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SOME INEQUALITIES ARISING FROM A BANACH ALGEBRA NORM

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Abstract

We derive some specific inequalities involving absolutely continuous functions and relate them to a norm inequality arising from Banach algebras of functions having bounded kth variation.

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We first obtain some integral inequalities involving absolutely continuous functions using elementary real variable theory. We then show that these and other inequalities are particular instances of an inequality satisfied by the norm of a Banach algebra of functions of bounded kth variation.

THEOREM 1. *Let f and g have absolutely continuous $(k-1)$th derivatives on $[a, b]$. In addition let $f_+^{(r)}(a) = 0 = g_+^{(r)}(a)$, $r = 0, 1, \ldots, k-1$. Then*

$$\text{(1)} \qquad \int_a^b |[f(t)g(t)]^{(k)}|\, dt \leq 2^{k-1}(b-a)^{k-1}\left(\int_a^b |f^{(k)}(t)|\, dt\right)\left(\int_a^b |g^{(k)}(t)|\, dt\right).$$

PROOF. We use an inductive argument. The case $k = 1$ is a special case of Theorem 1 in [7]. Assume now that (1) holds. Then

$$(2)\qquad \int_a^b |[f(x)g(x)]^{(k+1)}|\,dx = \int_a^b |\{[f(x)g(x)]'\}^{(k)}|\,dx$$
$$\leqslant \int_a^b |[f'(x)g(x)]^{(k)}|\,dx + \int_a^b |[f(x)g'(x)]^{(k)}|\,dx$$
$$\leqslant 2^{k-1}(b-a)^{k-1}\left(\int_a^b |f^{(k+1)}(x)|\,dx\right)\left(\int_a^b |g^{(k)}(x)|\,dx\right)$$
$$+ 2^{k-1}(b-a)^{k-1}\left(\int_a^b |f^{(k)}(x)|\,dx\right)\left(\int_a^b |g^{(k+1)}(x)|\,dx\right).$$

Since, under the given hypothesis,

$$f^{(k)}(x) = \int_a^x f^{(k+1)}(t)\,dt,$$

we obtain

$$\int_a^b |f^{(k)}(x)|\,dx \leqslant (b-a)\int_a^b |f^{(k+1)}(t)|\,dt.$$

Using the same inequality for g, the right hand side of (2) gives us the right hand side of (1), with k replaced by $k+1$, so the induction is complete.

Of course (1) extends by induction to more than two functions. Accordingly we present without proof the following result:

THEOREM 2. *Let f_i, $i = 1, 2, \ldots, n$, have absolutely continuous $(k-1)$th derivatives on $[a, b]$. If in addition*

$$f_{i+}^{(r)}(a) = 0, \qquad i = 1, 2, \ldots, n, r = 0, 1, 2, \ldots, k-1,$$

then

$$(3)\qquad \int_a^b \left| \left[\prod_{j=1}^n f_j(t) \right]^{(k)} \right| dt \leqslant [2(b-a)]^{(n-1)(k-1)} \prod_{j=1}^n \left(\int_a^b |f_j^{(k)}(t)|\,dt \right).$$

REMARKS. 1. The constant $2^{k-1}(b-a)^{k-1}$ in (1) is best possible when $k = 1$ and $k = 2$, as shown by putting $f(x) = g(x) = x$ and $f(x) = g(x) = x^{1+n^{-1}}$ respectively, and letting n tend to infinity in the second instance. The constant in (3) however is not best possible in general. It has recently been shown by Dr. C. J. F. Upton (see [8]) that the best possible constant is

$$\frac{(b-a)^{(n-1)(k-1)}}{[(k-1)!]^{n-1}} \binom{n(k-1)}{k-1}.$$

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2. The restrictions upon the functions at the end point a can be removed by considering $f(x) - \sum_{s=0}^{k-1}(x-a)^s f_+^{(s)}(a)/s!$. This, of course, complicates the left hand side of (3), but the right hand side will remain unchanged.

3. The inequality (1) first arose from consideration of a norm for a commutative Banach algebra of functions of generalized bounded variation. We now introduce this Banach algebra.

