) = \prod_{i \in I} g_i(x^\star) \subseteq \prod_{i \in I} G_i(x^\star).$$

∎

Next we improve Theorem 2.1 when our spaces E_i are Banach (in fact Theorem 2.2 can easily be extended to the Fréchet space setting).

THEOREM 2.2. *Let I be a countable index set and $\{Q_i\}_{i \in I}$ a family of nonempty, convex sets each in a Banach space E_i. For each $i \in I$, let $G_i \in DKT(Q, E_i)$ with $Q = \prod_{i \in I} Q_i$. Define $G : Q \to 2^E$ (here $E = \prod_{i \in I} E_i$ and note 2^E denotes the family of nonempty subsets of E) by*

$$G(x) = \prod_{i \in I} G_i(x) \quad \text{for each } \ x \in Q.$$

Suppose $G : Q \to 2^Q$ is a condensing map with $G(Q)$ a subset of a bounded set in Q. Then there exists $x^\star \in Q$ with $x^\star \in G(x^\star)$.

PROOF: Let $x_0 \in Q$. Note since Q is closed and convex then [**4**, p.19] implies that there exists a closed, convex set $X \subseteq E = \prod_{i \in I} E_i$ with $x_0 \in X$ and

$$X = \overline{co}\Big(G(Q \cap X) \cup \{x_0\}\Big).$$

Now since $G(Q \cap X) \cup \{x_0\} \subseteq G(Q) \cup \{x_0\} \subseteq Q$ we have $X \subseteq Q$ and so $Q \cap X = X$. Thus

$$X = \overline{co}\Big(G(X) \cup \{x_0\}\Big).$$

Since G is condensing we have, using the properties of measure of noncompactness, that X is compact. Thus $G : X \to X$ with X convex and compact.

Also $G_i \in DKT(X, Q_i)$ together with Theorem 1.1 (note X is paracompact) implies that G_i has a continuous selection $g_i : X \to Q_i$. Define the map $g : X \to Q$ by

$$g(x) = \prod_{i \in I} g_i(x) \quad \text{for each } \ x \in X.$$

Notice $g : X \to X$ since if $x \in X$ then

$$g(x) \subseteq \prod_{i \in I} G_i(x) = G(x) \subseteq X.$$

Thus $g : X \to X$ is continuous with X a convex, compact set. Then there exists $x^\star \in X \subseteq Q$ with $x^\star = g(x^\star) \in G(x^\star)$. $\square$

Our next fixed point result improves Theorem 2.1 if our spaces E_i are metrisable.

Theorem 2.3. *Let I be a countable index set and $\{Q_i\}_{i \in I}$ a family of nonempty, closed, convex sets each in a metrisable locally convex linear topological space E_i. Let $Q = \prod_{i \in I} Q_i$ and suppose $0 \in Q$. For each $i \in I$, let $G_i \in DKT(Q, E_i)$ be a compact map. Define $G : Q \to 2^E$ (here $E = \prod_{i \in I} E_i$) by*

$$G(x) = \prod_{i \in I} G_i(x) \quad \textit{for each} \quad x \in Q.$$

Suppose the following condition is satisfied:

$$\text{(2.1)} \qquad \begin{cases} \textit{if } \{(x_j, \lambda_j)\}_1^\infty \textit{ is a sequence in } \partial Q \times [0,1] \textit{ converging to } (x, \lambda) \\ \textit{with } x \in \lambda G(x) \textit{ and } 0 \leqslant \lambda < 1, \textit{ then there exists } j_0 \in \{1, 2, \dots\} \\ \textit{with } \{\lambda_j G(x_j)\} \subseteq Q \textit{ for each } j \geqslant j_0. \end{cases}$$

Then there exists $x^\star \in Q$ with $x^\star \in G(x^\star)$.

