4

Introducing the gauge/string duality

Chapters 4 and 5 together constitute a primer on gauge/string duality, written for a QCD audience.

Our goal in this section is to state what we mean by gauge/string duality, via a clear statement of the original example of such a duality [594, 392, 803], namely the conjectured equivalence between a certain conformal gauge theory and a certain gravitational theory in anti-de Sitter (AdS) spacetime. We shall do this in Section 4.3. In order to get there, in Section 4.1 we will first motivate from a gauge theory perspective why there must be such a duality. Then, in Section 4.2, we will give the reader a look at all that one needs to know about string theory itself in order to understand Section 4.3, and indeed to read this book.

Since some of the contents of this chapter are by now standard textbook material, in some cases we will not give specific references. The reader interested in a more detailed review of string theory may consult the many textbooks available such as [386, 587, 685, 501, 823, 532, 133, 321]. The reader interested in complementary aspects or extra details about the gauge/string duality may consult some of the many existing reviews, e.g. [29, 319, 596, 749, 672, 601, 340, 398, 730, 687].

4.1 Motivating the duality

Although the AdS/CFT correspondence was originally discovered [594, 392, 803] by studying D-branes and black holes in string theory, the fact that such an equivalence may exist can be directly motivated from certain aspects of gauge theories and gravity.[1] In this section we motivate such a direct path from gauge theory to string theory without going into any details about string theory and D-branes.

[1] Since string theory is a quantum theory of gravity and the standard Einstein gravity arises as the low energy limit of string theory, we will use the terms gravity and string theory interchangeably below.

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4.1.1 An intuitive picture: geometrizing the renormalization group flow

Consider a quantum field theory (more generally, a many-body system) in d-dimensional Minkowski spacetime with coordinates $(t, \vec{x})$, possibly defined with a short-distance cut-off ϵ. From the work of Kadanoff, Wilson and others in the 1960s, a good way to describe the system is to organize the physics in terms of length (or energy) scales, since degrees of freedom at widely separated scales are largely decoupled from each other. If one is interested in properties of the system at a length scale $z \gg \epsilon$, instead of using the bare theory defined at scale ϵ, it is more convenient to integrate out short-distance degrees of freedom and obtain an effective theory at length scale z. Similarly, for physics at an even longer length scale $z' \gg z$, it is more convenient to use the effective theory at scale z'. This procedure defines a renormalization group (RG) flow and gives rise to a continuous family of effective theories in d-dimensional Minkowski spacetime labeled by the RG scale z. One may now visualize this continuous family of d-dimensional theories as a single theory in a $(d+1)$-dimensional spacetime with the RG scale z now becoming a spatial coordinate.[2] By construction, this $(d+1)$-dimensional theory should have the following properties.

(1) The theory should be intrinsically non-local, since an effective theory at a scale z should only describe physics at scales longer than z. However, there should still be some degree of locality in the z-direction, since degrees of freedom of the original theory at different scales are not strongly correlated with each other. For example, the renormalization group equations governing the evolution of the couplings are local with respect to length scales.
(2) The theory should be invariant under reparametrizations of the z-coordinate, since the physics of the original theory is invariant under reparametrizations of the RG scale.
(3) All the physics in the region below the Minkowski plane at z (see Fig. 4.1) should be describable by the effective theory of the original system defined at a RG scale z. In particular, this $(d+1)$-dimensional description has only the number of degrees of freedom of a d-dimensional theory.

In practice, it is not yet clear how to 'merge' this continuous family of d-dimensional theories into a coherent description of a $(d+1)$-dimensional system, or whether this way of rewriting the renormalization group gives rise to something sensible or useful. Property number (3) above, however, suggests that if such a description is indeed sensible, the result may be a theory of quantum gravity. The clue comes from the holographic principle [771, 767] (for a review, see

[2] Arguments suggesting that the string dual of a Yang–Mills theory must involve an extra dimension were put forward in [694].

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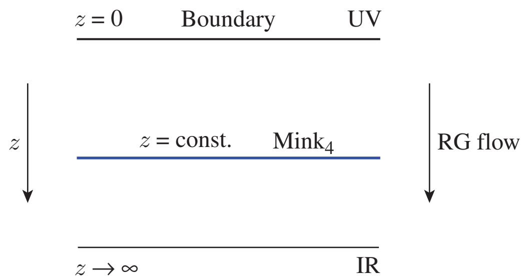

Figure 4.1 A geometric picture of AdS_5. Figure adapted from Ref. [601].

[183]), which says that a theory of quantum gravity in a region of space should be described by a *non-gravitational* theory living at the boundary of that region. In particular, one may think of the quantum field theory as living on the $z = 0$ slice, the boundary of the entire space.

We now see that the gauge/gravity duality, when interpreted as a geometrization of the RG evolution of a quantum field theory, appears to provide a specific realization of the holographic principle. An important organizing principle which follows from this discussion is the UV/IR connection [768, 671] between the physics of the boundary and the bulk systems. From the viewpoint of the bulk, physics near the $z = 0$ slice corresponds to physics near the boundary of the space, i.e. to large-volume or IR physics. In contrast, from the viewpoint of the quantum field theory, physics at small z corresponds to short-distance physics, i.e. UV physics.

4.1.2 The large-N_c expansion of a non-Abelian gauge theory vs. the string theory expansion

The heuristic picture of the previous section does not tell us for which many-body system such a gravity description is more likely to exist, or what kind of properties such a gravity system should have. A more concrete indication that a many-body theory may indeed have a gravitational description comes from the large-N_c expansion of a non-Abelian gauge theory.

