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ON THE HANKEL AND SOME RELATED TRANSFORMATIONS

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1. Introduction. The transformations we will discuss in this paper are the Hankel transformation H_υ, defined for $f \in \mathbf{C}_0$, the collection of continuous functions compactly supported in $(0, \infty)$, by

$$(1.1)\quad (H_\upsilon f)(x) = \int_0^\infty (xt)^{1/2} J_\upsilon(xt) f(t) dt,$$

and the $\mathscr{Y}_\upsilon$ and $\mathscr{H}_\upsilon$ transformations defined for such f by

$$(1.2)\quad (\mathscr{Y}_\upsilon f)(x) = \int_0^\infty (xt)^{1/2} Y_\upsilon(xt) f(t) dt,$$

and

$$(1.3)\quad (\mathscr{H}_\upsilon f)(x) = \int_0^\infty (xt)^{1/2} \mathbf{H}_\upsilon(xt) f(t) dt,$$

where J_υ and Y_υ are the Bessel functions of the first and second kinds respectively, and $\mathbf{H}_\upsilon$ is the Struve function; for the theory of these functions see [**1**, Chapter VII].

These transformations were studied extensively by one of us in [**5**] and [**6**] on the spaces $\mathscr{L}_{\mu,p}$ defined in [**7**; Sections 1 & 5]. In those papers the boundedness of the three transformations was fully given on the spaces $\mathscr{L}_{\mu,p}$ for $1 < p < \infty$, but not for $p = 1$. Also inversion formulae were given for the transformations only for portions of their respective ranges of boundedness.

In this paper we shall study the boundedness of the transformations on $\mathscr{L}_{\mu,1}$ and give inversion formulae for them for nearly their whole range of boundedness. The $\mathscr{L}_{\mu,1}$ boundedness will be studied in Section two, while Sections three, and four will be concerned with the inverses of H_υ, and $\mathscr{Y}_\upsilon$, respectively except for the case of $\mathscr{Y}_\upsilon$ on $\mathscr{L}_{1/2-\upsilon,p}$. It transpires that this is a special and more difficult case, for reasons that will be explained, and we shall treat this case in Section five. This may seem odd, since it includes the case of $\mathscr{Y}_0$ on $\mathscr{L}_{1/2,2}$ and *entre nous* $\mathscr{L}_{1/2,2} = L_2(0, \infty)$, where usually things are simpler. Finally in Section six we find an inverse for $\mathscr{H}_\upsilon$.

The notation here will be the same as that of [**6**] or [**7**]. In particular, if $1 \leqq p \leqq \infty$,

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$$\gamma(p) = \max(1/p,\ 1/p')$$

where, as usual, $p' = p/(p - 1)$.

2. Boundedness. Theorems 2.1, 2.2 and 2.3 below deal with the boundedness of H_ν, $\mathscr{Y}_\nu$ and $\mathscr{H}_\nu$ respectively on $\mathscr{L}_{\mu,1}$. However, we first need a lemma.

LEMMA 2.1. *Let*

$$M_1 = \int_0^\infty x^{1/2-\mu}|J_\nu(x)|dx,$$

$$M_2 = \int_0^\infty x^{1/2-\mu}|Y_0(x)|dx \quad \textit{and}$$

$$M_3 = \int_0^\infty x^{1/2-\mu}|\mathbf{H}_\nu(x)|dx.$$

Then (a) *if* $1 < \mu < \nu + 3/2$, $M_1 < \infty$; (b) *if* $1 < \mu < 3/2$, $M_2 < \infty$; (c) *if* $\nu + 1/2 < \mu < \nu + 5/2$ *and* $\mu > 1$, $M_3 < \infty$.

Proof. From the series for J_ν, $J_\nu(x) = O(x^\nu)$ as $x \to 0$. Also, from [**1**, 7.13.1(3)],

$$J_\nu(x) = O(x^{-1/2}) \quad \text{as } x \to \infty.$$

Hence $M_1 < \infty$ if and only if

$$\int_0^\delta x^{\nu-\mu+1/2}dx \quad \text{and} \quad \int_R^\infty x^{-\mu}dx$$

are both finite; that is if $1 < \mu < \nu + 3/2$. The proof for M_2 and M_3 is similar.

THEOREM 2.1. *If* $1 \leqq \mu \leqq \nu + 3/2$, $H_\nu \in [\mathscr{L}_{\mu,1}, \mathscr{L}_{1-\mu,\infty}]$. *If* $1 < \mu < \nu + 3/2$, *then for all* p, $1 \leqq p < \infty$, $H_\nu \in [\mathscr{L}_{\mu,1}, \mathscr{L}_{1-\mu,p}]$.

Proof. Since $J_\nu(x) = O(x^\nu)$ as $x \to 0$ and $J_\nu(x) = O(x^{-1/2})$ as $x \to \infty$, there is a constant K_ν so that for all $x > 0$,

$$|J_\nu(x)| \leqq K_\nu \cdot \min(x^\nu, x^{-1/2}) \leqq K_\nu x^{\mu-3/2},$$

if $1 \leqq \mu \leqq \nu + 3/2$. But then if $f \in \mathbf{C}_0$, $x > 0$ and $1 \leqq \mu \leqq \nu + 3/2$,

$$|(H_\nu f)(x)| \leqq K_\nu \int_0^\infty (xt)^{1/2}|J_\nu(xt)||f(t)|dt$$

$$\leqq K_\nu x^{\mu-1} \int_0^\infty t^{\mu-1}|f(t)|dt = K_\nu x^{\mu-1}||f||_{\mu,1}.$$

Thus

$$||H_\nu f||_{1-\mu,\infty} = \text{ess sup}_{x>0}\, x^{1-\mu}|(H_\nu f)(x)| \leqq K_\nu||f||_{\mu,1}$$

for $f \in \mathbf{C}_0$, so that H_υ can be extended to $\mathscr{L}_{\mu,1}$ for $1 \leqq \mu \leqq \upsilon + 3/2$, is in $[\mathscr{L}_{\mu,1}, \mathscr{L}_{1-\mu,\infty}]$ and is clearly given by (1.1) on $\mathscr{L}_{\mu,1}$.

If $1 < \mu < \upsilon + 3/2$,

$$\begin{aligned}
&\int_0^\infty x^{1-\mu} |(H_\upsilon f)(x)| dx/x \\
&\leqq \int_0^\infty x^{-\mu} dx \int_0^\infty (xt)^{1/2} |J_\upsilon(xt)| \, |f(t)| dt \\
&\leqq \int_0^\infty t^{1/2} |f(t)| dt \int_0^\infty x^{1/2-\mu} |J_\upsilon(tx)| dt \\
&= \int_0^\infty t^{\mu-1} |f(t)| dt \int_0^\infty x^{1/2-\mu} |J_\upsilon(x)| dx = M_1 \|f\|_{\mu,1}.
\end{aligned}$$

Hence, since by Lemma A, $M_1 < \infty$,

$$H_\upsilon \in [\mathscr{L}_{\mu,1}, \mathscr{L}_{1-\mu,1}].$$

But then by interpolation, using [**8**, Theorem 2], if $1 < \mu < \upsilon + 3/2$, $1 \leqq p \leqq \infty$,

$$H_\upsilon \in [\mathscr{L}_{\mu,1}, \mathscr{L}_{1-\mu,p}].$$

THEOREM 2.2. *If* $\upsilon \neq 0$ *and* $1 \leqq \mu \leqq 3/2 - |\upsilon|$, *then*

$$\mathscr{Y}_\upsilon \in [\mathscr{L}_{\mu,1}, \mathscr{L}_{1-\mu,\infty}].$$

If $\upsilon \neq 0$ *and* $1 < \mu < 3/2 - |\upsilon|$, *then for all* p, $1 \leqq p < \infty$,

$$\mathscr{Y}_\upsilon \in [\mathscr{L}_{\mu,1}, \mathscr{L}_{1-\mu,p}].$$

If $1 \leqq \mu < 3/2$,

$$\mathscr{Y}_0 \in [\mathscr{L}_{\mu,1}, \mathscr{L}_{1-\mu,\infty}].$$

If $1 < \mu < 3/2$, *then for all* p, $1 \leqq p < \infty$,

$$\mathscr{Y}_0 \in [\mathscr{L}_{\mu,1}, \mathscr{L}_{1-\mu,p}].$$

Proof. The results for $\upsilon \neq 0$ follow from Theorem 2.1 since from [**1**, 7.2.1(4)],

$$\mathscr{Y}_\upsilon = \cot(\pi\upsilon) \cdot H_\upsilon - \csc(\pi\upsilon) \cdot H_{-\upsilon}.$$