Proof: Note Q is paracompact since metrisable spaces are paracompact and closed subsets of paracompact spaces are paracompact. For each $i \in I$, by Theorem 1.1, there exists a continuous selection $g_i : Q \to E_i$ of G_i. Define $g : Q \to E = \prod_{i \in I} E_i$ by

$$g(x) = \prod_{i \in I} g_i(x).$$

Notice g is continuous. Let $r : E \to Q$ be a continuous retraction with $r(z) \in \partial Q$ for $z \in E \backslash Q$ (see [5]). Consider

$$X = \left\{ x \in E : \ x = g\,r\,(x) \right\}.$$

Notice $X \neq \emptyset$; this follows from the Schauder–Tychonoff Theorem since $g\,r : E \to E$ is a continuous, compact map (note $g\,r\,(E) \subseteq g(Q) = \prod_{i \in I} g_i\,(Q)$). Also X is closed and in fact compact since $X \subseteq g\,r\,(X) \subseteq g(Q) = \prod_{i \in I} g_i(Q)$. It remains to show $X \cap Q \neq \emptyset$. Suppose $X \cap Q = \emptyset$. Then there exists $\delta > 0$ with $dist\,(X, Q) > \delta$. Choose $m \in \{1, 2, \dots\}$ such that $1 < \delta\, m$. Fix $i \in \{m, m+1, \dots\}$ and let

$$U_i = \left\{ x \in E : \ d(x, Q) \ < \ \frac{1}{i} \right\};$$

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here d is the metric associated with E. Then $X \cap \overline{U_i} = \emptyset$. Now the nonlinear alternative for compact single valued maps (see [3]) implies (since $X \cap \overline{U_i} = \emptyset$) that there exists $(y_i, \lambda_i) \in \partial U_i \times (0,1)$ with $y_i = \lambda_i\, g\, r\,(y_i)$. Now $\lambda_i\, g\, r\,(y_i) \notin Q$ for $i \in \{m, m+1, \dots\}$ and so

$$\left\{\lambda_i\, G\, r\,(y_i)\right\} \not\subseteq Q \quad \text{for each} \quad i \in \{m, m+1, \dots\}. \tag{2.2}$$

Let

$$D = \{x \in E : \; x = \lambda\, g\, r\,(x) \;\; \text{for some} \;\; \lambda \in [0,1]\}.$$

Now D is compact (since $D \subseteq \overline{co}\,(g(r\,(D)) \cup \{0\})$) and so we may assume without loss of generality that $\lambda_j \to \lambda^\star$ and $y_j \to y^\star \in \partial Q$. Also $y^\star = \lambda^\star\, g\, r\,(y^\star)$. Now $\lambda^\star \neq 1$ since $X \cap Q = \emptyset$. Thus $0 \leqslant \lambda^\star < 1$. But in this case (2.1), with $x_j = r(y_j)$, $x = y^\star = r(y^\star)$, implies that there exists $j_0 \in \{1, 2, \dots\}$ with $\{\lambda_j\, G\, r\,(y_j)\} \subseteq Q$ for each $j \geqslant j_0$. This contradicts (2.2). Thus $X \cap Q \neq \emptyset$ that is, there exists $w \in Q$ with $w \in G\, r\,(w) = G(w)$. ∎

In Theorem 2.3 the compactness assumption on the maps G_i may be replaced by a condensing assumption on the map G. We follow the reasoning in Theorem 2.3. We need however our spaces E_i to be Hilbert spaces; choose the map $r : E \to Q$ in this case to be the nearest point projection (which we know to be nonexpansive). For completeness we state the result (the minor adjustments in the proof of Theorem 2.3 are left to the reader).

THEOREM 2.4. *Let I be a countable index set and $\{Q_i\}_{i \in I}$ a family of nonempty, closed, convex sets each in a Hilbert space E_i. Let $Q = \prod_{i \in I} Q_i$ and suppose $0 \in Q$. For each $i \in I$, let $G_i \in DKT(Q, E_i)$. Define $G : Q \to 2^E$ (here $E = \prod_{i \in I} E_i$) by*

$$G(x) = \prod_{i \in I} G_i(x) \quad \textit{for each} \quad x \in Q.$$

Suppose $G : Q \to 2^E$ is a condensing map with $G(Q)$ a subset of a bounded set in E and assume (2.1) holds. Then there exists $x^\star \in Q$ with $x^\star \in G(x^\star)$.

