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A CHARACTERIZATION OF PROJECTIVE METRIC SPACES

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ABSTRACT. A projective metric space is a pappian projective space together with a quadric and a certain equivalence relation on the pairs of those points which do not belong to the quadric. This equivalence relation is defined by means of the corresponding quadratic form and satisfies a condition which is a projective version of Miquel's theorem. We characterize the projective metric spaces of dimension at least two over fields of order at least 13.

§1. **Introduction**. Let V be a vector space over a commutative field K, and let Q: $V \rightarrow K$ be a quadratic form with the corresponding bilinear form f_Q. The pair (V, Q) is called a *metric vector space*. Let $\Pi(V)$ denote the projective space corresponding to V, the points of which are the one-dimensional subspaces of V, and let $\mathcal{P}_Q$ denote the set of those points of $\Pi(V)$ for which the quadratic form Q is not zero. On $\mathcal{P}_Q \times \mathcal{P}_Q$ we define an equivalence relation $\equiv_Q$ by $(A, B) \equiv_Q (C, D)$: $\Leftrightarrow$ There exist vectors $a, b, c, d \in V$ satisfying $A = Ka, \ldots, D = Kd$, such that one of the following statements holds:

(i) $a = b$ and $c = d$;

(ii) $a = c$ and $b = d$;

(iii) $a = d$ and $c = Q(a)b - f_Q(a, b)a$;

(iv) $b = a + d$ and $c = Q(d)a + Q(a)d$.

By [5, Lemma 3.1] $\equiv_Q$ is the linear congruence relation defined by Schröder [7]. If (V, Q) is regular, then $(A, B) \equiv_Q (C, D)$ iff $\sigma_A \circ \sigma_B = \sigma_C \circ \sigma_D$, where σ_X is the reflection in the hyperplane perpendicular to X (see [5, Lemma 1.1]). Therefore $(A, B) \equiv_Q (C, D)$ implies that A, B, C, D are on a common line and that the angle from A to B equals the angle from C to D. This justifies the name "linear congruence relation". The pair $(\Pi(V), \equiv_Q)$ is called a *projective metric space*. In [8] Schröder characterizes the projective metric spaces, starting with a subset $\mathcal{P}$ of the point set of a projective space and an equivalence relation on $\mathcal{P} \times \mathcal{P}$. In the present paper, we start with a set $\mathcal{P}$ and

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an equivalence relation $\equiv$ on $\mathcal{P} \times \mathcal{P}$. It turns out that the properties of the equivalence relation used by Schröder can also be used to embed $\mathcal{P}$ into a projective space Π such that $(\Pi, \equiv)$ is a projective metric space.

§2. **Result**. Let $\mathcal{P}$ be a set and $\mathcal{L}$ a set of subsets of $\mathcal{P}$ satisfying $|l| \geqq 2$ for every $l \in \mathcal{L}$. The elements of $\mathcal{P}$ are called *points*, those of $\mathcal{L}$ are called *lines*. The pair $(\mathcal{P}, \mathcal{L})$ is a *linear space*, iff for every pair of distinct points $A, B \in \mathcal{P}$ there exists one and only one line $l \in \mathcal{L}$ with $A, B, \in l$. A subset $\mathcal{T}$ of $\mathcal{P}$ is a *subspace*, iff it contains with every pair of distinct points the line through these points. If $\mathcal{M}$ is a set of points, then there exists a smallest subspace containing $\mathcal{M}$, called the *hull* of $\mathcal{M}$. A *plane* is the hull of three noncollinear points. For any subset $\mathcal{M}$ of $\mathcal{P}$ we define $\mathcal{L}_{\mathcal{M}}: = \{l \cap \mathcal{M}: l \in \mathcal{L}$ and $|l \cap \mathcal{M}| \geqq 2\}$. Then $(\mathcal{M}, \mathcal{L}_{\mathcal{M}})$ is a linear space, which is *embedded* into $(\mathcal{P}, \mathcal{L})$. We say that $(\mathcal{M}, \mathcal{L}_{\mathcal{M}})$ is *locally completely embedded* into $(\mathcal{P}, \mathcal{L})$, iff $|l \cap \mathcal{M}| \neq l$ for every line $l \in \mathcal{L}$ (see [1, p. 346]).