That it ought to be possible to reformulate a non-Abelian gauge theory as a string theory can be motivated at different levels. After all, string theory was first invented to describe strong interactions. Different vibration modes of a string provided an economical way to explain many resonances discovered in the 1960s which obey the so-called Regge behavior, i.e. the relation $M^2 \propto J$ between the mass and the angular momentum of a particle. After the formulation of QCD as the microscopic theory for the strong interactions, confinement provided a physical picture for possible stringy degrees of freedom in QCD. Owing to confinement,

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gluons at low energies behave to some extent like flux tubes which can close on themselves or connect a quark-antiquark pair, which naturally suggests a possible string formulation. Such a low-energy effective description, however, does not extend to high energies if the theory becomes weakly coupled, or to non-confining gauge theories.

A strong indication that a fundamental (as opposed to effective) string theory description may exist for any non-Abelian gauge theory (confining or not) comes from 't Hooft's large-N_c expansion [770]. Owing to space limitations, here we will not give a self-contained review of the expansion (see e.g. [801, 297, 599] for reviews) and will only summarize the most important features. The basic idea of 't Hooft was to treat the number of colors N_c for a non-Abelian gauge theory as a parameter, take it to be large, and expand physical quantities in $1/N_c$. For example, consider the Euclidean partition function for a $U(N_c)$ pure gauge theory with gauge coupling g:

$$Z = \int DA_\mu \, \exp\left(-\frac{1}{4g^2}\int d^4x \, \mathrm{Tr} F^2\right). \tag{4.1}$$

Introducing the 't Hooft coupling

$$\lambda = g^2 N_c \,, \tag{4.2}$$

one finds that the vacuum-to-vacuum amplitude $\log Z$ can be expanded in $1/N_c$ as

$$\log Z = \sum_{h=0}^{\infty} N_c^{2-2h} f_h(\lambda) = N_c^2 f_0(\lambda) + f_1(\lambda) + \frac{1}{N_c^2} f_2(\lambda) + \cdots \,, \tag{4.3}$$

where $f_h(\lambda), h = 0, 1, \ldots$ are functions of the 't Hooft coupling λ only. What is remarkable about the large-N_c expansion (4.3) is that, at a fixed λ, Feynman diagrams are organized by their topologies. For example, diagrams which can be drawn on a plane without crossing any lines ("planar diagrams") all have the same N_c dependence, proportional to N_c^2, and are included in $f_0(\lambda)$. Similarly, $f_h(\lambda)$ includes the contributions of all diagrams which can be drawn on a two-dimensional surface with h holes without crossing any lines. Given that the topology of a two-dimensional compact orientable surface is classified by its number of holes, the large-N_c expansion (4.3) can be considered as an expansion in terms of the topology of two-dimensional compact surfaces.

This is in remarkable parallel with the perturbative expansion of a closed string theory, which expresses physical quantities in terms of the propagation of a string in spacetime. The worldsheet of a *closed* string is a two-dimensional compact surface[3]

[3] With external legs contracted to points, as can be done thanks to the conformal invariance of the string worldsheet. For details see any of the standard textbooks on string theory cited above.

and the string perturbative expansion is given by a sum over the topologies of two-dimensional surfaces. For example, the vacuum-to-vacuum amplitude in a string theory can be written as

$$\mathcal{A} = \sum_{h=0}^{\infty} g_s^{2h-2} F_h(\alpha') = \frac{1}{g_s^2} F_0(\alpha') + F_1(\alpha') + g_s^2 F_2(\alpha') + \cdots , \tag{4.4}$$

where g_s is the string coupling, $2\pi\alpha'$ is the inverse string tension, and $F_h(\alpha')$ is the contribution of two-dimensional surfaces with h holes.

Comparing (4.3) and (4.4), it is tempting to identify (4.3) as the perturbative expansion of some string theory with

$$g_s \sim \frac{1}{N_c} \tag{4.5}$$

and the string tension given as some function of the 't Hooft coupling λ. Note that the identification of (4.3) and (4.4) is more than just a mathematical analogy. Consider for example $f_0(\lambda)$, which is given by the sum over all Feynman diagrams which can be drawn on a plane (which is topologically a sphere). Each planar Feynman diagram can be thought of as a discrete triangulation of the sphere. Summing all planar diagrams can then be thought of as summing over all possible discrete triangulations of a sphere, which in turn can be considered as summing over all possible embeddings of a two-dimensional surface with the topology of a sphere in some ambient spacetime. This motivates the conjecture of identifying $f_0(\lambda)$ with F_0 for some closed string theory, but leaves open what the specific string theory is.

One can also include quarks, or more generally matter in the fundamental representation. Since quarks have N_c degrees of freedom, in contrast with the N_c^2 carried by gluons, including quark loops in the Feynman diagrams will lead to $1/N_c$ suppressions. For example, in a theory with N_f flavors, the single-quark loop planar-diagram contribution to the vacuum amplitude scales as $\log Z \sim N_f N_c$ rather than as N_c^2 as in (4.3). In the large-N_c limit with finite N_f, the contribution from quark loops is thus suppressed by powers of N_f/N_c. Feynman diagrams with quark loops can also be classified by using topologies of two-dimensional surfaces, now with inclusion of surfaces with boundaries. Each boundary can be identified with a quark loop. On the string side, two-dimensional surfaces with boundaries describe worldsheets of a string theory containing both closed and open strings, with boundaries corresponding to the worldlines of the endpoints of the open strings.