From [**1**, 7.2.4(33)], $Y_0(x) = O(\log x)$ as $x \to 0$, and from [**1**, 7.13.1(4)], $Y_0(x) = O(x^{-1/2})$ as $x \to \infty$. Hence there are constants A, B and C such that

$$|Y_0(x)| \leqq A|\log x| + B, \quad 0 < x < 1, \text{ and}$$
$$|Y_0(x)| \leqq Cx^{-1/2}, \quad x > 1.$$

But then if $f \in \mathbf{C}_0$ and $x > 0$,

$$|(\mathscr{Y}_0 f)(x)| \leqq \int_0^\infty (xt)^{1/2} |Y_0(xt)| \, |f(t)| dt$$

$$\leqq A \int_0^{1/x} (xt)^{1/2}|\log(xt)|\,|f(t)|dt$$
$$+ B \int_0^{1/x} (xt)^{1/2}|f(t)|dt + C \int_{1/x}^{\infty} |f(t)|dt$$
$$= AI_1 + BI_2 + CI_3,$$

say. Now, if $\mu \geqq 1$,

$$I_3 \leqq x^{\mu-1} \int_{1/x}^{\infty} t^{\mu-1}|f(t)|dt \leqq x^{\mu-1}||f||_{\mu,1}.$$

Also, if $\mu \leqq 3/2$,

$$I_2 = x^{1/2} \int_0^{1/x} t^{3/2-\mu}t^{\mu-1}|f(t)|dt$$
$$\leqq x^{\mu-1} \int_0^{1/x} t^{\mu-1}|f(t)|dt \leqq x^{\mu-1}||f||_{\mu,1}.$$

Let

$$h(t) = t^{3/2-\mu}|\log (xt)|.$$

Then if $\mu < 3/2$, $h(t) \to 0$ as $t \to 0$, $h(1/x) = 0$, and if $0 < t < 1/x$, $h(t) > 0$. Hence the maximum value of h occurs at the point t_0 in that interval where $h'(t_0) = 0$. Since

$$h'(t) = -t^{1/2-\mu}((3/2 - \mu)\log(xt) + 1),$$

$t_0 = K/x$, where

$$K = e^{-1/(3/2-\mu)} < 1,$$

so that $0 < t_0 < 1/x$, and thus

$$h(t_0) = Lx^{\mu-3/2},$$

where $L = e^{-1}/(3/2 - \mu)$. Hence if $0 < t \leqq 1/x$,

$$0 \leqq h(t) \leqq Lx^{\mu-3/2}.$$

Thus if $\mu < 3/2$,

$$I_1 = x^{1/2} \int_0^{1/x} h(t)t^{\mu-1}|f(t)|dt \leqq Lx^{\mu-1} \int_0^{1/x} t^{\mu-1}|f(t)|dt$$
$$\leqq Lx^{\mu-1}||f||_{\mu,1}.$$

Hence

$$|(\mathcal{Y}_0 f)(x)| \leqq (AL + B + C)x^{\mu-1}||f||_{\mu,1},$$

and thus, as for H_ν in the proof of Theorem 2.1, $\mathcal{Y}_0$ can be extended to $\mathcal{L}_{\mu,1}$ as a member of $[\mathcal{L}_{\mu,1}, \mathcal{L}_{1-\mu,\infty}]$ if $1 \leqq \mu \leqq 3/2$, and clearly formula (2.1) remains valid. The proof that $\mathcal{Y}_0 \in [\mathcal{L}_{\mu,1}, \mathcal{L}_{1-\mu,p}]$ if $1 \leqq p < \infty$, $1 < \mu < 3/2$ is practically the same as for H_ν in Theorem 2.1 using that $M_2 < \infty$ from Lemma 2.1.

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THEOREM 2.3. *If* $\upsilon + 1/2 \leqq \mu \leqq \upsilon + 5/2$ *and* $\mu \geqq 1$, *then*

$$\mathscr{H}_\upsilon \in [\mathscr{L}_{\mu,1}, \mathscr{L}_{1-\mu,\infty}].$$

If $\upsilon + 1/2 < \mu < \upsilon + 5/2$ *and* $\mu > 1$, *then for all* p, $1 \leqq p < \infty$,

$$\mathscr{H}_\upsilon \in [\mathscr{L}_{\mu,1}, \mathscr{L}_{1-\mu,p}].$$

Proof. Much the same as the proof of Theorem 2.1, using $\mathbf{H}_\upsilon(x) = O(x^{\upsilon+1})$ as $x \to 0$, from [**1**, 7.5.4(55)], $\mathbf{H}_\upsilon(x) = O(x^{\upsilon-1})$ as $x \to \infty$ if $\upsilon \geqq 1/2$ and $\mathbf{H}_\upsilon(x) = O(x^{-1/2})$ as $x \to \infty$ if $\upsilon < 1/2$, from [**1**, 7.5.4(63) & 7.13.1(4)], and from Lemma 2.1, $M_3 < \infty$.

3. Inversion of the Hankel transformation. In [**6**, Theorem 2.3] we found an inverse for the Hankel transformation H_υ on $\mathscr{L}_{\mu,p}$, but with the restriction that $\mu < 1$. In the theorem below we find an inverse without this restriction.

THEOREM 3.1. *Suppose that* $f \in \mathscr{L}_{\mu,p}$ *where either* $1 < p < \infty$ *and* $\gamma(p) \leqq \mu < \upsilon + 3/2$, *or* $p = 1$ *and* $1 \leqq \mu \leqq \upsilon + 3/2$. *Choose* $n > \mu$. *Then for almost all* $x > 0$

$$f(x) = x^{-\upsilon+1/2}\left[\frac{1}{x}\cdot\frac{d}{dx}\right]^n x^{\upsilon+n-1/2}\int_0^\infty (xt)^{1/2}J_{\upsilon+n}(xt) \times (H_\upsilon f)(t)dt/t^n.$$

Proof. Suppose first that $1 < p < \infty$, and for $x > 0$ let

$$g_x(t) = t^{-n+1/2}J_{\upsilon+n}(xt).$$

Then $g_x(t) = O(t^{\upsilon+1/2})$ as $t \to 0$, and $g_x(t) = O(t^{-n})$ as $t \to \infty$. But the hypotheses imply that $\mu \geqq 1/2$ since $\gamma(p) \geqq 1/2$, and that $\upsilon > -1$, for the same reason. Thus, since also $\mu < n$, $g_x \in \mathscr{L}_{\mu,p'}$. From [**2**, 8.11.(7)],

$$(H_\upsilon g_x)(t) = \begin{cases} t^{\upsilon+1/2}(x^2 - t^2)^{n-1}/2^{n-1}x^{\upsilon+n}\Gamma(n), & 0 < t < x,\\ 0, & t > x.\end{cases}$$

Hence, from [**6**, Theorem 2.1],

$$\begin{aligned}(3.1)\quad & x^{\upsilon+n-1/2}\int_0^\infty (xt)^{1/2}J_{\upsilon+n}(xt)\cdot(H_\upsilon f)(t)dt/t^n\\ &= x^{\upsilon+n}\int_0^\infty g_x(t)\cdot(H_\upsilon f)(t)dt = x^{\upsilon+n}\int_0^\infty (H_\upsilon g_x)(t)\cdot f(t)dt\\ &= (2^{n-1}\Gamma(n))^{-1}\int_0^x t^{\upsilon+1/2}(x^2 - t^2)^{n-1}f(t)dt.\end{aligned}$$

Now (3.1) also holds if $p = 1$. For it holds if $f \in \mathbf{C}_0$, which is dense in $\mathscr{L}_{\mu,1}$, and for each $x > 0$ both sides of (3.1) represent bounded linear functionals on $\mathscr{L}_{\mu,1}$ if $1 \leqq \mu \leqq \upsilon + 3/2$; for, if $f \in \mathscr{L}_{\mu,1}$ where $1 \leqq \mu \leqq 3/2$, then

$$\left|\int_0^x t^{\upsilon+1/2}(x^2 - t^2)^{n-1}f(t)dt\right|$$
$$\leqq \int_0^x t^{\upsilon+3/2-\mu}(x^2 - t^2)^{n-1}|t^\mu f(t)|dt/t \leqq x^{\upsilon+2n-\mu-1/2}||f||_{\mu,1},$$

so that the right-hand side of (3.1) represents a bounded linear functional; and

$$\left|\int_0^\infty t^{-n+1/2}J_{\upsilon+n}(xt)\cdot(H_\upsilon f)(t)dt\right|$$
$$\leqq \int_0^\infty t^{\mu-n-1/2}|J_{\upsilon+n}(xt)|\,|t^{1-\mu}(H_\upsilon f)(t)|dt$$
$$\leqq ||H_\upsilon f||_{1-\mu,\infty}\int_0^\infty t^{\mu-n-1/2}|J_{\upsilon+n}(xt)|dt$$
$$\leqq x^{n-1/2-\mu}K_\upsilon||f||_{\mu,1}\int_0^\infty t^{\mu-n-1/2}|J_{\upsilon+n}(t)|dt < \infty$$

by Lemma 2.1, K_υ being the norm of H_υ. Hence (3.1) is valid on $\mathscr{L}_{\mu,1}$ if $1 \leqq \mu \leqq \upsilon + 3/2$. But if we call the right hand side of (3.1) $(T_n f)(x)$, then it is obvious that

$$\frac{1}{x}\cdot\frac{d}{dx}(T_n f)(x) = (T_{n-1}f)(x) \quad \text{if } n > 1, \quad \text{and}$$

$$\frac{d}{dx}(T_1 f)(x) = x^{\upsilon+1/2}f(x) \quad \text{a.e.}$$

Hence

$$\left[\frac{1}{x}\cdot\frac{d}{dx}\right]^n(T_n f)(x) = x^{\upsilon-1/2}f(x) \text{ a.e.},$$

and the result follows.