Let $BV_1[a, b]$ denote the linear space of functions of bounded variation on the closed interval $[a, b]$, and let

$$BV_1^*[a, b] = \{f: f \in BV_1[a, b], f(a) = 0\}.$$

We shall denote the total variation of f on $[a, b]$ by $V_1(f; a, b)$ or just $V_1(f)$ when no confusion can arise. If pointwise operations are employed it is known (see [1], [2] or Section 17.35 of [3]) that $BV_1^*[a, b]$ is a commutative Banach algebra with a unit element under the norm $\|\cdot\|_1^*$, where $\|f\|_1^* = V_1(f)$.

More recently, classes of functions of bounded kth variation have been studied, and have also been shown to be commutative Banach algebras under certain norms. (For the definition of bounded kth variation, the reader is referred to [4].) More specifically, let $BV_k[a, b]$ denote the linear space of functions of bounded kth variation on $[a, b]$, and let

$$BV_k^*[a, b] = \{f: f \in BV_k[a, b], f(a) = f'_+(a) = \cdots + f_+^{(k-1)}(a) = 0\}.$$

If total kth variation is denoted by $V_k(f)$, and pointwise operations are employed, it has been shown in [6] that for each $k \geqslant 1$, $BV_k^*[a, b]$ is a commutative Banach algebra under the norm $\|\cdot\|_k^*$, where $\|f\|_k^* = \alpha_k V_k(f)$, and

$$\alpha_k = 2^{k-1}(b-a)^{k-1}(k-1)!$$

The norm inequality,

$$\|fg\|_k^* \leqslant \|f\|_k^* \|g\|_k^*$$

can, of course, be written as the following inequality involving total kth variation:

(4) $$V_k(fg) \leqslant \alpha_k V_k(f) V_k(g).$$

It is well known (see Section 18 of [3]) that when f is absolutely continuous on $[a, b]$, $V_1(f) = \int_a^b |f'(t)|\, dt$. If in addition $f(a) = 0 = g(a)$, then (4) gives (1) with $k = 1$. Furthermore, it has been shown in [5] that when $f^{(k-1)}$ is absolutely continuous on $[a, b]$, then the total kth variation has the integral representation

$$(k-1)!V_k(f; a, x) = \int_a^x |f^{(k)}(t)|\, dt, \qquad a \leqslant x \leqslant b.$$

Thus if f and g belong to $BV_k^*[a, b]$ and have absolutely continuous $(k-1)$th derivatives on $[a, b]$, then (4) gives (1). Consequently (1) and (2) are particular cases of (4). We conclude with another case of (4) which does not involve integrals.

THEOREM 3. *Let $x_0, x_1, \ldots, x_n$ and $y_0, y_1, \ldots, y_n$ be two sets of complex numbers such that $x_0 = 0 = y_0$. Then*

$$\sum_{i=1}^{n} |x_i y_i - x_{i-1} y_{i-1}| \leq \left(\sum_{i=1}^{n} |x_i - x_{i-1}| \right) \left(\sum_{i=1}^{n} |y_i - y_{i-1}| \right). \tag{5}$$

A proof of (5) appears in [7], so will not be repeated here.

References

[1] S. K. Berberian, 'The character space of the algebra of regulated functions,' *Pacific J. Math.* **74** (1978), 15–36.

[2] G. Dickmeis and W. Dickmeis, *Beste Approximation in Räumen von beschränkter p-Variation*, Forschungberichte Des Landes Bordrhein-Westfalen Nr. 2697 Fachgruppe Mathematik/Informatik (Westdeutscher Verlag, 1977).

[3] E. Hewitt and K. Stromberg, *Real and abstract analysis* (Springer-Verlag, Berlin, Heidelberg, New York, 1969).

[4] A. M. Russell, 'Functions of bounded kth variation,' *Proc. London Math. Soc.* (3) **26** (1973), 547–563.

[5] A. M. Russell, 'Further results on an integral representation of functions of generalized variation,' *Bull. Austral. Math. Soc.* **3** (1978), 407–420.

[6] A. M. Russell, 'A commutative Banach algebra of functions of generalized variation', *Pacific J. Math.* **84** (1979), 455–463.

[7] A. M. Russell, 'A commutative Banach algebra of functions of bounded variation,' *Amer. Math. Monthly* **87** (1980), 39–40.

[8] C. J. F. Upton, 'An integral inequality associated with generalized variation', *Applicable Analysis*, to appear.

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