4.1.3 Why AdS?

Assuming that a d-dimensional field theory can be described by a $(d+1)$-dimensional string or gravity theory, we can try to derive some properties of the $(d+1)$-dimensional spacetime. The most general metric in $d+1$ dimensions consistent with d-dimensional Poincare symmetry can be written as

$$ds^2 = \Omega^2(z)\left(-dt^2 + d\vec{x}^2 + dz^2\right), \tag{4.6}$$

where z is the extra spatial direction. Note that in order to have translational symmetries in the $t, \vec{x}$ directions, the warp factor $\Omega(z)$ can depend on z only. At this stage not much can be said of the form of $\Omega(z)$ for a general quantum field theory. However, if we consider field theories which are conformal (CFTs), then we can determine $\Omega(z)$ using the additional symmetry constraints! A conformally invariant theory is invariant under the scaling

$$(t, \vec{x}) \to C(t, \vec{x}) \tag{4.7}$$

with C a constant. For the gravity theory formulated in (4.6) to describe such a field theory, the metric (4.6) should respect the scaling symmetry (4.7) with the simultaneous scaling of the z coordinate $z \to Cz$, since z represents a length scale in the boundary theory. For this to be the case we need $\Omega(z)$ to scale as

$$\Omega(z) \to C^{-1}\Omega(z) \quad \text{under} \quad z \to Cz\,. \tag{4.8}$$

This uniquely determines

$$\Omega(z) = \frac{R}{z}, \tag{4.9}$$

where R is a constant. The metric (4.6) can now be written as

$$ds^2 = \frac{R^2}{z^2}\left(-dt^2 + d\vec{x}^2 + dz^2\right), \tag{4.10}$$

which is precisely the line element of $(d+1)$-dimensional anti-de Sitter spacetime, AdS_{d+1}. This is a maximally symmetric spacetime with curvature radius R and constant negative curvature proportional to $1/R^2$. See e.g. [436] for a detailed discussion of the properties of AdS space.

In addition to Poincare symmetry and the scaling (4.7), a conformal field theory in d dimensions is also invariant under d special conformal transformations, which altogether form the d-dimensional conformal group $SO(2, d)$. It turns out that the isometry group[4] of (4.10) is also $SO(2, d)$, precisely matching that of the field theory. Thus one expects that a conformal field theory should have a string theory description in AdS spacetime!

[4] Namely the spacetime coordinate transformations that leave the metric invariant.

Note that it is not possible to use the discussion of this section to deduce the precise string theory dual of a given field theory, nor the precise relations between their parameters. In next section we will give a brief review of some essential aspects of string theory which will then enable us to arrive at a precise formulation of the duality, at least for some gauge theories.

4.2 All you need to know about string theory

Here we will review some basic concepts of string theory and D-branes, which will enable us to establish an equivalence between IIB string theory in $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM theory. Although some of the contents of this section are not indispensable to understanding some of the subsequent chapters, they are important for building the reader's intuition about the AdS/CFT correspondence.

4.2.1 Strings

Unlike quantum field theory, which describes the dynamics of point particles, string theory is a quantum theory of interacting, relativistic one-dimensional objects. It is characterized by the string tension, $T_{\rm str}$, and by a dimensionless coupling constant, g_s, that controls the strength of interactions. It is customary to write the string tension in terms of a fundamental length scale ℓ_s, called string length, as

$$T_{\rm str} \equiv \frac{1}{2\pi\alpha'} \qquad \text{with} \qquad \alpha' \equiv \ell_s^2\,. \tag{4.11}$$

We now describe the conceptual steps involved in the definition of the theory, in a first-quantized formulation, i.e. we consider the dynamics of a single string propagating in a fixed spacetime. Although perhaps less familiar, an analogous first-quantized formulation also exists for point particles [693, 242], whose second-quantized formulation is a quantum field theory. In the case of string theory, the corresponding second-quantized formulation is provided by string field theory, which contains an infinite number of quantum fields, one for each of the vibration modes of a single string. In this book we will not need to consider such a formulation. For the moment we also restrict ourselves to closed strings – we will discuss open strings in the next section.

A string will sweep out a two-dimensional worldsheet which, in the case of a closed string, has no boundary. We postulate that the action that governs the dynamics of the string is simply the area of this worldsheet. This is a natural generalization of the action for a relativistic particle, which is simply the length of its worldline. In order to write down the string action explicitly, we parametrize the worldsheet with local coordinates σ^α, with $\alpha = 0, 1$. For fixed worldsheet time,

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$\sigma^0 =$ const., the coordinate σ^1 parametrizes the length of the string. Let x^M, with $M = 0, \ldots, D-1$, be spacetime coordinates. The trajectory of the string is then described by specifying x^M as a function of σ^α. In terms of these functions, the two-dimensional metric $g_{\alpha\beta}$ induced on the string worldsheet has components

$$g_{\alpha\beta} = \partial_\alpha x^M \partial_\beta x^N \, g_{MN} \,, \tag{4.12}$$

where g_{MN} is the spacetime metric. (If x^M are Cartesian coordinates in flat spacetime then $g_{MN} = \eta_{MN} = \mathrm{diag}(- + \cdots +)$.) The action of the string is then given by

$$S_{\mathrm{str}} = -T_{\mathrm{str}} \int d^2\sigma \sqrt{-\det g} \,. \tag{4.13}$$

In order to construct the quantum states of a single string, one needs to quantize this action. It turns out that the quantization imposes strong constraints on the spacetime one started with; not all spacetimes allow a consistent string propagation – see e.g. [386]. For example, if we start with a D-dimensional Minkowski spacetime, then a consistent bosonic string theory (4.13) exists only for $D = 26$. Otherwise the spacetime Lorentz group becomes anomalous at the quantum level and the theory contains negative norm states.

Physically, different states in the spectrum of the two-dimensional worldsheet theory correspond to different vibration modes of the string. From the spacetime viewpoint, each of these modes appears as a particle of a given mass and spin. The spectrum typically contains a finite number of massless modes and an infinite tower of massive modes with masses of order $m_s \equiv \ell_s^{-1}$. A crucial fact about a closed string theory is that one of the massless modes is a particle of spin two, i.e. a graviton. This is the reason why string theory is, in particular, a theory of quantum gravity. The graviton describes small fluctuations of the spacetime metric, implying that the fixed spacetime that we started with is actually dynamical.