It seems worth remarking that for a particular value of υ one can find a value of n, namely $[\upsilon + 3/2] + 1$, so that the inversion given in Theorem 3.1 is valid for all μ for which H_υ is bounded. However, for a particular value of μ, this value of n may well be unnecessarily large.

4. Inversion of the $\mathscr{Y}_\upsilon$ transformation. The inverse for the $\mathscr{Y}_\upsilon$ transformation found in [**6**, Theorem 6.2] was only valid on $\mathscr{L}_{\mu,p}$ for $\mu < 1/2 - \upsilon$, which, since $\mu \geqq \gamma(p) \geqq 1/2$, necessarily entails that $\upsilon < 0$. In this section we find formulae for the inverse of the $\mathscr{Y}_\upsilon$ transformation valid for $\mu > 1/2 - \upsilon$. The case when $\mu = 1/2 - \upsilon$ will be treated in the next section. First, however, we need a definition and several lemmas.

Definition. Let

(4.1) $m_\upsilon(s) = 2^{s-1/2}\Gamma((\upsilon + s + 1/2)/2)/\Gamma((\upsilon - s + 3/2)/2)$,

(4.2) $g_\upsilon(s) = m_\upsilon(s) \cdot \tan(\pi(\upsilon + s + 1/2)/2)/(\upsilon - s + 3/2)$,

(4.3) $A_\upsilon = 1/(2^\upsilon \pi^{1/2}\Gamma(\upsilon + 3/2))$,

(4.4) $h_\upsilon(t) = t^{-1/2}\{\mathbf{H}_{\upsilon+1}(t) - A_\upsilon t^\upsilon\}$,

(4.5) $$r_{\upsilon,x}(t) = \begin{cases} t^{\upsilon+1/2}/x^{\upsilon+3/2}, & 0 < t < x \\ 0, & t > x. \end{cases}$$

We also use the operator D_a defined for $a > 0$ in [7, (2.12)].

LEMMA 4.1. *If* $1/2 \leqq \mu < 3/2$ *and* $1/2 - \mu < \upsilon < 1$, *then*

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{\mu-iR}^{\mu+iR} x^{-s} g_\upsilon(s)ds = h_\upsilon(x).$$

Proof. For $n = 1, 2, \ldots$, consider

$$I_{n,R} = \frac{1}{2\pi i} \int_{\Gamma_{n,R}} x^{-s} g_\upsilon(s)ds,$$

where $\Gamma_{n,R}$ is the rectangle with vertices $\mu \pm iR$ and $\sigma_n \pm iR$, where $\sigma_n = -2n - 1/2 - \upsilon$, oriented counterclockwise. Note that, using

$$\Gamma(z) \cdot \Gamma(1 - z) = \pi/\sin(\pi z) \quad \text{and} \quad \Gamma(z + 1) = z\Gamma(z),$$

$$g_\upsilon(s) = \pi \cdot 2^{s-3/2}/(\Gamma((3/2 - s - \upsilon)/2)\Gamma((7/2 - s + \upsilon)/2) \times \cos(\pi(s + \upsilon + 1/2)/2))$$

and thus the integrand has simple poles at the points

$$s_m = -2m + 1/2 - \upsilon, \quad m = 0, \pm 1, \pm 2, \ldots,$$

and since $1/2 - \upsilon < \mu < 3/2 < 5/2 - \upsilon$, s_m is within $\Gamma_{n,R}$ if and only if $m = 0, 1, 2, \ldots, n$; some of the poles are cancelled by some of the zeroes if $\upsilon = \pm 1/2$, but these are all outside $\Gamma_{n,R}$. The residue of $x^{-s}g_\upsilon(s)$ at s_m is

$$\begin{aligned} R_m &= (\pi \cdot x^{\upsilon+2m-1/2}2^{-(\upsilon+2m+1)}/(\Gamma(m + 1/2)\Gamma(\upsilon + m + 3/2)) \\ &\quad \times (\lim_{s\to s_m} (s - s_m)/\cos(\pi(s + \upsilon + 1/2)/2) \\ &= (-1)^{m+1}x^{\upsilon+2m-1/2}/(2^{\upsilon+2m}\Gamma(m + 1/2)\Gamma(\upsilon + m + 3/2)), \end{aligned}$$

and hence

$$I_{n,R} = \sum_{m=0}^{n} R_m.$$

On top of $\Gamma_{n,R}$, $s = \sigma + iR$, $-2n - 1/2 - \upsilon \leqq \sigma \leqq \mu$, and as $R \to \infty$, from [**1**, 1.18(6)],

$$|m_\upsilon(s)/(\upsilon + 3/2 - s)| \sim R^{\sigma - 3/2},$$

uniformly in σ for σ in any finite interval, so that for sufficiently large R, on top of $\Gamma_{n,R}$,

$$|m_\upsilon(s)/(\upsilon + 3/2 - s)| \leqq 2R^{\sigma - 3/2} \leqq 2R^{\mu - 3/2}.$$

Also, if $|\mathrm{Im}\ z| > 1$, $|\tan z| < K$, and hence on the top of $\Gamma_{n,R}$,

$$|x^{-s}g_\upsilon(s)| \leqq 2Kx^{-\sigma}R^{\mu - 3/2}.$$

But $x^{-\sigma} \leqq x^{-\mu}$ if $0 < x \leqq 1$, and $x^{-\sigma} \leqq x^{2n+\upsilon+1/2}$, $1 \leqq x < \infty$, and thus on top of $\Gamma_{n,R}$,

$$x^{-s}g_\upsilon(s) = O(R^{\mu - 3/2}),$$

uniformly in σ, so that the integral along the top of $\Gamma_{n,R}$ tends to zero as $R \to \infty$, as does the integral along the bottom in a similar fashion. Hence letting $R \to \infty$, it follows that

$$\lim_{R\to\infty} \frac{1}{2\pi i}\left[\int_{\mu - iR}^{\mu + iR} x^{-s}g_\upsilon(s)ds - \int_{\sigma_n - iR}^{\sigma_n + iR} x^{-s}g_\upsilon(s)ds\right] = \sum_{m=0}^{n} R_m.$$

But on $\mathrm{Re}\ s = \sigma_n$, $s = \sigma_n + i\tau$, and from [**1**, 1.18(6)], as $|\tau| \to \infty$,

$$\begin{aligned} |x^{-s}g_\upsilon(s)| &= x^{2n+\upsilon+1/2}2^{-2n-\upsilon-2}|\Gamma((-2n + i\tau)/2) \\ &\quad \times \tanh(\pi\tau/2)/\Gamma((2\upsilon + 2n + 4 - i\tau)/2)| \\ &\sim x^{2n+\upsilon+1/2}|\tau|^{-\upsilon-2n-2}, \end{aligned}$$

so that since $\upsilon < 1$ and $n \geqq 0$, $x^{-s}g_\upsilon(s)$, with $s = \sigma_n + i\tau$, is in $L_1(\mathbf{R})$, and thus

$$\begin{aligned} \lim_{R\to\infty} \frac{1}{2\pi i} \int_{\mu - iR}^{\mu + iR} x^{-s}g_\upsilon(s)ds &= \sum_{m=0}^{n} R_m + \frac{1}{2\pi i}\int_{\sigma_n - i\infty}^{\sigma_n + i\infty} x^{-s}g_\upsilon(s)ds \\ &= \sum_{m=0}^{n} R_m + I_n, \end{aligned}$$

say. Now, setting $s = \sigma_n + 2it$,

$$\begin{aligned} |I_n| &\leqq x^{2n+\upsilon+1/2}2^{-2n-\upsilon-2} \\ &\quad \times \pi^{-1}\int_{-\infty}^{\infty} |\Gamma(-n + it)\tanh(\pi t)/\Gamma(\upsilon + n + 2 - it)|dt. \end{aligned}$$