Next we apply Theorem 2.1 and Theorem 2.3 to give equilibrium theorems for an abstract economy. It is worth remarking here that we could also apply Theorem 2.2 and Theorem 2.4 to obtain other equilibrium theorems for an abstract economy. Let I be a countable or uncountable set of agents. We shall describe an abstract economy by $\Gamma = (Q_i, F_i, G_i, P_i)_{i \in I}$ where for each $i \in I$, $Q_i\, (\subseteq E_i)$ is the choice (or strategy) set, $F_i, G_i : \prod_{i \in I} Q_i = Q \to 2^{Q_i}$ are contraint correspondences and $P_i : Q \to 2^{Q_i}$ is a preference correspondence (see [8, 9]). In the case of an abstract economy being given by $\Gamma = (Q_i, F_i, G_i, P_i)_{i \in I}$ a point $x^\star \in Q$ is called an *equilibrium point* for Γ (or a generalised Nash equilibrium point) if for each $i \in I$, $x_i^\star \in cl_{E_i}\, G_i(x^\star)$ and $F_i(x^\star) \cap P_i(x^\star) = \emptyset$ (here $x_i^\star$ is the projection of $x^\star$ on E_i); if such an $x^\star$ exists we say Γ has an equilibrium.

THEOREM 2.5. *Let $\Gamma = (Q_i, F_i, G_i, P_i)_{i\in I}$ be an abstract economy such that for each $i \in I$ the following conditions hold:*

$$\left\{\begin{array}{l} Q_i \text{ is a nonempty, convex set in a locally convex} \\ \text{Hausdorff linear topological space } E_i \end{array}\right. \tag{2.3}$$

$$\text{for each } x \in Q = \prod_{i\in I} Q_i, \quad F_i(x) \neq \emptyset \text{ and } co\left(F_i(x)\right) \subseteq G_i(x) \tag{2.4}$$

$$\left\{\begin{array}{l} \text{for each } y_i \in Q_i, \text{ the set } \left[(co\,(P_i))^{-1}(y_i) \cup M_i\right] \cap F_i^{-1}(y_i) \\ \text{is open in } Q; \text{ here } M_i = \left\{x \in Q : F_i(x) \cap P_i(x) = \emptyset\right\} \end{array}\right. \tag{2.5}$$

$$G_i : Q \to 2^{Q_i} \text{ is a compact map} \tag{2.6}$$

and

$$\text{for each } x \in Q,\ x_i \notin co\left(P_i(x)\right); \text{ here } x_i \text{ is the projection of } x \text{ on } E_i. \tag{2.7}$$

Then Γ has an equilibrium point $x^\star \in Q$, that is, for each $i \in I$,

$$x_i^\star \in G_i(x^\star) \quad \text{and} \quad F_i(x^\star) \cap P_i(x^\star) = \emptyset;$$

here $x_i^\star$ is the projection of $x^\star$ on E_i.

PROOF: For each $i \in I$ let

$$N_i = \left\{x \in Q : F_i(x) \cap P_i(x) \neq \emptyset\right\}$$

and for each $x \in Q$ let

$$I(x) = \left\{i \in I : F_i(x) \cap P_i(x) \neq \emptyset\right\}.$$

For each $i \in I$, define the correspondences A_i, $B_i : Q \to 2^{Q_i}$ by

$$A_i(x) = \left\{\begin{array}{ll} co\,P_i(x) \cap F_i(x) & \text{if } i \in I(x) \text{ (that is, } x \in N_i) \\ F_i(x) & \text{if } i \notin I(x) \end{array}\right.$$

and

$$B_i(x) = \left\{\begin{array}{ll} co\,P_i(x) \cap G_i(x) & \text{if } i \in I(x) \\ G_i(x) & \text{if } i \notin I(x). \end{array}\right.$$

It is easy to see (use (2.4) and the definition of $I(x)$) for each $i \in I$ and $x \in Q$ that

$$co\left(A_i(x)\right) \subseteq B_i(x) \quad \text{and} \quad A_i(x) \neq \emptyset.$$