PROPOSITION 1: *Let* Q *be a quadratic form on a vector space. We write* $\mathcal{P}$ *and* $\equiv$ *for* $\mathcal{P}_{\mathrm{Q}}$ *and* $\equiv_{\mathrm{Q}}$. *Then the following statements hold*:

(1) $(A, A) \equiv (B, B)$ *for all* $A, B \in \mathcal{P}$.

(2) *Given* $A, B, C \in \mathcal{P}$, *there is at most one* $X \in \mathcal{P}$, *denoted by* ABC, *such that* $(A, B) \equiv (X, C)$ holds. *If* ABC *exists, then so does* $\pi(A)\pi(B)\pi(C)$ *for every permutation* π *of* $\{A, B, C\}$.

(3) *Let* A, B, C, D *be elements of* $\mathcal{P}$ *such that* $A \neq B$. *If* ABC *and* ABD *exist, then so does* ACD.

Because of (1), (2), (3) $R: = \{(A, B, C): A, B, C \in \mathcal{P}$ *and* ABC *exists*$\}$ *is a ternary equivalence relation* (*see* [1, *pp*. 64-65]). *Therefore we get a linear space with point set* $\mathcal{P}$, *if for every pair of distinct points* $A, B \in \mathcal{P}$ *we define the line* $A + B: = \{X \in \mathcal{P}: ABX$ *exists*$\}$ *through* A *and* B. *We denote this linear space by* $L(\mathcal{P}, \equiv)$ *and the set of all its lines by* $\mathcal{L}$. *For the linear space* $L(\mathcal{P}, \equiv)$ *the following statements hold*:

(4) *Let* $a, b, c \in \mathcal{L}$ *be pairwise intersecting lines contained in a plane* ϵ. *Then every line contained in* ϵ *meets at least one of them.*

(5) *Let* ϵ *be a plane containing points* A, B, C, D, *no three of them collinear, such that* $C(B(AXA)B)C = DXD$ *for every* $X \in \epsilon$. *Then for every line* $l \in \mathcal{L}_{\epsilon}$ *and every point* $X \in \epsilon$ *there is at most one line* $m \in \mathcal{L}_{\epsilon}$ *through* X, *which does not meet* l. (*This means that* $(\epsilon, \mathcal{L}_{\epsilon})$ *is a semi-affine plane as defined by Dembowski* [3].)

(6) (*Hexagram condition of* [8, *Theorem* 7]) *Let* $A, \ldots, G$ *be elements of* $\mathcal{P}$. *If each of the sets* $\{A, B, C\}$, $\{C, D, E\}$, $\{E, F, G\}$, $\{B, D, F\}$, $\{A, BDF, G\}$ *is collinear, then the set* $\{ABC, CDE, EFG\}$ *is also collinear, and* $(ABC)(CDE)(EFG) = A(BDF)G$.

PROOF: The validity of (1), (2), (3), (4) and (6) follows from [8, Theorem 7]. We show that (5) is true. Let ϵ be a plane containing points A, B, C, D, no three of them collinear, such that $C(B(AXA)B)C = DXD$ for every $X \in \epsilon$. There is a three-dimensional metric vector space (V, q) corresponding to ϵ. If the underlying field K has

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only two elements, then (5) is obviously true. Therefore we may assume $|K| \geqq 3$. We choose vectors $a, b, c, d \in V$ such that $A = Ka, \ldots, D = Kd$ and define a map σ_a: $V \to V; x \mapsto x - q(a)^{-1} f_q(a, x) a$. Now σ_a and $-\sigma_a$ are the only isometries of V to induce the map $\tilde{A}$: $\epsilon \to \epsilon; X \mapsto AXA$ (see [6, Lemma 3.4]). Therefore $\sigma_a \circ \sigma_b \circ \sigma_c = \sigma_d$. Because a, b, c are linearly independent, this implies $\dim(\text{Rad } V) \geqq 2$ (see [6, Proposition 3.5]). In case $\dim(\text{Rad } V) = 2$ we have char $K \neq 2$, and hence $q(x) = 0$ is equivalent to $f_q(x, x) = 0$ which in turn is equivalent to $x \in \text{Rad } V$. If $\dim(\text{Rad } V) = 3$, then $q(x + y) = q(x) + q(y)$ for all $x, y \in V$. In each case the set $\{x \in V: q(x) = 0\}$ is a subspace of V. The assertion of (5) follows. □