But,

$$\begin{aligned}
&|\Gamma(-n + it)| \\
&= |\Gamma(it)|/|(-1 + it)(-2 + it)\dots(-n + it)| \\
&= |\Gamma(it)|/((1 + t^2)(4 + t^2)\dots(n^2 + t^2))^{1/2} \\
&\leqq |\Gamma(it)|/n!, \quad \text{and,} \\
&|\Gamma(v + n + 2 - it)| \\
&= |(v + n + 1 - it)(v + n - it) \\
&\qquad\qquad \dots(v + 2 - it)\Gamma(v + 2 - it)| \\
&\geqq (v + n + 1)(v + n)\dots(v + 2)|\Gamma(v + 2 - it)| \\
&= \Gamma(v + n + 2)|\Gamma(v + 2 - it)|/\Gamma(v + 2),
\end{aligned}$$

and thus

$$|I_n| \leqq (x^{2n+v+1/2}\Gamma(v + 2)/(2^{2n+v+1}n! \cdot \pi \cdot \Gamma(v + n + 2))) \times \int_{-\infty}^{\infty} |\Gamma(it)\tanh(\pi t)/\Gamma(v + 2 - it)|dt \to 0$$

as $n \to \infty$. Thus, letting $n \to \infty$,

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{\mu-iR}^{\mu+iR} x^{-s}g_v(s)ds = \sum_{m=0}^{\infty} R_m = h_v(x).$$

LEMMA 4.2. *If* $1/2 \leqq \mu < 1$ *and* $1/2 - \mu < v < 1$, *then as a function of* t,

$$g_v(\mu + it) \in L_2(\mathbf{R}).$$

Proof. From (4.1), $m_v(s)$ is holomorphic in $\operatorname{Re} s > -(v + 1/2)$, and clearly $\mu > -(v + 1/2)$, so that $m_v(\mu + it)$ is continuous on $\mathbf{R}$. Also, $\tan(\pi(s + v + 1/2)/2)$ is holomorphic in the strip $1/2 - v < \operatorname{Re} s < 5/2 - v$, and thus since

$$1/2 < \mu + v < 5/2,$$

it follows that $\tan(\pi(\mu + it + v + 1/2)/2)$ is continuous on $\mathbf{R}$. Hence, $g_v(\mu + it)$ is continuous on $\mathbf{R}$. In addition,

$$|\tan(\pi(\mu + it + v + 1/2)/2)| \to 1 \quad \text{as } |t| \to \infty,$$

while from [**1**, 1.18(6)],

$$|m_v(\mu + it)| \sim |t|^{\mu-1/2} \quad \text{as } |t| \to \infty,$$

and thus

$$g_v(\mu + it) \sim |t|^{\mu-3/2} \quad \text{as } |t| \to \infty,$$

and hence since $\mu < 1$, $g_v(\mu + it) \in L_2(\mathbf{R})$.

LEMMA 4.3. *If* $1/2 - \upsilon < \mu < 1$, *then* $h_\upsilon \in \mathscr{L}_{\mu,p}$ *for* $1 \leqq p < \infty$. *If* $1/2 \leqq \mu < 1$ *and* $1/2 - \mu < \upsilon < 1$, *then for* $\text{Re } s = \mu$, $(\mathscr{M}h_\upsilon)(s) = g_\upsilon(s)$.

Proof. From [**1**, 7.5.4(63) & 7.13.1(4)], $h_\upsilon(x) = O(x^{-1})$ as $x \to \infty$, and from [**1**, 7.5.4(55)] and (4.4), $h_\upsilon(x) = O(x^{\upsilon-1/2})$ as $x \to 0$. Thus for $1 \leqq p < \infty$, since $1/2 - \upsilon < \mu < 1$, $h_\upsilon \in \mathscr{L}_{\mu,p}$.

By [**4**, Lemma 4.1], $\mathscr{M}$ is a unitary transformation of $\mathscr{L}_{\mu,2}$ onto $L_2(\mathbf{R})$. Hence by Lemma 4.2 there is a function $G_\upsilon \in \mathscr{L}_{\mu,2}$ so that when $\text{Re } s = \mu$, $(\mathscr{M}G_\upsilon)(s) = g_\upsilon(s)$. By [**6**, (1.9)],

$$G_\upsilon(x) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{\mu-iR}^{\mu+iR} x^{-s} g_\upsilon(s)ds,$$

where the limit is in the topology of $\mathscr{L}_{\mu,2}$. But by Lemma 4.1, this limit exists pointwise everywhere for $x > 0$ and equals $h_\upsilon(x)$. Thus $G_\upsilon(x) = h_\upsilon(x)$ a.e., and hence if $\text{Re } s = \mu$, $\mathscr{M}h_\upsilon(s) = g_\upsilon(s)$.

LEMMA 4.4. *If* $x > 0$, $1/2 \leqq \mu < 1$ *and* $1/2 \leqq \mu + \upsilon < 3/2$, *then*

$$\mathscr{Y}_\upsilon D_x h_\upsilon = r_{\upsilon,x}.$$

Proof. Since $1/2 \leqq \mu < 3/2 - \upsilon$, $\upsilon < 1$, and hence from Lemma 4.3, $h_\upsilon \in \mathscr{L}_{\mu,2}$. Also, since $\mu < 1$, $\upsilon > 1/2 - \mu > -1/2$ and thus $\upsilon + 3/2 > 1 > \mu$. Hence since $\mu < \upsilon + 3/2$, and from the hypotheses $\mu < 3/2 - \upsilon$, it follows that $\mu < 3/2 - |\upsilon|$, and thus by [**6**, Theorem 4.2] and by [**7**, (2.16)], $\mathscr{Y}_\upsilon D_x h_\upsilon$ exists, is in $\mathscr{L}_{1-\mu,2}$, and if $\text{Re } s = 1 - \mu$,

$$\begin{aligned}(\mathscr{M}\mathscr{Y}_\upsilon D_x h_\upsilon)(s) &= -m_\upsilon(s) \cot(\pi(s + 1/2 - \upsilon)/2) \cdot (\mathscr{M}D_x h_\upsilon)(1 - s)\\ &= -m_\upsilon(s) \cot(\pi(s + 1/2 - \upsilon)/2)x^{s-1}(\mathscr{M}h_\upsilon)(1 - s)\\ &= -m_\upsilon(s) \cot(\pi(s + 1/2 - \upsilon)/2)x^{s-1}g_\upsilon(1 - s)\\ &= x^{s-1}/(\upsilon + 1/2 + s).\end{aligned}$$

But clearly since $\mu < 1 < \upsilon + 3/2$, $r_{\upsilon,x} \in \mathscr{L}_{1-\mu,2}$ and thus if $\text{Re } s = 1 - \mu$,

$$(\mathscr{M}r_{\upsilon,x})(s) = x^{-(\upsilon+3/2)} \int_0^x t^{s+\upsilon-1/2}dt = x^{s-1}/(\upsilon + 1/2 + s).$$

Hence $\mathscr{Y}_\upsilon D_x h_\upsilon = r_{\upsilon,x}$.

THEOREM 4.1. *Suppose* $f \in \mathscr{L}_{\mu,p}$ *where* $1 < p < \infty$ *and* $\gamma(p) \leqq \mu < 1$ *and* $1/2 < \mu + \upsilon < 3/2$. *Then for almost all* $x > 0$,

$$f(x) = x^{-(\upsilon+1/2)}\frac{d}{dx}x^{\upsilon+1/2} \times \int_0^\infty (xt)^{1/2}[\mathbf{H}_{\upsilon+1}(xt) - A_\upsilon(xt)^\upsilon](\mathscr{Y}_\upsilon f)(t)dt/t.$$

Proof. As in the proof of Lemma 4.4, $\upsilon < 1$ and thus by Lemma 4.3. $h_\upsilon \in \mathscr{L}_{\mu,p'}$, and $\mu < 3/2 - |\upsilon|$, so that by [**6**, Theorem 4.2] $\mathscr{Y}_\upsilon f$ exists and from [**6**, Theorem 4.3] and Lemma 4.4,

$$x^{\upsilon+1/2}\int_0^\infty (xt)^{1/2}[\mathbf{H}_{\upsilon+1}(xt) - A_\upsilon(xt)^\upsilon](\mathscr{Y}_\upsilon f)(t)dt/t$$

$$= x^{\upsilon+3/2}\int_0^\infty (D_x h_\upsilon)(t)\cdot(\mathscr{Y}_\upsilon f)(t)dt$$

$$= x^{\upsilon+3/2}\int_0^\infty (\mathscr{Y}_\upsilon D_x h_\upsilon)(t)\cdot f(t)dt = x^{\upsilon+3/2}\int_0^\infty r_{\upsilon,x}(t)\cdot f(t)dt$$

$$= \int_0^x t^{\upsilon+1/2}f(t)dt,$$

and the result follows on differentiating.