Also for each $i \in I$ and $y_i \in Q_i$ we have

$$\begin{aligned} A_i^{-1}(y_i) &= \left\{\left[(co\,(P_i))^{-1}(y_i) \cap F_i^{-1}(y_i)\right] \cap N_i\right\} \cup \left[F_i^{-1}(y_i) \cap M_i\right] \\ &= \left[(co\,(P_i))^{-1}(y_i) \cap F_i^{-1}(y_i)\right] \cup \left[F_i^{-1}(y_i) \cap M_i\right] \\ &= \left[(co\,(P_i))^{-1}(y_i) \cup M_i\right] \cap F_i^{-1}(y_i) \end{aligned}$$

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which is open in Q by (2.5). Thus $B_i \in DKT(Q, Q_i)$. In addition (2.6) implies that B_i is a compact map. Now Theorem 2.1 implies that there exists $x^\star \in Q$ with $x_i^\star \in B_i(x^\star)$ for all $i \in I$ (here $x_i^\star$ is the projection of $x^\star$ on E_i). Note if $i \in I(x^\star)$ for some $i \in I$, then $F_i(x^\star) \cap P_i(x^\star) \neq \emptyset$ and $x_i^\star \in co\,(P_i(x^\star)) \cap G_i(x^\star)$ (in particular $x_i^\star \in co\,(P_i(x^\star))$). This contradicts (2.7). Thus $i \notin I(x^\star)$ for all $i \in I$. Consequently $F_i(x^\star) \cap P_i(x^\star) = \emptyset$ and $x_i^\star \in G_i(x^\star)$ for all $i \in I$. ∎

Theorem 2.6. *Let $\Gamma = (Q_i, F_i, G_i, P_i)_{i \in I}$ be an abstract economy (here I is countable) such that for each $i \in I$ the following conditions hold:*

$$\left\{\begin{array}{l} Q_i \text{ is a nonempty, closed, convex set in a metrisable} \\ \text{locally convex linear topological space } E_i \end{array}\right. \tag{2.8}$$

$$0 \in Q = \prod_{i \in I} Q_i \tag{2.9}$$

$$\text{for each } x \in Q, \quad F_i(x) \neq \emptyset \quad \text{and} \quad co\left(F_i(x)\right) \subseteq G_i(x) \tag{2.10}$$

$$\left\{\begin{array}{l} \text{for each } y_i \in E_i, \text{ the set } \left[(co\,(P_i))^{-1}(y_i) \cup M_i\right] \cap F_i^{-1}(y_i) \\ \text{is open in } Q; \text{ here } M_i = \left\{x \in Q : F_i(x) \cap P_i(x) = \emptyset\right\} \end{array}\right. \tag{2.11}$$

$$G_i : Q \to 2^{E_i} \text{ is a compact map} \tag{2.12}$$

and

$$\text{for each } x \in Q,\ x_i \notin co\,(P_i(x)); \text{ here } x_i \text{ is the projection of } x \text{ on } E_i. \tag{2.13}$$

Let $I(x) = \left\{i \in I : F_i(x) \cap P_i(x) \neq \emptyset\right\}$ and $B_i : Q \to 2^{E_i}$ be

$$B_i(x) = \left\{\begin{array}{ll} co\,P_i(x) \cap G_i(x) & \text{if } i \in I(x) \\ G_i(x) & \text{if } i \notin I(x). \end{array}\right.$$

Define $B : Q \to 2^E$ (here $E = \prod_{i \in I} E_i$) by

$$B(x) = \prod_{i \in I} B_i(x) \quad \text{for each } x \in Q.$$

Assume

$$\left\{\begin{array}{l} \text{if } \{(x_j, \lambda_j)\}_1^\infty \text{ is a sequence in } \partial Q \times [0,1] \text{ converging to } (x, \lambda) \\ \text{with } x \in \lambda\, B(x) \text{ and } 0 \leqslant \lambda < 1, \text{ then there exists } j_0 \in \{1, 2, \ldots\} \\ \text{with } \{\lambda_j\, B(x_j)\} \subseteq Q \text{ for each } j \geqslant j_0 \end{array}\right. \tag{2.14}$$

holds. Then Γ has an equilibrium.