The hexagram condition was stated first by Schröder [8]. The following example illustrates its significance for elementary geometry. Let V be the three-dimensional real vector space and Q the square of the Euclidean length. Then the affine geometry corresponding to (V, Q) is the three-dimensional Euclidean space. Let eight points of V be attached to the vertices of a cube in such a way that for five of the six faces of the cube the vertices correspond to points on a circle. The vertices of any quadrangle lie on a circle iff opposite inner angles add up to 180°. Consequently four points A, B, C, D, no three of them collinear, lie on a circle iff $(\mathbb{R}(A-B), \mathbb{R}(C-B)) \equiv_Q (\mathbb{R}(A-D), \mathbb{R}(C-D))$ in the projective metric space corresponding to (V, Q). Therefore the hexagram condition implies that the vertices of the sixth face of the cube correspond to points on a circle too, which is the assertion of the theorem of Miquel (see [2, p. 131]). Schröder [8] calls the hexagram condition a projective version of the theorem of Miquel. By [2, pp. 236–238] a circle plane is projectively embeddable if the theorem of Miquel holds. In view of this fact, Theorem 2 confirms Schröder's interpretation.

THEOREM 2: *Let $\mathcal{P}$ be a set and $\equiv$ an equivalence relation on $\mathcal{P} \times \mathcal{P}$ satisfying conditions* (1)–(6) *stated in Proposition* 1. *Let the linear space $L(\mathcal{P}, \equiv)$ contain at least two lines, on every line at least three points and on one line at least* 13 *points.*

Then $L(\mathcal{P}, \equiv)$ is locally completely embeddable into a projective space Π, and $(\Pi, \equiv)$ is a projective metric space.

§3. **Towards the Proof of Theorem 2**. Throughout this paragraph, $\mathcal{P}$ is a set and $\equiv$ is an equivalence relation on $\mathcal{P} \times \mathcal{P}$ satisfying conditions (1)–(6) stated in Proposition 1. ϵ is a plane of $L(\mathcal{P}, \equiv)$ containing at least three points on every line $l \in \mathcal{L}_\epsilon$. Our aim is the proof of the following proposition.

PROPOSITION 3: *The linear space $(\epsilon, \mathcal{L}_\epsilon)$ is locally completely embeddable into a projective plane.*

For every $A \in \epsilon$ we define a map $\tilde{A}$: $\epsilon \to \epsilon; X \mapsto AXA$. By [8, (22) and (24)] $\tilde{A}$ is a collineation satisfying $\tilde{A} \circ \tilde{A} = \text{id}_\epsilon$. For collinear points $A, B, C \in \epsilon$ we have $\tilde{A} \circ \tilde{B} \circ \tilde{C} = \widetilde{ABC}$ by [8, (23)]. We remark that although Schröder proves (22)–(24) in [8] under stronger assumptions, his proof remains valid without changes in our more general situation. If for $A, B, C \in \epsilon$ there is a point $D \in \epsilon$ such that $\tilde{A} \circ \tilde{B} \circ \tilde{C} = \tilde{D}$ only if A, B, C are collinear, then the assertion of Proposition 3 follows from [6, Main Theorem 6.31]. Hence we may assume that ϵ contains non-collinear points A_0, B_0, C_0 and a point D_0 such that $\tilde{A}_0 \circ \tilde{B}_0 \circ \tilde{C}_0 = \tilde{D}_0$.

LEMMA 4: *Let $(\epsilon, \mathscr{L}_\epsilon)$ be a semi-affine plane, g and h disjoint elements of $\mathscr{L}_\epsilon$, and X a point of ϵ not on g. Then there is a line $l \in \mathscr{L}_\epsilon$ such that $X \in l$ and $l \cap g = \emptyset$.*

PROOF: We may assume $X \notin h$, as otherwise the assertion is obvious. We choose two points H_1 and H_2 on h. The line $X + H_1$ meets g in a point G_1. The point $Z := H_1 G_1 X$ is collinear with X and H_1 and distinct from both. The line $Z + H_2$ meets g in a point G_2. The point $Y := G_2 H_2 Z$ is collinear with Z and G_2. Also $(X + H_1) \cap (Z + G_2) = \{Z\}$ and $X \neq Z$ imply $X \neq Y$. We show that the line $l := X + Y$ is disjoint to g. Assume there is a point $S \in l \cap g$. Then each of the sets $\{Z, X, G_1\}$, $\{G_1, S, G_2\}$, $\{G_2, Y, Z\}$, $\{X, S, Y\}$, $\{Z, XSY, Z\}$ is collinear, and the hexagram condition implies that the set $\{ZXG_1, G_1SG_2, G_2YZ\}$ is also collinear. Because $ZXG_1 = H_1$ and $G_2YZ = H_2$ we have $G_1SG_2 \in h$, a contradiction to $G_1SG_2 \in g$ and $g \cap h = \emptyset$. □