One can construct other string theories by adding degrees of freedom to the string worldsheet. The theory that will be of interest here is a supersymmetric theory of strings, the so-called type IIB superstring theory [385, 731], which can be obtained by adding two-dimensional worldsheet fermions to the action (4.13). Although we will of course be interested in eventually breaking supersymmetry in order to obtain a dual description of QCD, it will be important that the underlying theory be supersymmetric, since this will guarantee the stability of our constructions. For a superstring, absence of negative-norm states requires the dimension of spacetime to be $D = 10$. In addition to the graviton, the massless spectrum of IIB superstring theory includes two scalars, a number of antisymmetric tensor fields, and various fermionic partners as required by supersymmetry. One of the scalars, the so-called dilaton Φ, will play an important role here.

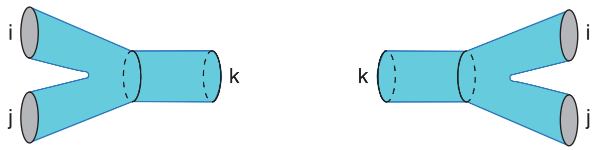

Figure 4.2 'Geometrical' interactions in string theory: two strings in initial states i and j can join into one string in a state k (left) or vice versa (right).

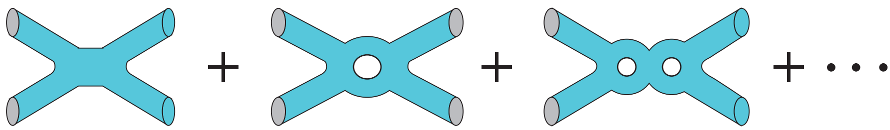

Figure 4.3 Sum over topologies contributing to the two-to-two amplitude.

Interactions can be introduced geometrically by postulating that two strings can join together and that one string can split into two through a vertex of strength g_s – see Fig. 4.2. Physical observables like scattering amplitudes can be found by summing over string propagations (including all possible splittings and joinings) between initial and final states. After fixing all the gauge symmetries on the string worldsheets, such a sum reduces to a sum over the topologies of two-dimensional surfaces, with contributions from surfaces of h holes weighted by a factor g_s^{2h-2}. This is illustrated in Fig. 4.3 for the two-to-two amplitude.

At low energies $E \ll m_s$, one can integrate out the massive string modes and obtain a low energy effective theory for the massless modes. Since the massless spectrum of a closed string theory always contains a graviton, to second order in derivatives, the low energy effective action has the form of Einstein gravity coupled to other (massless) matter fields, i.e.

$$S = \frac{1}{16\pi G} \int d^D x \sqrt{-g} \mathcal{R} + \cdots , \tag{4.14}$$

where $\mathcal{R}$ is the Ricci scalar for the metric with D spacetime dimensions and where the dots stand for additional terms associated with the rest of massless modes. For type IIB superstring theory, the full low energy effective action at the level of two derivatives is given by the so-called IIB supergravity [733, 732], a supersymmetric generalization of (4.14) (with $D = 10$). The higher order corrections to (4.14) take the form of a double expansion, in powers of $\alpha' E^2$ from integrating out the massive stringy modes, and in powers of the string coupling g_s from loop corrections.

We conclude this section by making two important comments. First, we note that the ten-dimensional Newton's constant G in type IIB supergravity can be expressed in terms of the string coupling and the string length as

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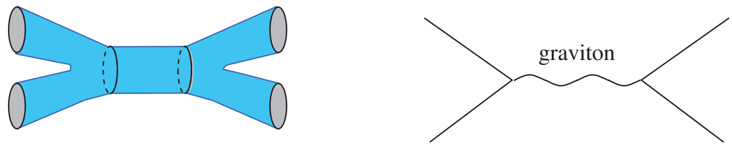

(a) (b)

Figure 4.4 (a) Tree-level contribution, of order g_s^2, to a two-to-two scattering process in string theory. The low energy limit of this tree-level diagram must coincide with the corresponding field theory diagram depicted in (b), which is proportional to Newton's constant, G.

$$16\pi G = (2\pi)^7 g_s^2 \ell_s^8 \,. \tag{4.15}$$

The dependence on ℓ_s follows from dimensional analysis, since in D dimensions Newton's constant has dimension (length)$^{D-2}$. The dependence on g_s follows from considering two-to-two string scattering. The leading string theory diagram, depicted in Fig. 4.4a, is proportional to g_s^2, since it is obtained by joining together the two diagrams of Fig. 4.2. The corresponding diagram in supergravity is drawn in Fig. 4.4b, and is proportional to G. The requirement that the two amplitudes yield the same result at energies much lower than the string scale implies $G \propto g_s^2$.

Second, the string coupling constant g_s is not a free parameter, but is given by the expectation value of the dilaton field Φ as $g_s = e^{\Phi}$. As a result, g_s may actually vary over space and time. Under these circumstances we may still speak of the string coupling constant, e.g. in formulas like (4.15) or (4.19), meaning the asymptotic value of the dilaton at infinity, $g_s = e^{\Phi_\infty}$.