PROOF: For each $i \in I$ let N_i, A_i be as in Theorem 2.5. Essentially the same reasoning as in Theorem 2.5 implies that $B_i \in DKT(Q, E_i)$ is a compact map. Now Theorem 2.3 implies that there exists $x^\star \in Q$ with $x_i^\star \in B_i(x^\star)$ for all $i \in I$. The conclusion follows as in Theorem 2.5. ∎

We conclude the paper by extending the main result in [9] if the spaces E_i are Fréchet (that is, complete, metrisable locally convex linear topological space). In the analysis we shall need the following result of the author [**6**, **7**].

THEOREM 2.7. *Let E be a Fréchet space with Q a closed, convex subset of E and $0 \in Q$. Assume $G : Q \to CK(E)$ is a upper semicontinuous, compact map; here $CK(E)$ denotes the family of nonempty, convex, compact subsets of E. In addition suppose*

$$\text{(2.15)} \qquad \begin{cases} \text{if } \{(x_j, \lambda_j)\}_1^\infty \text{ is a sequence in } \partial Q \times [0,1] \text{ converging to } (x, \lambda) \\ \text{with } x \in \lambda G(x) \text{ and } 0 \leqslant \lambda < 1, \text{ then there exists } j_0 \in \{1, 2, \ldots\} \\ \text{with } \{\lambda_j G(x_j)\} \subseteq Q \text{ for each } j \geqslant j_0 \end{cases}$$

holds. Then G has a fixed point in Q.

REMARK. In fact Theorem 2.7 holds true if E is just a metrisable, locally convex linear topological space (follow the ideas in [**6**, **7**] using the fixed point theorem of Himmelberg for compact multifunctions).

We shall now use Theorem 2.7 together with some of the ideas in [9] to establish the following theorem.

THEOREM 2.8. *Let $\Gamma = (Q_i, F_i, G_i, P_i)_{i \in I}$ be an abstract economy (here I is countable) such that for each $i \in I$ the following conditions hold:*

$$\text{(2.16)} \qquad Q_i \text{ is a nonempty, closed, convex set in a Fréchet space } E_i$$

$$\text{(2.17)} \qquad 0 \in Q = \prod_{i \in I} Q_i$$

$$\text{(2.18)} \qquad \begin{cases} \text{for each } x \in Q, \ F_i(x) \neq \emptyset \text{ with } F_i(x) \subseteq G_i(x) \\ \text{and } G_i(x) \text{ is convex} \end{cases}$$

$$\text{(2.19)} \qquad \begin{cases} \text{the correspondence } \overline{G_i} : Q \to 2^{E_i} \text{ defined by} \\ \overline{G_i}(x) = cl_{E_i} G_i(x) \text{ is upper semicontinuous} \end{cases}$$

$$\text{(2.20)} \qquad \text{for each } y_i \in E_i, \ F_i^{-1}(y_i) \text{ is open in } Q$$

$$\text{(2.21)} \qquad \text{for each } y_i \in E_i, \ P_i^{-1}(y_i) \text{ is open in } Q$$

$$\text{(2.22)} \qquad \overline{G_i} : Q \to 2^{E_i} \text{ is a compact map}$$

and

$$\text{(2.23)} \qquad \text{for each } x \in Q, \ x_i \notin co(P_i(x)); \text{ here } x_i \text{ is the projection of } x \text{ on } E_i.$$

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Now for each $i \in I$ let

$$\phi_i(x) = F_i(x) \cap co\,(P_i(x)) \quad \text{for } x \in Q \quad \text{and} \quad U_i = \left\{x \in Q : \ \phi_i(x) \neq \emptyset\right\}$$

and

$$H_i(x) = \begin{cases} \left\{f_i(x)\right\} & \text{if } x \in U_i \\ \overline{G_i}\,(x) & \text{if } x \notin U_i \end{cases}$$

where $f_i : U_i \to 2^{E_i}$ is a continuous selection of $\phi_i|_{U_i}$ (see the proof for the existence of f_i). Define the map $H : Q \to 2^E$ (here $E = \prod_{i \in I} E_i$) by