LEMMA 5: *If ϵ does not contain distinct points U and V such that $\tilde{U} = \tilde{V}$, then we have*:

(*i*) *Let $A, \ldots, D$ be elements of ϵ satisfying $\tilde{A} \circ \tilde{B} = \tilde{D} \circ \tilde{C}$. If A, B, C are non-collinear, then $(A + B) \cap (D + C) = \emptyset = (A + D) \cap (B + C)$.*

(*ii*) *The linear space $(\epsilon, \mathscr{L}_\epsilon)$ is a semi-affine plane.*

(*iii*) *For every line $l \in \mathscr{L}_\epsilon$ there is a line $m \in \mathscr{L}_\epsilon$ such that $l \cap m = \emptyset$.*

PROOF: (*i*) Assume there is a point $X \in (A + B) \cap (D + C)$. Then we have $\widetilde{ABX} = \tilde{A} \circ \tilde{B} \circ \tilde{X} = \tilde{D} \circ \tilde{C} \circ \tilde{X} = \widetilde{DCX}$ which implies $ABX = DCX$ and therefore $X, ABX \in (A + B) \cap (D + C)$. This contradicts $A + B \neq D + C$, for $X = ABX$ would imply $A = B$. Hence $(A + B) \cap (D + C) = \emptyset$ is true. Similarly $(A + D) \cap (B + C) = \emptyset$ follows from $\tilde{A} \circ \tilde{D} = \tilde{B} \circ \tilde{C}$.

(*ii*) By (*i*) no three of the points A_0, B_0, C_0, D_0 are collinear. Therefore (*ii*) follows from condition (5).

(*iii*) We may assume that l meets $A_0 + B_0$. Then by (*i*) and (*ii*) l meets $D_0 + C_0$ too. We call the points of intersection A_1 and D_1. There are points $B_1 \in A_0 + B_0$ and $C_1 \in D_0 + C_0$ such that $\tilde{A}_1 \circ \tilde{B}_1 = \tilde{A}_0 \circ \tilde{B}_0 = \tilde{D}_0 \circ \tilde{C}_0 = \tilde{D}_1 \circ \tilde{C}_1$. Because A_1, B_1, C_1 are noncollinear, (*i*) implies $(A_1 + D_1) \cap (B_1 + C_1) = \emptyset$. Together with $l = A_1 + D_1$ this proves (*iii*). □

LEMMA 6: *If for a point $A \in \epsilon$ the collineation $\tilde{A}$ fixes three noncollinear points $X, Y, Z \in \epsilon - \{A\}$, then $\tilde{A}$ is the identity map on ϵ.*

PROOF: Because $\tilde{A}$ fixes every line through A, the following is obvious: If $\tilde{A}$ fixes points $R, S \in \epsilon$ not collinear with A, then $\tilde{A}$ fixes the line $R + S$ pointwise. We will frequently make use of this fact. Because every line of $\mathscr{L}_\epsilon$ contains at least three points, we may assume $A \notin X + Y, Y + Z, Z + X$. We choose points $U \in X + Y$, $V \in Y +$

Z, $W \in Z + X$ distinct from X, Y, Z and may assume $A \notin U + V$, $U + W$. For every point $P \in \epsilon - \{Z\}$ the line $P + Z$ meets at least one of the lines $U + V$, $V + W$, $X + Y$ in a point distinct from Z. Hence $\tilde{A}$ fixes every point of $P + Z$ if P and Z are not collinear with A. Therefore $\tilde{A}$ fixes all points of ϵ, except perhaps the points on the line $A + Z$. But then $\tilde{A}$ must be the identity map on ϵ. □

LEMMA 7: *If ϵ contains distinct points U and V such that $\tilde{U} = \tilde{V}$, then we have:*
(i) $\tilde{A}$ is the identity map on ϵ for every $A \in \epsilon$.
(ii) $(\epsilon, \mathcal{L}_\epsilon)$ is a semi-affine plane.
(iii) The diagonals of any parallelogram in $(\epsilon, \mathcal{L}_\epsilon)$ do not intersect.