4.2.2 D-branes and gauge theories

Perturbatively, string theory is a theory of strings. Non-perturbatively, the theory also contains a variety of higher-dimensional solitonic objects. D-branes [686] are a particularly important class of solitons. To be definite, let us consider a superstring theory (e.g. type IIA or IIB theory) in a ten-dimensional flat Minkowski spacetime, labeled by time $t \equiv x_0$ and spatial coordinates $x_1, \ldots, x_9$. A Dp-brane is then a "defect" where closed strings can break and open strings can end that occupies a p-dimensional subspace – see Fig. 4.5, where the x-directions are parallel to the branes and the y-directions are transverse to them. When closed strings break, they become open strings. The end points of the open strings can move freely along the directions of the D-brane, but cannot leave it by moving in the transverse directions. Just like domain walls or cosmic strings in a quantum field theory, D-branes are dynamical objects which can move around. A Dp-brane then sweeps out a $(p+1)$-dimensional worldvolume in spacetime. D0-branes are particle-like objects, D1-branes are string-like, D2-branes are membrane-like, etc. Stable Dp-branes in Type

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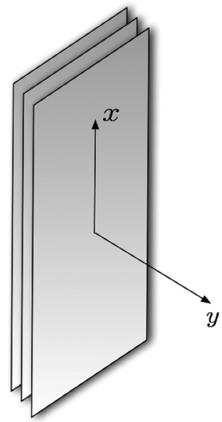

Figure 4.5 Stack of D-branes. Figure adapted from Ref. [601].

IIA superstring theory exist for $p = 0, 2, 4, 6, 8$, whereas those in Type IIB have $p = 1, 3, 5, 7$ [686].[5] This can be seen in a variety of (not unrelated) ways, two of which are: (i) in these cases the corresponding Dp-branes preserve a fraction of the supersymmetry of the underlying theory; (ii) in these cases the corresponding Dp-branes are the lightest states that carry a conserved charge.

Introducing a D-brane adds an entirely new sector to the theory of closed strings, consisting of open strings whose endpoints must satisfy the boundary condition that they lie on the D-brane. Recall that in the case of closed strings we started with a fixed spacetime and discovered, after quantization, that the close string spectrum corresponds to dynamical fluctuations of the spacetime. An analogous situation holds for open strings on a D-brane. Suppose we start with a Dp-brane extending in the $x^\mu = (x^0, x^1, \ldots, x^p)$ directions, with transverse directions labeled as $y^i = (x^{p+1}, \ldots, x^9)$. Then, after quantization, one obtains an open string spectrum which can be identified with fluctuations of the D-brane.

More explicitly, the open string spectrum consists of a finite number of massless modes and an infinite tower of massive modes with masses of order $m_s = 1/\ell_s$. For a single Dp-brane, the massless spectrum consists of an Abelian gauge field $A_\mu(x)$, $\mu = 0, 1, \ldots, p$, $9-p$ scalar fields $\phi^i(x)$, $i = 1, \ldots, 9-p$, and their superpartners. Since these fields are supported on the D-brane, they depend only on the x^μ coordinates along the worldvolume, but not on the transverse coordinates. The $9-p$ scalar excitations ϕ^i describe fluctuations of the D-brane in the transverse directions y^i, including deformations of the brane's shape and linear motions. They are the exact parallel of familiar collective coordinates for a domain wall or a cosmic string in

[5] D9-branes also exist, but additional consistency conditions must be imposed in their presence. We will not consider them here.

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a quantum field theory, and can be understood as the Goldstone bosons associated to the subset of translational symmetries spontaneously broken by the brane. The presence of a $U(1)$ gauge field $A_\mu(x)$ as part of collective excitations lies at the origin of many fascinating properties of D-branes, which (as we will discuss below) ultimately lead to the gauge/string duality. Although this gauge field is less familiar in the context of quantum field theory solitons (see e.g. [298, 489, 703]), it can nevertheless be understood as a Goldstone mode associated to large gauge transformations spontaneously broken by the brane [231, 512, 22].

Another striking new feature of D-branes, which has no parallel in field theory, is the appearance of a non-Abelian gauge theory when multiple D-branes become close to one another [802]. In addition to the degrees of freedom pertaining to each D-brane, now there are new sectors corresponding to open strings stretched between different branes. For example, consider two parallel branes separated from each other by a distance r, as shown in Fig. 4.6. Now there are four types of open strings, depending on which brane their endpoints lie. The strings with both endpoints on the same brane give rise, as before, to two massless gauge vectors, which can be denoted by $(A_\mu)^1{}_1$ and $(A_\mu)^2{}_2$, where the upper (lower) numeric index labels the brane on which the string starts (ends). Open strings stretching between different branes give rise to two additional vector fields $(A_\mu)^1{}_2$ and $(A_\mu)^2{}_1$, which have a mass given by the tension of the string times the distance between the branes, i.e. $m = r/2\pi\alpha'$. These become massless when the branes lie on top of each other, $r = 0$. In this case there are four massless vector fields altogether, $(A_\mu)^a{}_b$ with $a, b = 1, 2$, which precisely correspond to the gauge fields of a non-Abelian $U(2)$ gauge group. Similarly, one finds that the $9-p$ massless scalar fields also become 2×2 matrices $(\phi^i)^a{}_b$, which transform in the adjoint representation of the $U(2)$ gauge group. In the general case of N_c parallel coinciding branes one finds a $U(N_c)$ multiplet of non-Abelian gauge fields with $9 - p$ scalar fields in the adjoint representation of $U(N_c)$. The low-energy dynamics of these modes can

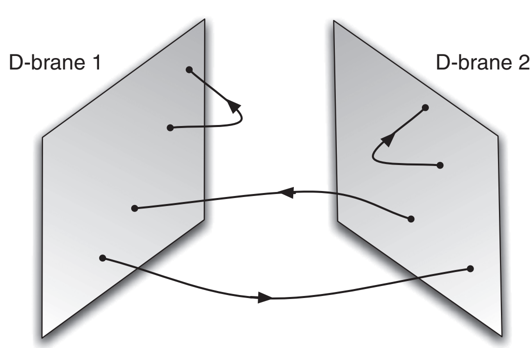

Figure 4.6 Strings stretching between two D-branes.