COROLLARY. *Under the hypotheses of the theorem, if* $x > 0$, *then*

$$\int_0^x t^{\upsilon+1/2}f(t)dt$$

$$= x^{\upsilon+1/2}\int_0^\infty (xt)^{1/2}[\mathbf{H}_{\upsilon+1}(xt) - A_\upsilon(xt)^\upsilon](\mathscr{Y}_\upsilon f)(t)dt/t.$$

Proof. This was proved in the course of the proof of the theorem.

THEOREM 4.2. *Suppose* $f \in \mathscr{L}_{\mu,p}$ *where either* (a) $1 < p < \infty$ *and* $\gamma(p) \leqq \mu < 3/2 - |\upsilon|$ *and* $\mu > 1/2 - \upsilon$, *or* (b) $p = 1$, $\upsilon \neq 0$, *and* $1 \leqq \mu \leqq 3/2 - |\upsilon|$, *or* (c) $p = 1$, $\upsilon = 0$ *and* $1 \leqq \mu < 3/2$. *Then for almost all* $x > 0$,

$$f(x) = x^{-(\upsilon+1/2)}\left(\frac{1}{x}\cdot\frac{d}{dx}\right)^2 x^{\upsilon+3/2}$$

$$\times \int_0^\infty (xt)^{1/2}[\mathbf{H}_{\upsilon+2}(xt) - A_{\upsilon+1}(xt)^{\upsilon+1}](\mathscr{Y}_\upsilon f)(t)dt/t^2.$$

Proof. By the Corollary to Theorem 4.1, if $f \in \mathbf{C}_0$, and υ satisfies hypotheses (a), (b) or (c), and if $u > 0$, then

$$\int_0^u t^{\upsilon+1/2}f(t)dt$$

$$= u^{\upsilon+1/2}\int_0^\infty (ut)^{1/2}[\mathbf{H}_{\upsilon+1}(ut) - A_\upsilon(ut)^\upsilon](\mathscr{Y}_\upsilon f)(t)dt/t;$$

for example, with hypotheses (a), $3/2 - |\upsilon| > 1/2 - \upsilon$ implies $\upsilon > -1/2$ and thus $1/2 - \upsilon < 1$, and $3/2 - |\upsilon| > \gamma(p)$ implies $3/2 - |\upsilon| > 1/2$ so that we can choose μ_1,

$$\max(1/2 - \upsilon, 1/2) < \mu_1 < \min(3/2 - |\upsilon|, 1),$$

and then clearly $f \in \mathscr{L}_{\mu_1,2}$.

Multiply both sides of the above equation by u and integrate from 0 to x. The left-hand side becomes

$$\int_0^x u\,du \int_0^u t^{\upsilon+1/2}f(t)dt = \int_0^x t^{\upsilon+1/2}f(t)dt \int_t^x u\,du$$
$$= \frac{1}{2}\int_0^x t^{\upsilon+1/2}(x^2 - t^2)f(t)dt,$$

while the right-hand side becomes, using [**1**, 7.5.4(48)],

$$\int_0^x u^{\upsilon+3/2}du \int_0^\infty (ut)^{1/2}[\mathbf{H}_{\upsilon+1}(ut) - A_\upsilon(ut)^\upsilon](\mathscr{Y}_\upsilon f)(t)dt/t$$
$$= \int_0^\infty t^{-1/2}(\mathscr{Y}_\upsilon f)(t)dt \int_0^x u^{\upsilon+2}[\mathbf{H}_{\upsilon+1}(tu) - A_\upsilon(tu)^\upsilon]du$$
$$= \int_0^\infty t^{-\upsilon-7/2}(\mathscr{Y}_\upsilon f)(t)dt \int_0^{xt} u^{\upsilon+2}[\mathbf{H}_{\upsilon+1}(u) - A_\upsilon u^\upsilon]du$$
$$= \int_0^\infty t^{-\upsilon-7/2}(\mathscr{Y}_\upsilon f)(t)[\,(xt)^{\upsilon+2}(\mathbf{H}_{\upsilon+2}(xt) - A_\upsilon(xt)^{\upsilon+1}/(2\upsilon+3))\,]dt$$
$$= x^{\upsilon+3/2}\int_0^\infty (xt)^{1/2}[\mathbf{H}_{\upsilon+2}(xt) - A_{\upsilon+1}(xt)^{\upsilon+1}](\mathscr{Y}_\upsilon f)(t)dt/t^2,$$

provided we justify the interchange of the order of integrations. For this we note that by [**6**, Theorem 4.2], $\mathscr{Y}_\upsilon f \in \mathscr{L}_{1-\mu_1,2}$, and thus,

$$\int_0^x u^{\upsilon+3/2}du \int_0^\infty (ut)^{1/2}|\mathbf{H}_{\upsilon+1}(ut) - A_\upsilon(ut)^\upsilon|\,|(\mathscr{Y}f)(t)|dt/t$$
$$= \int_0^x u^{\upsilon+5/2}du \int_0^\infty |t^{\mu_1}h_\upsilon(ut)|\,|t^{1-\mu_1}(\mathscr{Y}_\upsilon f)(t)|dt/t$$
$$\leqq \int_0^x u^{\upsilon+5/2}\left[\int_0^\infty |t^{\mu_1}h_\upsilon(ut)|^2 dt/t\right]^{1/2} du \cdot \|\mathscr{Y}_\upsilon f\|_{1-\mu_1,2}$$
$$= \|h_\upsilon\|_{\mu_1,2}\cdot\|\mathscr{Y}_\upsilon f\|_{1-\mu_1,2}\int_0^x u^{\upsilon-\mu_1+5/2}du < \infty,$$

by Lemma 4.3 since $1/2 < \mu_1 < 1$ and since $1/2 - \mu_1 < \upsilon < 1$. Thus by Fubini's theorem, the interchange of the order of integrations is justified.

Hence if $f \in \mathbf{C}_0$ then for all $x > 0$,

$$(4.6)\quad \frac{1}{2}\int_0^x t^{\upsilon+1/2}(x^2 - t^2)f(t)dt$$
$$= x^{\upsilon+3/2}\int_0^\infty (xt)^{1/2}[\mathbf{H}_{\upsilon+2}(xt) - A_{\upsilon+1}(xt)^{\upsilon+1}](\mathscr{Y}_\upsilon f)(t)dt/t^2.$$

But for each $x > 0$, under the hypotheses of the theorem both sides of (4.6) represent bounded linear functionals on $\mathscr{L}_{\mu,p}$. For from [**6**, Theorem 4.2], if $1 < p < \infty$, then

$$\mathscr{Y}_\upsilon \in [\mathscr{L}_{\mu,p}, \mathscr{L}_{1-\mu,p}]$$

since $1/p' \leqq \gamma(p) \leqq \mu$, and hence if $f \in \mathscr{L}_{\mu,p}$,

$$\left|\int_0^\infty (\mathbf{H}_{\upsilon+2}(xt) - A_{\upsilon+1}(xt)^{\upsilon+1})(\mathscr{Y}_\upsilon f)(t)dt/t^{3/2}\right|$$
$$= x^{1/2}\left|\int_0^\infty h_{\upsilon+1}(xt) \cdot (\mathscr{Y}_\upsilon f)(t)dt/t\right|$$
$$\leqq x^{1/2}\left[\int_0^\infty |t^{\mu-1}h_{\upsilon+1}(xt)|^{p'}dt/t\right]^{1/p'} ||\mathscr{Y}_\upsilon f||_{1-\mu,p}$$
$$\leqq K_\upsilon x^{3/2-\mu}||f||_{\mu,p} \cdot ||h_{\upsilon+1}||_{\mu-1,p'},$$

where K_υ is the norm of $\mathscr{Y}_\upsilon$. But a hypothesis of this theorem in the present case is that $\mu > 1/2 - \upsilon$, and thus $\mu - 1 > 1/2 - (\upsilon + 1)$, so that by Lemma 4.3,

$$||h_{\upsilon+1}||_{\mu-1,p'} < \infty.$$

Hence the right-hand side of (4.6) represents a bounded linear functional on $\mathscr{L}_{\mu,p}$ if $1 < p < \infty$, and the proof is similar if $p = 1$, using, from Theorem 2.2, that

$$\mathscr{Y}_\upsilon \in [\mathscr{L}_{\mu,1}, \mathscr{L}_{1-\mu,\infty}].$$

The proof that the left-hand side of (4.6) represents a bounded linear functional on $\mathscr{L}_{\mu,p}$ is almost trivial from Holder's inequality, and hence (4.6) holds under the hypotheses of the theorem, and differentiating twice, as in the proof of Theorem 3.1, we obtain the conclusion.