PROOF: (i) Let X be a point of ϵ not on $U + V$. The map $\tilde{U}$ $(= \tilde{V})$ fixes the lines $U + X$ and $V + X$, and hence fixes X. Therefore $\tilde{U}$ is the identity map on ϵ. Let A be a point of ϵ not on $U + V$. The map $\tilde{A}$ fixes U and V. We choose a point $W \in A + U$ distinct from A and U. Because $AWA = AW(UAU) = (AWU)AU = (UWA)AU = UW(AAU) = UWU = W$, $\tilde{A}$ fixes W too. Hence $\tilde{A}$ is the identity map by Lemma 6. If B is a point on $U + V$ distinct from U and V, then $\tilde{B}$ fixes the noncollinear points U, V, A and hence is the identity map by Lemma 6.

(ii) ϵ contains four points $A_1, \ldots, A_4$ such that no three of them are collinear. By (i) we have $\tilde{A}_1 \circ \tilde{A}_2 \circ \tilde{A}_3 = \tilde{A}_4$. Hence (ii) follows from condition (5).

(iii) Let A, B, C, D be distinct points of ϵ such that $(A + B) \cap (C + D) = \emptyset = (A + D) \cap (B + C)$. We show that $A + C$ and $B + D$ do not intersect. Assume there is a point $X \in (A + C) \cap (B + D)$. Then $(XAC + XBD) \cap (A + D) = \emptyset$; for if there is a point $Y \in (XAC + XBD) \cap (A + D)$, the hexagram condition implies $C((XAC)Y(XBD))B = (XA(XAC))((XAC)Y(XBD))((XBD)DX) = X(AYD)X = AYD$, and hence $AYD \in (A + D) \cap (B + C)$. Similarly we get $(XAC + XBD) \cap (A + B) = \emptyset$. But now there are two lines (namely $A + D$ and $A + B$) containing A which do not meet the line $XAC + XBD$. By (ii) this is not possible. □

If ϵ does not contain distinct points U and V such that $\tilde{U} = \tilde{V}$, then we deduce from Lemma 4 and Lemma 5 that $(\epsilon, \mathcal{L}_\epsilon)$ is an affine plane. Hence Proposition 3 is true in this case. If ϵ does contain such points, define an ideal point to be a set of pairwise nonintersecting lines which fill out ϵ. An ideal point is to lie on each of its lines. If there exist at least two ideal points, then we define the set of all ideal points to be a new line. We deduce from Lemma 4 and Lemma 7 that in this way we get a projective plane. This concludes the proof of Proposition 3.

PROPOSITION 8: *Let $(\mathcal{P}, \mathcal{L})$ be a linear space containing at least two lines, on every line at least three points and on one line at least 13 points. If every plane ϵ of $(\mathcal{P}, \mathcal{L})$ is embeddable into a projective plane $\Pi(\epsilon)$ such that $(\Pi(\epsilon), \equiv_\epsilon)$ is a projective metric plane for a suitable equivalence relation $\equiv_\epsilon$ on $\epsilon \times \epsilon$, then $(\mathcal{P}, \mathcal{L})$ is locally completely embeddable into a projective space Π.*

PROOF: The proof of Proposition 8 is contained in the proof of Theorem 2.3 in [4].

§4. **Proof of Theorem 2**. Let ϵ be any plane of the linear space $L(\mathcal{P}, \equiv)$ and denote by $\equiv_\epsilon$ the restriction of $\equiv$ to $\epsilon \times \epsilon$. By Proposition 3 the linear space $(\epsilon, \mathcal{L}_\epsilon)$ is locally completely embeddable into a projective plane $\Pi(\epsilon)$. Now [8, Theorem 7] yields that $(\Pi(\epsilon), \equiv_\epsilon)$ is a projective metric plane. We deduce from Proposition 8 that $L(\mathcal{P}, \equiv)$ is locally completely embeddable into a projective space Π. The pair $(\Pi, \equiv)$ satisfies the conditions of Theorem 7 in [8] and hence is a projective metric space.

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