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be determined by integrating out the massive open string modes, and it turns out to be governed by a non-Abelian gauge theory [802]. To be more specific, let us consider N_c D3-branes in type IIB theory. The massless spectrum consists of a gauge field A_μ, six scalar fields ϕ^i, $i = 1, \ldots, 6$ and four Weyl fermions, all of which are in the adjoint representation of $U(N_c)$ and can be written as $N_c \times N_c$ matrices. At the two-derivative level the low energy effective action for these modes turns out to be precisely [802] the $\mathcal{N} = 4$ super-Yang–Mills theory with gauge group $U(N_c)$ in (3+1) dimensions [199, 380] (for reviews see e.g. [545, 319]), the bosonic part of whose Lagrangian can be written as

$$\mathcal{L} = -\frac{1}{g^2}\mathrm{Tr}\left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}D_\mu\phi^i D^\mu\phi^i + [\phi^i, \phi^j]^2\right), \tag{4.16}$$

with the Yang–Mills coupling constant given by

$$g^2 = 4\pi g_s\,. \tag{4.17}$$

Equation (4.16) is in fact the (bosonic part of the) most general renormalizable Lagrangian consistent with $\mathcal{N} = 4$ global supersymmetry. Owing to the large number of supersymmetries the theory has many interesting properties, including the fact that the beta function vanishes exactly [86, 387, 745, 259, 466, 598, 200] (see section 4.1 of [687] for a one-paragraph proof). Consequently, the coupling constant is scale-independent and the theory is conformally invariant.

Note that the $U(1)$ part of (4.16) is free and can be decoupled. Physically, the reason for this is as follows. Excitations of the overall, diagonal $U(1)$ subgroup of $U(N_c)$ describe motion of the branes' centre of mass, i.e. rigid motion of the entire system of branes as a whole. Because of the overall translation invariance, this mode decouples from the remaining $SU(N_c) \subset U(N_c)$ modes that describe motion of the branes relative to one another. This is the reason why, as we will see, IIB strings in $\mathrm{AdS}_5 \times S^5$ are dual to $\mathcal{N} = 4$ super-Yang–Mills theory with gauge group $SU(N_c)$.

The Lagrangian (4.16) receives higher-derivative corrections suppressed by $\alpha' E^2$. The full system also contains closed string modes (e.g. gravitons) which propagate in the bulk of the ten-dimensional spacetime (see Fig. 4.7) and the full theory contains interactions between closed and open strings. The strength of interactions of closed string modes with each other is controlled by Newton's constant G, so the dimensionless coupling constant at an energy E is GE^8. This vanishes at low energies and so in this limit closed strings become noninteracting, which is essentially the statement that gravity is infrared-free. Interactions between closed and open strings are also controlled by the same parameter, since gravity couples universally to all forms of matter. Therefore at low energies closed strings decouple from open strings. We thus conclude that at low energies the interacting sector reduces to an $SU(N_c)$ $\mathcal{N} = 4$ SYM theory in four dimensions.

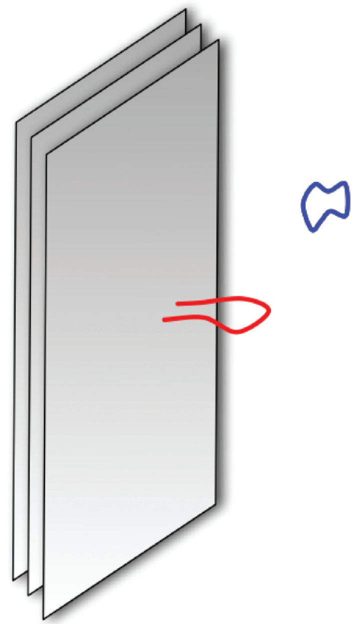

Figure 4.7 Open and closed string excitations of the full system. Figure adapted from Ref. [601].

Before closing this section we note that, for a single Dp-brane with constant $F_{\mu\nu}$ and $\partial_\mu \phi^i$, all higher order α'-corrections to (4.16) (or its p-dimensional generalizations) can be resummed exactly into the so-called Dirac–Born–Infeld (DBI) action [575]

$$S_{\text{DBI}} = -T_{\text{Dp}} \int d^{p+1}x\, e^{-\Phi} \sqrt{-\det\left(g_{\mu\nu} + 2\pi\ell_s^2\, F_{\mu\nu}\right)}\,, \tag{4.18}$$

where

$$T_{\text{Dp}} = \frac{1}{(2\pi)^p\, g_s \ell_s^{p+1}} \tag{4.19}$$

is the tension of the brane, namely its mass per unit spatial volume. In this action, Φ is the dilaton and $g_{\mu\nu}$ denotes the induced metric on the brane. In flat space, the latter can be written more explicitly as

$$g_{\mu\nu} = \eta_{\mu\nu} + (2\pi\ell_s^2)^2 \partial_\mu\phi^i \partial_\nu\phi^i\,. \tag{4.20}$$

The first term in (4.20) comes from the flat spacetime metric along the worldvolume directions, and the second term arises from fluctuations in the transverse directions. Expanding the action (4.18) to quadratic order in F and $\partial\phi$ one recovers the Abelian version of Eq. (4.16). The non-Abelian generalization of the DBI action (4.18) is not known in closed form – see, for example, [781] for a review. Corrections to (4.18) beyond the approximation of slowly varying fields have been considered in [54, 536, 93, 384].