5. Inversion of the $\mathscr{Y}_\upsilon$ transformation on $\mathscr{L}_{1/2-\upsilon,p}$. From [**6**, Theorem 4.2], if $1 < p < \infty$ and $\gamma(p) \leqq \mu < 3/2 - |\upsilon|$, then except when $\mu = 1/2 - \upsilon$,

$$\mathscr{Y}_\upsilon(\mathscr{L}_{\mu,p}) = H_\upsilon(\mathscr{L}_{\mu,p}).$$

However, when $\mu = 1/2 - \upsilon$ the situation changes radically; for, as Theorem 5.1 and its Corollary below show,

$$\mathscr{Y}_\upsilon(\mathscr{L}_{1/2-\upsilon,p}) \neq H_\upsilon(\mathscr{L}_{1/2-\upsilon,p}).$$

Note that when $\mu = 1/2 - \upsilon$, then $p > 1$; for $\mu < \upsilon + 3/2$ gives $\upsilon > -1/2$, while $\mu \geqq \gamma(1) = 1$ gives $\upsilon \leqq -1/2$.

We use the notations $\int^{\to\infty}$ and $\int_{\to 0}$ which are explained in [**9**, Section 1.7].

THEOREM 5.1. *If $f \in \mathscr{L}_{1/2-\upsilon,p}$ where $1 < p < \infty$, $-1/2 < \upsilon \leqq 1/2 - \gamma(p)$, then*

$$\int_{\to 0}^{\to\infty} t^{\upsilon-1/2}(\mathscr{Y}_\upsilon f)(t)dt$$

converges and equals zero.

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Proof. Note that $\gamma(p) \leqq 1/2 - \upsilon < 1 \leqq 3/2 - |\upsilon|$ and thus from [**6**, Theorem 4.2], $\mathscr{Y}_\upsilon f$ exists and

$$\mathscr{Y}_\upsilon f = -(M_{-(\upsilon-1/2)}H_- M_{\upsilon-1/2}H_\upsilon)f.$$

But if $f \in \mathscr{L}_{1/2-\upsilon,p}$ then

$$H_\upsilon f \in \mathscr{L}_{\upsilon+1/2,p}$$

and thus

$$(M_{\upsilon-1/2}H_\upsilon)f \in M_{\upsilon-1/2}(\mathscr{L}_{\upsilon+1/2,p}) = \mathscr{L}_{1,p},$$

so that

$$(M_{\upsilon-1/2}\mathscr{Y}_\upsilon)f \in H_-(\mathscr{L}_{1,p}).$$

Hence, by [**3**, Corollary 4.3],

$$\int_{\to 0}^{\to\infty} t^{\upsilon-1/2}(\mathscr{Y}_\upsilon f)(t)dt = \int_{\to 0}^{\to\infty} (M_{\upsilon-1/2}\mathscr{Y}_\upsilon f)(t)dt$$

converges and equals zero.

COROLLARY. *If* $1 < p < \infty$, $-1/2 < \upsilon \leqq 1/2 - \gamma(p)$, *then*

$$\mathscr{Y}_\upsilon(\mathscr{L}_{1/2-\upsilon,p}) \neq H_\upsilon(\mathscr{L}_{1/2-\upsilon,p}).$$

Proof. Let

$$f(x) = x^{\upsilon+1/2}(x^2 + 1)^{-1}.$$

Then for all υ and for any p, $1 < p < \infty$, $f \in \mathscr{L}_{1/2-\upsilon,p}$ since

$$\int_0^\infty (x/(x^2 + 1))^p dx/x < \infty.$$

But from [**2**, 8.5(12)],

$$(H_\upsilon f)(x) = x^{1/2}K_\upsilon(x),$$

and since $K_\upsilon(x) > 0$ for $x > 0$, we cannot have

$$\int_{\to 0}^{\to\infty} x^{\upsilon-1/2}(H_\upsilon f)(x)dx = 0.$$

(Actually, from [**2**, 6.8(26)],

$$\int_0^\infty x^{\upsilon-1/2}(H_\upsilon f)(x)dx = 2^{\upsilon-1}\pi^{1/2}\Gamma(\upsilon + 1/2).)$$

The case $\upsilon = 0$, $p = 2$, of this result says, since $\mathscr{L}_{1/2,2} = L_2(0, \infty)$, that

$$\mathscr{Y}_0(L_2(0, \infty)) \neq H_0(L_2(0, \infty)).$$

Since, from [**6**, Theorem 2.3] *et seq.*, $H_0(L_2(0, \infty)) = L_2(0, \infty)$, we thus have that $\mathscr{Y}_0(L_2(0, \infty))$ is a proper subset of $L_2(0, \infty)$. In this case we can characterize $\mathscr{Y}_0(L_2(0, \infty))$, which is done in the following theorem.

THEOREM 5.2. *A function g is in* $\mathscr{Y}_0(L_2(0, \infty))$ *if and only if*
(a) $g \in L_2(0, \infty)$,
(b) $\int_1^{\to\infty} t^{-1/2}g(t)dt$ *converges, and*
(c) $k \in L_2(0, \infty)$, *where*

$$k(x) = x^{-1/2} \int_x^{\to\infty} t^{-1/2}g(t)dt.$$

Proof. Suppose $f \in L_2(0, \infty)$ and $g = \mathscr{Y}_0 f$, and let $h = M_{-1/2}g$ and $l = M_{-1/2}k$. Then from [**6**, Theorem 4.2], (a) $g \in L_2(0, \infty)$, and from Theorem 5.1, (b) is satisfied. Also, from [**6**, Theorem 4.2],

$$g = -M_{1/2}H_{-}M_{-1/2}H_0 f,$$

and thus since $H_0(L_2(0, \infty)) = L_2(0, \infty)$, $h \in \mathscr{L}_{1,2}$, so that from [**3**, Theorem 4.4] $l \in \mathscr{L}_{1,2}$, which is equivalent to (c).

Conversely, suppose g satisfies (a), (b) and (c). Then if $h = M_{-1/2}g$, h satisfies (a), (b) and (c) of [**3**, Theorem 4.4] with $p = 2$, and thus there is a function $r \in \mathscr{L}_{1,2}$ so that $h = H_{-}r$. But then since $M_{1/2}r \in L_2(0, \infty)$, and $H_0(L_2(0, \infty)) = L_2(0, \infty)$, there is a function $f \in L_2(0, \infty)$ so that $M_{1/2}r = -H_0 f$. Thus,

$$\begin{aligned} g &= M_{1/2}h = M_{1/2}H_{-}r = M_{1/2}H_{-}M_{-1/2}M_{1/2}r \\ &= -M_{1/2}H_{-}M_{-1/2}H_0 f = \mathscr{Y}_0 f, \end{aligned}$$

and hence $g \in \mathscr{Y}_0(L_2(0, \infty))$.

In our final theorem of this section and its corollary we find that either the inverse for $\mathscr{Y}_\upsilon$ given for $\mu < 1/2 - \upsilon$ in [**6**, Theorem 6.2] or the one given for $\mu > 1/2 - \upsilon$ in Theorem 4.1 works for $\mu = 1/2 - \upsilon$ provided one takes the correct limit of integration to be an arrow limit.

THEOREM 5.3. *If* $f \in \mathscr{L}_{1/2-\upsilon,p}$ *where* $1 < p < \infty$, $-1/2 < \upsilon \leqq 1/2 - \gamma(p)$, *then for almost all positive x,*

$$f(x) = x^{-(\upsilon+1/2)}\frac{d}{dx}x^{\upsilon+1/2} \int_0^{\to\infty} (xt)^{1/2}\mathbf{H}_{\upsilon+1}(xt)(\mathscr{Y}_\upsilon f)(t)dt/t.$$

Proof. As in the proof of Theorem 5.1, $\mathscr{Y}_\upsilon f$ exists. There are two cases to distinguish, $1/2 - \upsilon > \gamma(p)$ and $1/2 - \upsilon = \gamma(p)$.