$$H(x) = \prod_{i \in I} H_i(x) \quad \text{for each } x \in Q.$$

Assume

$$\text{(2.24)} \qquad \begin{cases} \text{if } \left\{(x_j, \lambda_j)\right\}_1^\infty \text{ is a sequence in } \partial Q \times [0,1] \text{ converging to } (x, \lambda) \\ \text{with } x \in \lambda\, H(x) \text{ and } 0 \leqslant \lambda < 1, \text{ then there exists } j_0 \in \{1, 2, \ldots\} \\ \text{with } \left\{\lambda_j\, H(x_j)\right\} \subseteq Q \text{ for each } j \geqslant j_0 \end{cases}$$

holds. Then Γ has an equilibrium $x^\star \in Q$, that is, for each $i \in I$,

$$x_i^\star \in \overline{G_i}\,(x^\star) \quad \text{and} \quad F_i(x^\star) \cap P_i(x^\star) = \emptyset;$$

here $x_i^\star$ is the projection of $x^\star$ on E_i.

REMARK. Theorem 2.8 remains valid if for each $i \in I$, E_i is just a metrisable, locally convex linear topological space.

REMARK If $G_i(Q) \subseteq Q_i$ for each $i \in I$ then (2.24) is clearly satisfied. Consequently in [**9**, Theorem 6.1] is a special case of Theorem 2.8.

REMARK More generally if $\overline{G_i}\,(\partial Q) \subseteq Q_i$ for each $i \in I$ then (2.24) is automatically true since $0 \in Q$ and

$$H(\partial Q) = \prod_{i \in I} H_i(\partial Q) \subseteq \prod_{i \in I} Q_i = Q.$$

PROOF: Fix $i \in I$. Let $\phi_i : Q \to 2^{E_i}$ and U_i be as in the statement of Theorem 2.8. Note (2.20), (2.21) and [**9**, Lemma 5.1] implies for each $y \in E_i$ that $\phi_i^{-1}(y)$ is open in Q. Also notice it is easy to see that

$$U_i = \bigcup_{y \in E_i} \phi_i^{-1}(y),$$

and so U_i is open in Q. Now $E = \prod_{i \in I} E_i$ is metrisable and U_i is paracompact. In addition we have (see the definitions of ϕ_i and U_i) that

$$\psi_i = \phi_i|_{U_i} : U_i \to 2^{E_i}$$

has nonempty, convex values and for each $y \in E_i$,

$$\psi_i^{-1}(y) = (\phi_i)^{-1}(y) \cap U_i$$

is open in U_i. Now [**9**, Theorem 3.1] guarantees that there exists a continuous selection $f_i : U_i \to 2^{E_i}$ such that $f_i(x) \in \psi_i(x)$ for all $x \in U_i$. Let H_i be as described in the statement of Theorem 2.8. Notice (2.19) together with [**9**, Lemma 6.1] implies that $H_i : Q \to 2^{E_i}$ is upper semicontinuous Also from (2.18) it is easy to see for each $x \in Q$ that $H_i(x)$ is a nonempty, closed and convex subset of E_i. Moreover (2.22) implies $H_i : Q \to CK(E_i)$ is a upper semicontinuous, compact map. Define H as in the statement of Theorem 2.8. Clearly $H : Q \to CK(E)$ is a upper semicontinuous (see [**1**, p.472]), compact map. Now Theorem 2.8 implies that there exists $x^\star \in Q$ with $x^\star \in H(x^\star)$. If $x^\star \in U_i$ for some $i \in I$ then $x_i^\star = f_i(x^\star) \in F_i(x^\star) \cap co\,(P_i(x^\star)) \subseteq co\,(P_i(x^\star))$ (here $x_i^\star$ is the projection of $x^\star$ on E_i). This contradicts (2.23). Thus for each $i \in I$ we must have $x^\star \notin U_i$ so that $x_i^\star \in \overline{G_i}\,(x_i)$ and $F_i(x^\star) \cap co\,(P_i(x^\star)) = \emptyset$. $\square$

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Department of Mathematics
Nationional University of Ireland
Galway
Ireland