4.2.3 D-branes as spacetime geometry

Owing to their infinite extent along the x-directions, the total mass of a Dp-brane is infinite. However, the mass per unit p-volume, known as the tension, is finite and is given in terms of fundamental parameters by Eq. (4.19). The dependence of the tension on the string length is dictated by dimensional analysis. The inverse dependence on the coupling g_s is familiar from solitons in quantum field theory (see e.g. [298, 489, 703]) and signals the nonperturbative nature of D-branes, since it implies that they become infinitely massive (even per unit volume) and hence decouple from the spectrum in the perturbative limit $g_s \to 0$. The crucial difference is that the D-branes' tension scales as $1/g_s$ instead of the $1/g^2$ scaling that is typical of field theory solitons. This dependence can be anticipated based on the divergences of string perturbation theory [736] and, as we will see, is of great importance for the gauge/string duality.

In a theory with gravity, all forms of matter gravitate. D-branes are no exception, and their presence deforms the spacetime metric around them. The spacetime metric sourced by N_c Dp-branes can be found by explicitly solving the supergravity equations of motion [377, 369, 462]. For illustration we again use the example of D3-branes in type IIB theory, for which one finds:

$$ds^2 = H^{-1/2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + H^{1/2}\left(dr^2 + r^2 d\Omega_5^2\right) . \tag{4.21}$$

The metric inside the parentheses in the second term is just the flat metric in the y-directions transverse to the D3-branes written in spherical coordinates, with radial coordinate $r^2 = y_1^2 + \cdots + y_6^2$. The function $H(r)$ is given by

$$H = 1 + \frac{R^4}{r^4} , \tag{4.22}$$

where

$$R^4 = 4\pi g_s N_c \ell_s^4 . \tag{4.23}$$

Let us gain some physical intuition regarding this solution. Since D3-branes extend along three spatial directions, their gravitational effect is similar to that of a point particle with mass $M \propto N_c T_{\mathrm{D3}}$ in the six transverse directions. Thus the metric (4.21) only depends on the radial coordinate r of the transverse directions. For $r \gg R$ we have $H \simeq 1$ and the metric reduces to that of flat space with a small correction proportional to

$$\frac{R^4}{r^4} \sim \frac{N_c g_s \ell_s^4}{r^4} \sim \frac{GM}{r^4}, \tag{4.24}$$

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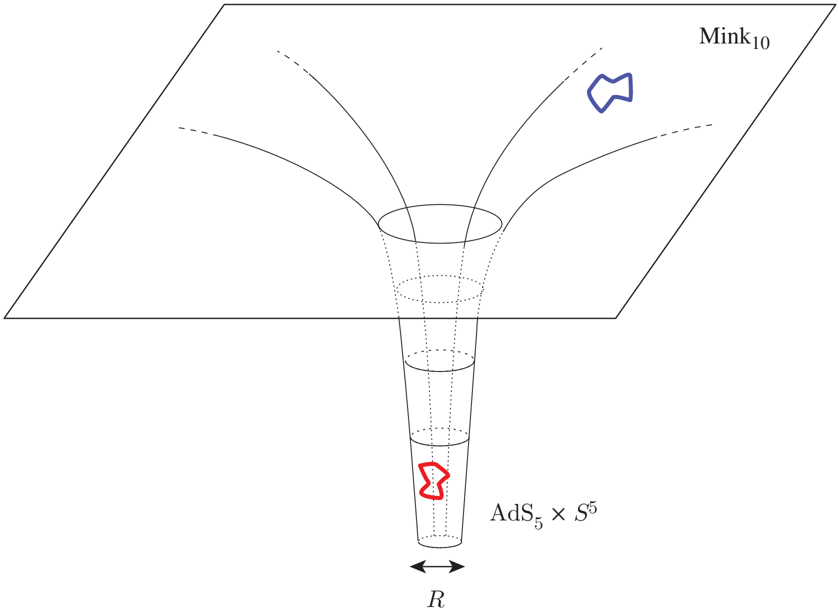

Figure 4.8 Excitations of the system in the closed string description. Figure adapted from Ref. [601].

which can be interpreted as the gravitational potential due to a massive object of mass M in six spatial dimensions.[6] Note that in the last step in Eq. (4.24) we have used the fact that $G \propto g_s^2 \ell_s^8$ and $M \propto N_c T_{\rm D3} \propto N_c/g_s \ell_s^4$ – see (4.15) and (4.19).

The parameter R can thus be considered as the length scale characteristic of the range of the gravitational effects of N_c D3-branes. These effects are weak for $r \gg R$, but become strong for $r \ll R$. In the latter limit, we may neglect the "1" in Eq. (4.22), in which case the metric (4.21) reduces to

$$ds^2 = ds^2_{\rm AdS_5} + R^2 d\Omega_5^2 \,, \tag{4.25}$$

where

$$ds^2_{\rm AdS_5} = \frac{r^2}{R^2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \frac{R^2}{r^2} dr^2 \tag{4.26}$$

is the metric (4.10) of five-dimensional anti-de Sitter spacetime written in terms of $r = R^2/z$. We thus see that in the strong gravity region the ten-dimensional metric factorizes into AdS$_5$ × S^5.

We conclude that the geometry sourced by the D3-branes takes the form displayed in Fig. 4.8: far away from the branes the spacetime is flat, ten-dimensional

[6] Recall that a massive object of mass M in D spatial dimensions generates a gravitational potential GM/r^{D-2} at a distance r from its position.

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Minkowski space, whereas close to the branes a "throat" geometry of the form $AdS_5 \times S^5$ develops. The size of the throat is set by the length-scale R, given by (4.23). As we will see, the spacetime geometry (4.21) can be considered as providing an *alternative* description of the D3-branes.

4.3 The AdS/CFT conjecture

In the previous two sections we have seen two descriptions of D3-branes. In the description of Section 4.2.2, which we will refer to as the open string description, D-branes correspond to a hyperplane in a *flat* spacetime. In this description the D-branes' excitations are open strings living on the branes, and closed strings propagate outside the branes – see Fig. 4.7.