If $1/2 - \upsilon > \gamma(p)$, let $f_1 = f \cdot \chi_{(1,\infty)}$ where χ_E is the characteristic function of the set E, and let $f_2 = f - f_1$. Also, choose $\epsilon > 0$ so that $1/2 - \upsilon - \epsilon > \gamma(p)$. Then with $\mu_1 = 1/2 - \upsilon - \epsilon$, and $\mu_2 = 1/2 - \upsilon + \epsilon$, $f_i \in \mathscr{L}_{\mu_i,p}$, and if $g_i = \mathscr{Y}_\upsilon f_i$, $i = 1, 2$, from [**6**, Theorem 6.2],

$$f_1(x) = x^{-(\upsilon+1/2)}\frac{d}{dx}x^{\upsilon+1/2} \int_0^{\infty} (xt)^{1/2}\mathbf{H}_{\upsilon+1}(xt) \cdot g_1(t)dt/t,$$

and from Theorem 4.1,

$$f_2(x) = x^{-(v+1/2)}\frac{d}{dx}x^{v+1/2}\int_0^\infty (xt)^{1/2}[\mathbf{H}_{v+1}(xt) - A_v(xt)^v]g_2(t)dt/t.$$

But from Theorem 5.1,

$$\int_{\to 0}^{\to\infty} t^{v-1/2}g_2(t)dt = 0,$$

and thus adding we obtain, since $g_1 + g_2 = \mathscr{Y}_v f$,

$$f(x) = x^{-(v+1/2)}\frac{d}{dx}x^{v+1/2}\int_0^{\to\infty} (xt)^{1/2}\mathbf{H}_{v+1}(xt)\cdot(\mathscr{Y}_v f)(t)dt/t,$$

the arrow being unnecessary at 0, as is easy to see.

Suppose $1/2 - v = \gamma(p)$. Let x be positive and $a > 1$, and let

$$\Phi_x(t) = t^{-1/2}(\mathbf{H}_{v+1}(xt) - A_v(xt)^v\chi_{(a,\infty)}(t)).$$

By [**1**, 7.5.4(63) & 7.13.1(4)], as $t \to \infty$,

$$\mathbf{H}_{v+1}(xt) = A_v(xt)^v + O(t^{-1/2}),$$

so that

$$\Phi_x(t) = O(t^{-1}).$$

Also, as $t \to 0$, from [**1**, 7.5.4(55)],

$$\Phi_x(t) = O(t^{v+3/2}).$$

Thus $\Phi_x \in \mathscr{L}_{1/2-v,p'}$.

Let f_i and g_i be as before, $i = 1, 2$. Then from [**6**, Theorem 4.3],

$$\begin{aligned}(5.1)\quad \int_0^\infty g_1(t)\cdot\Phi_x(t)dt &= \int_0^\infty (\mathscr{Y}_v f_1)(t)\cdot\Phi_x(t)dt \\ &= \int_0^\infty f_1(t)\cdot(\mathscr{Y}_v\Phi_x)(t)dt \\ &= \int_1^\infty f(t)\cdot(\mathscr{Y}_v\Phi_x)(t)dt.\end{aligned}$$

Now from Theorem 5.1,

$$\int_a^{\to\infty} t^{v-1/2}g_1(t)dt$$

converges, and thus we can write the left-hand side of (5.1) as

$$\int_0^{\to\infty} t^{-1/2}\mathbf{H}_{v+1}(xt)\cdot g_1(t)dt - A_v x^v\int_a^{\to\infty} t^{v-1/2}g_1(t)dt.$$

Also from [**2**, 9.4(37)],

$$\int_0^{\to\infty} (tu)^{1/2}Y_v(tu)\cdot u^{-1/2}\mathbf{H}_{v+1}(xu)du = x^{-v-1}t^{v+1/2}\chi_{(0,x)}(t).$$

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Further, it is clear from [**1**, 7.13.1(4)] that, since $\upsilon \leqq 0$, for all $t > 0$,

$$\int_a^{\rightarrow\infty} u^\upsilon Y_\upsilon(tu)du$$

converges. Hence

$$\int_0^{\rightarrow\infty} (tu)^{1/2}Y_\upsilon(tu) \cdot \Phi_x(u)du$$

converges for all $t > 0$ and equals

$$x^{-\upsilon-1}t^{\upsilon+1/2}\chi_{(0,x)}(t) - A_\upsilon x^\upsilon t^{-\upsilon-1/2} \int_{at}^{\rightarrow\infty} u^\upsilon Y_\upsilon(u)du.$$

But it is obvious that if $h \in \mathscr{L}_{1/2-\upsilon,p'}$, and

$$\int_0^{\rightarrow\infty} (tu)^{1/2}Y_\upsilon(tu) \cdot h(u)du$$

converges for almost all $t > 0$, then the value of this integral is $(\mathscr{Y}_\upsilon h)(t)$ a.e. Hence, for almost all $t > 0$,

$$(\mathscr{Y}_\upsilon \Phi_x)(t) = x^{-\upsilon-1}t^{\upsilon+1/2}\chi_{(0,x)}(t) - A_\upsilon x^\upsilon t^{-\upsilon-1/2} \int_{at}^{\rightarrow\infty} u^\upsilon Y_\upsilon(u)du,$$

and substituting this into (5.1), we obtain

$$(5.2) \quad \int_0^{\rightarrow\infty} t^{-1/2}\mathbf{H}_{\upsilon+1}(xt) \cdot g_1(t)dt - A_\upsilon x^\upsilon \int_a^{\rightarrow\infty} t^{\upsilon-1/2} g_1(t)dt$$
$$= x^{-\upsilon-1} \int_0^x t^{\upsilon+1/2}f_1(t)dt$$
$$- A_\upsilon x^\upsilon \int_1^\infty t^{-\upsilon-1/2}f(t)dt \int_{at}^{\rightarrow\infty} u^\upsilon Y_\upsilon(u)du.$$

Now from [**1**, 7.2.8(52)],

$$\frac{d}{dz}z^{\upsilon+1}Y_{\upsilon+1}(z) = z^{\upsilon+1}Y_\upsilon(z).$$

Hence, integrating by parts,

$$\int_{at}^{\rightarrow\infty} u^\upsilon Y_\upsilon(u)du = \int_{at}^{\rightarrow\infty} u^{-1}u^{\upsilon+1}Y_\upsilon(u)du$$
$$= -(at)^\upsilon Y_{\upsilon+1}(at) + \int_{at}^\infty u^{\upsilon-1}Y_{\upsilon+1}(u)du,$$

so that the last term on the right of (5.2) is

$$A_\upsilon x^\upsilon\Big[a^\upsilon \int_1^\infty t^{-\upsilon-1/2}f(t) \cdot t^\upsilon Y_{\upsilon+1}(at)dt$$
$$- \int_1^\infty t^{-\upsilon-1/2}f(t)dt \int_{at}^\infty u^{\upsilon-1}Y_{\upsilon+1}(u)du\Big].$$

Now, by Holder's inequality,

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$$\left|a^{\upsilon}\int_1^{\infty} t^{1/2}f(t)Y_{\upsilon+1}(at)dt/t\right|$$
$$\leqq a^{\upsilon}\left[\int_1^{\infty}|t^{1/2-\upsilon}f(t)|^p dt/t\right]^{1/p}\cdot\left[\int_1^{\infty}|t^{\upsilon}Y_{\upsilon+1}(at)|^{p'}dt/t\right]^{1/p'}$$
$$\leqq ||f||_{1/2-\upsilon,p}\cdot\left[\int_a^{\infty}|t^{\upsilon}Y_{\upsilon+1}(t)|^{p'}dt/t\right]^{1/p'}\to 0$$

as $a \to \infty$ since $\upsilon < 1/2$ and, from [**1**, 7.13.1(4)],

$$t^{\upsilon}Y_{\upsilon+1}(t) = O(t^{\upsilon-1/2}) \quad \text{as } t \to \infty.$$

Also, since $at > 1$, if $u > at$, then $|Y_{\upsilon+1}(u)| \leqq C/u^{1/2}$ for some constant C and thus,

$$\left|\int_1^{\infty} t^{-\upsilon-1/2}f(t)dt\int_{at}^{\infty}u^{\upsilon-1}Y_{\upsilon+1}(u)du\right|$$
$$\leqq C\int_1^{\infty}t^{-\upsilon-1/2}|f(t)|dt\int_{at}^{\infty}u^{\upsilon-3/2}du$$
$$= C'a^{\upsilon-1/2}\int_1^{\infty}|f(t)|dt/t,$$

where $C' = C/(1/2 - \upsilon)$. This expression tends to zero as $a \to \infty$ since $\upsilon < 1/2$ and since from Holder's inequality,

$$\int_1^{\infty}|f(t)|dt/t$$
$$\leqq \left[\int_1^{\infty}|t^{1/2-\upsilon}f(t)|^p dt/t\right]^{1/p}\cdot\left[\int_1^{\infty}t^{p'(\upsilon-1/2)-1}dt\right]^{1/p'} < \infty.$$

Hence letting $a \to \infty$ in (5.2) we obtain,

$$\int_0^{\to\infty} t^{-1/2}\mathbf{H}_{\upsilon+1}(xt)\cdot g_1(t)dt = x^{-\upsilon-1}\int_0^x t^{\upsilon+1/2}f_1(t)dt$$

or

$$(5.3)\quad \int_0^x t^{\upsilon+1/2}f_1(t)dt = x^{\upsilon+1/2}\int_0^{\to\infty}(xt)^{1/2}\mathbf{H}_{\upsilon+1}(xt)g_1(t)dt/t.$$

Since $f \in \mathcal{L}_{1/2-\upsilon,p}$ and $f_2 = 0$ if $t > 1$, $f_2 \in \mathcal{L}_{\mu,p}$ for every $\mu > 1/2 - \upsilon$. Since $\gamma(p) = 1/2 - \upsilon < 1$, we can choose ϵ, $0 < \epsilon < 1$, so that $1/2 - \upsilon + \epsilon < 1$, and it then follows from the corollary to Theorem 4.1, with $\mu = 1/2 - \upsilon + \epsilon$ that

$$(5.4)\quad \int_0^x t^{\upsilon+1/2}f_2(t)dt$$
$$= x^{\upsilon+1/2}\int_0^{\infty}(xt)^{1/2}[\mathbf{H}_{\upsilon+1}(xt) - A_{\upsilon}(xt)^{\upsilon}]g_2(t)dt/t.$$

But by Theorem 5.1, since $g_2 = \mathcal{Y}_{\upsilon}f_2$,

$$\int_{\to 0}^{\to\infty} t^{\upsilon-1/2}g_2(t)dt = 0,$$

and thus the right-hand side of (5.4) reduces to

$$x^{\upsilon+1/2}\int_0^{\to\infty}(xt)^{1/2}\mathbf{H}_{\upsilon+1}(xt)\cdot g_2(t)dt/t,$$

the arrow at 0 being unnecessary, as is easily seen. Adding this to (5.3) we find that

$$\int_0^{x,} t^{\upsilon+1/2}f(t)dt = x^{\upsilon+1/2}\int_0^{\to\infty}(xt)^{1/2}\mathbf{H}_{\upsilon+1}(xt)(\mathscr{Y}_\upsilon f)(t)dt/t,$$

and on differentiating, we arrive at the conclusion.

COROLLARY. *If* $f \in \mathscr{L}_{1/2-\upsilon,p}$ *where* $-1/2 < \upsilon \leqq 1/2 - \gamma(p)$ *then for almost all* x,

$$f(x) = x^{-(\upsilon+1/2)}\frac{d}{dx}x^{\upsilon+1/2}$$
$$\times\int_{\to 0}^{\infty}(xt)^{1/2}[\mathbf{H}_{\upsilon+1}(xt) - A_\upsilon(xt)^\upsilon](\mathscr{Y}_\upsilon f)(t)dt/t.$$

Proof. By Theorem 5.1,

$$\int_{\to 0}^{\infty}(xt)^{1/2}[\mathbf{H}_{\upsilon+1}(xt) - A_\upsilon(xt)^\upsilon](\mathscr{Y}_\upsilon f)(t)dt/t$$
$$= \int_0^{\to\infty}(xt)^{1/2}\mathbf{H}_{\upsilon+1}(xt)\cdot(\mathscr{Y}_\upsilon f)(t)dt/t,$$

an arrow at zero again being unnecessary.

6. Inversion of the $\mathscr{H}_\upsilon$ transformation. In [**6**, Theorem 6.3] we found an inverse for the $\mathscr{H}_\upsilon$ transformation on $\mathscr{L}_{\mu,p}$ but only valid under the conditions $\mu \geqq \gamma(p)$ and $\upsilon + 1/2 < \mu < \min(1, \upsilon + 3/2)$. In Theorem 6.1 below we shall find a formula for the inverse of $\mathscr{H}_\upsilon$ for nearly the full range of boundedness of the transformation. However, as in Section four, we first need some definitions and some lemmas.

The proofs of the various lemmas and of Theorem 6.1 are very like the proofs of the corresponding lemmas and theorems of Section four, so we will largely omit the proofs, just calling attention to any special points that arise.

Definition. 6.1. For $s \in \mathbf{C}$, let

$$(6.1)\quad \Psi_{\upsilon,n}(s) = 2^{s-3/2}(\upsilon - s + 3/2)\cot(\pi(\upsilon - s + 3/2)/2)$$
$$\times\,\Gamma((\upsilon + s + 1/2)/2)/\Gamma((7/2 - s + \upsilon + 2n)/2).$$

For $x > 0$ let

$$(6.2)\quad \Phi_{\upsilon,n}(x)$$
$$= (2/x)^n\left[Y_{\upsilon+n}(x) + \pi^{-1}\sum_{k=0}^{n-1}(x/2)^{2k-\upsilon-n}\Gamma(\upsilon + n - k)/k!\right]$$

(6.3) $\lambda_{v,n}(x) = x^{1/2}[\Phi_{v,n}(x) - n\Phi_{v,n+1}(x)]$.

It should be noticed that (6.2) doesn't make sense when $v = -1$ since the last term in the finite sum in (6.2) contains $\Gamma(v + 1)$. However $\lambda_{v,n}$ does make sense since the $\Gamma(v + 1)$'s in the two terms cancel each other.

LEMMA 6.1. *Suppose that* $\max(v + 1/2, -(v + 1/2)) < \mu < v + 5/2$, *and* $1/2 \leqq \mu < n + 1/2$. *Then*

$$\lim_{R\to\infty} \frac{1}{2\pi i}\int_{\mu-iR}^{\mu+iR} x^{-s}\Psi_{v,n}(s)ds = \lambda_{v,n}(s).$$

LEMMA 6.2. *Suppose that* $\max(v + 1/2, -(v + 1/2)) < \mu < v + 5/2$ *and that* $1/2 \leqq \mu < n$. *Then as a function of* t, $\lambda_{v,n}(\mu + it) \in L_2(\mathbf{R})$.

LEMMA 6.3. *Suppose that* $1/2 \leqq \mu < n$. *Then*

(a) *if* $\max(v - 1/2, -(v + 1/2)) < \mu < v + 7/2$, $\lambda_{v,n} \in \mathscr{L}_{\mu,p}$ *for* $1 \leqq p < \infty$; *and*

(b) *if* $\max(v + 1/2, -(v + 1/2)) < \mu < v + 5/2$, *then for* $\mathrm{Re}\, s = \mu$,

$$(\mathscr{M}\lambda_{v,n})(s) = \Psi_{v,n}(s).$$

Proof. The proof is essentially that of Lemma 4.3, but for the first part note that

$$\lambda_{v,n}(x) = O(x^{-\rho}) \quad \text{as } x \to \infty,$$

where $\rho = \min(n, v + 7/2)$, since the terms of highest order arising from the finite sums in $\Phi_{v,n}$ and $\Phi_{v,n+1}$ cancel each other.

LEMMA 6.4. *If* $x > 0$, $\max(v + 1/2, -(v + 1/2)) < \mu < v + 5/2$ *and* $1/2 \leqq \mu < n$, *then*

$$(\mathscr{H}_v D_x \lambda_{v,n})(t) = (2/\Gamma(n))x^{-v-2n-3/2}t^{v+5/2}(x^2 - t^2)^{n-1}, \quad 0 < t < x,$$
$$= 0, \quad t \geqq x.$$

THEOREM 6.1. *Suppose that* $f \in \mathscr{L}_{\mu,p}$ *where* $1 \leqq p < \infty$,

$\max(v + 1/2, -(v + 1/2)) < \mu < v + 5/2$ *and* $\gamma(p) \leqq \mu < n$.

Then for almost all $x > 0$,

$$f(x) = 2^{-n}x^{-v-5/2}\left[\frac{1}{x}\cdot\frac{d}{dx}\right]^n x^{v+2n+3/2}$$
$$\times \int_0^\infty (xt)^{1/2}(\Phi_{v,n}(xt) - n\Phi_{v,n+1}(xt))(\mathscr{H}_v f)(t)dt.$$

We note that even for $n = 1$ this result is an improvement over [**6**, Theorem 6.3], which had the hypothesis $\mu < v + 3/2$, whereas Theorem 6.1 replaces this with $\mu < v + 5/2$, and the implicit hypothesis of [**6**, Theorem 6.3] that $v > -1$ assures that

$$-(\upsilon + 1/2) < 1/2 \leqq \mu.$$

However, if $\mu < \upsilon + 3/2$, one can obtain a rather simpler version of the theorem in which only one $\Phi_{\upsilon,n}$ appears. One can also choose n so large that the inversion given by Theorem 6.1 is valid for all μ for which $\mathscr{H}_\upsilon$ is bounded, but as in the case of H_υ, this will often be unnecessarily large.

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