In contrast, in the description of Section 4.2.3, which we will call the closed string description, D-branes correspond to a spacetime geometry in which only *closed* strings propagate, as displayed in Fig. 4.8. In this description there are no open strings. In this case the low energy limit consists of focusing on excitations that have arbitrarily low energy with respect to an observer in the asymptotically flat Minkowski region. We have here two distinct sets of degrees of freedom, those propagating in the Minkowski region and those propagating in the throat – see Fig. 4.8. In the Minkowski region the only modes that remain are those of the massless ten-dimensional graviton supermultiplet. Moreover, at low energies these modes decouple from each other, since their interactions are proportional to GE^8. In the throat region, however, the whole tower of massive string excitations survives. This is because a mode in the throat must climb up a gravitational potential in order to reach the asymptotically flat region. Consequently, a closed string of arbitrarily high proper energy in the throat region may have an arbitrarily low energy as seen by an observer at asymptotic infinity, provided the string is located sufficiently deep down the throat. As we focus on lower and lower energies these modes become supported deeper and deeper in the throat, and so they decouple from those in the asymptotic region. We thus conclude that in the closed string description, the interacting sector of the system at low energies reduces to closed strings in $AdS_5 \times S^5$.

These two representations are tractable in different parameter regimes. For $g_s N_c \ll 1$, we see from Eq. (4.23) that $R \ll \ell_s$, i.e. the radius characterizing the gravitational effect of the D-branes becomes small in string units, and closed strings feel a flat spacetime everywhere except very close to the hyperplane where the D-branes are located. In this regime the closed string description is not useful since one would need to understand sub-string-scale geometry. In the opposite regime, $g_s N \gg 1$, we find that $R \gg \ell_s$ and the geometry becomes weakly curved. In this limit the closed string description simplifies and essentially reduces

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to classical gravity. Instead, the open string description becomes intractable, since $g_s N_c$ controls the loop expansion of the theory and one would need to deal with strongly coupled open strings. Note that both representations exist in both limiting regimes, and in between.

To summarize, the two descriptions of N_c D3-branes that we have discussed and their low energy limits are as follows.

(1) A hyperplane in a *flat* spacetime with open strings attached. The low energy limit is described by $\mathcal{N} = 4$ SYM theory (4.16) with a gauge group $SU(N_c)$.
(2) A curved spacetime geometry (4.21) where only closed strings propagate. The low energy limit is described by closed IIB string theory in $\mathrm{AdS}_5 \times S^5$.

It is natural to conjecture that these two descriptions are equivalent. Equating in particular their low energy limits, we are led to conjecture that

$$\Big\{\mathcal{N} = 4\ SU(N_c)\ \text{SYM theory}\Big\} = \Big\{\text{IIB string theory in } \mathrm{AdS}_5 \times S_5\Big\}. \tag{4.27}$$

From Eqs. (4.17) and (4.23) we find how the parameters of the two theories are related to one another:

$$g_s = \frac{g^2}{4\pi}, \qquad \frac{R}{\ell_s} = (g^2 N_c)^{1/4}. \tag{4.28}$$

One can also use the ten-dimensional Newton constant (4.15) in place of g_s in the first equation above and obtain equivalently

$$\frac{G}{R^8} = \frac{\pi^4}{2N_c^2}, \qquad \frac{R}{\ell_s} = (g^2 N_c)^{1/4}. \tag{4.29}$$

Note that, in particular, the first equation in (4.28) implies that the criterion that $g_s N_c$ be large or small translates into the criterion that the gauge theory 't Hooft coupling $\lambda = g^2 N_c$ be large or small. Therefore the question of which representation of the D-branes is tractable becomes the question of whether the gauge theory is strongly or weakly coupled. We will come back to this in Chapter 5.

The discussion above relates string theory to $\mathcal{N} = 4$ SYM theory at zero temperature, as we were considering the ground state of the N_c D3-branes. On the supergravity side this corresponds to the so-called extremal solution. The above discussion can easily be generalized to a nonzero temperature T by exciting the degrees of freedom on the D3-branes to a finite temperature, which corresponds [391, 804] to the so-called non-extremal solution [462]. It turns out that the net effect of this is solely to modify the AdS part of the metric, replacing (4.26) by

$$ds^2 = \frac{r^2}{R^2}\left(-f dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \frac{R^2}{r^2 f} dr^2, \tag{4.30}$$

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where

$$f(r) = 1 - \frac{r_0^4}{r^4}\,. \tag{4.31}$$

Equivalently, in terms of the z-coordinate of Section 4.1 we replace (4.10) by

$$ds^2 = \frac{R^2}{z^2}\left(-f dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + \frac{dz^2}{f}\right)\,, \tag{4.32}$$

where

$$f(z) = 1 - \frac{z^4}{z_0^4}\,. \tag{4.33}$$

These two metrics are related by the simple coordinate transformation $z = R^2/r$, and represent a black brane in AdS spacetime with a horizon located at $r = r_0$ or $z = z_0$ which extends in all three spatial directions of the original brane. As we will discuss in more detail in the next chapter, r_0 and z_0 are related to the temperature of the $\mathcal{N} = 4$ SYM theory as

$$r_0 \propto \frac{1}{z_0} \propto T\,. \tag{4.34}$$

Thus we conclude that $\mathcal{N} = 4$ SYM theory at finite temperature is described by string theory in an AdS black brane geometry.

To summarize this section, we have arrived at a duality (4.27) of the type anticipated in Section 4.1, that is, an equivalence between a conformal field theory with zero β-function and trivial RG-flow and string theory on a scale-invariant metric that looks the same at any z. In the finite-temperature case, Eqn. (4.34) provides an example of the expected relationship between energy scale in the gauge theory, set in this case by the temperature, and position in the fifth dimension, set by the location of the horizon.