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ON A THEOREM OF SOBCZYK

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In this paper the result of Sobczyk about complemented copies of c_0 is extended to a class of Banach spaces X such that the unit ball of their dual endowed with the weak* topology has a certain topological property satisfied by every Corson-compact space. By means of a simple example it is shown that if Corson-compact is replaced by Rosenthal-compact, this extension does not hold. This example gives an easy proof of a result of Phillips and an easy solution to a question of Sobczyk about the existence of a Banach space E, $c_0 \subset E \subset \ell_\infty$, such that E is not complemented in ℓ_∞ and c_0 is not complemented in E. Assuming the continuum hypothesis, it is proved that there exists a Rosenthal-compact space K such that $C(K)$ has no projectional resolution of the identity.

There are two results that play an important role in deciding whether a copy of c_0 is complemented in a Banach space. The first one, due to Sobczyk, asserts that c_0 is complemented in every separable Banach space and the second one, due to Phillips, that c_0 is not complemented in ℓ_∞. In this paper the result of Sobczyk is extended to a class of Banach spaces X such that the unit ball U^* of their dual, endowed with the weak* topology w^*, has a certain topological property. In particular, every X such that (U^*, w^*) is a Corson-compact space, belongs to this class. By means of a simple example, it is shown that if Corson-compact is replaced by Rosenthal-compact, this extension does not hold. Moreover that example gives an easy proof of the result of Phillips. In fact a Banach space E is obtained such that $c_0 \subset E \subset \ell_\infty$; E is not complemented in ℓ_∞ and c_0 is not complemented in E. This gives an easy solution to a problem raised in [10] by Sobczyk. At the end of the paper, assuming the continuum hypothesis, it is proved that there exists a Rosenthal-compact space K such that $C(K)$ has no projectional resolution of the identity.

If X is a Banach space, we denote its dual by X^*, by U^* the unit ball of X^* and by $\sigma(X^*, X)$ the weak* topology. Given a subset A of X^*, (A, w^*) stands for the

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topological space obtained when A is endowed with the topology induced by $\sigma(X^*, X)$. The cardinal of a set J is denoted by $|J|$. A compact topological space K is said to be Rosenthal-compact if there is a Polish space X such that K is homeomorphic to a subspace of $(\mathcal{B}_1(X), \mathcal{J}_p)$ where $\mathcal{B}_1(X)$ is the set of functions of the first Baire class on X and $\mathcal{J}_p$ the topology of the pointwise convergence [5].

DEFINITION 1: Let X be a topological space and $\{F_n\}$ a sequence of disjoint subsets of X. A point x in X is said to be cofinitely near to $\{F_n\}$ if for every neighbourhood V of x, the set

$$\{n \in \mathbb{N}: V \cap F_n = \emptyset\}$$

is finite.

DEFINITION 2: A topological space is called cofinitely sequential if for every x_0 cofinitely near to a disjoint sequence of closed $\mathcal{G}_\delta$ sets $\{F_n\}$ there exists a sequence $\{y_n\}$ such that $y_n \in F_n$ and $\{y_n\}$ converges to x_0.

It is easily seen that every metric space is cofinitely sequential. Let X be a Fréchet space (that is, each point in the closure of a set A in X is the limit of a convergent sequence of points in A [4]); given a point x cofinitely near to a disjoint sequence of sets $\{F_n\}$, there exists a subsequence $\{F_{n_k}\}$ and points $x_k \in F_{n_k}$ such that $\lim x_k = x$. Nevertheless a Fréchet compact space will be constructed that is not cofinitely sequential.

THEOREM 3. *If c_0 is a closed linear subspace of a Banach space X such that (U^*, ω^*) is cofinitely sequential, then there is a continuous projection of X onto c_0 with norm not greater than two.*

PROOF: Let $\psi: c_0 \to X$ be the inclusion mapping, that is, $\psi(x) = x$, $\forall x \in c_0$. If $\psi^*: X^* \to c_0^*$ is the conjugate mapping, let us take

$$F_n = \psi^{-1}(\{e_n^*\}) \cap V^*,$$

where e_n^* is the nth coordinate functional and

$$V^* = \{x^* \in X^*: \|x^*\| \leqslant 2\}.$$

According to the Hahn-Banach theorem, no $F_n \cap U^*$ is empty. Moreover each F_n is a closed $\mathcal{G}_\delta$ set in (V^*, w^*) since each e_n^* is a closed $\mathcal{G}_\delta$ in (c_0^*, w^*). We will show that 0, the null functional, is cofinitely near to $\{F_n\}$. Indeed, otherwise there would exist a neighbourhood W of 0 and an infinite sequence of natural numbers $\{n_k\}$ such that

$$W \cap F_{n_k} = \emptyset, \quad \forall k \in \mathbb{N}; \tag{1}$$

then

$$W \cap U^* \cap \psi^{-1}\Big(\{e^*_{n_k}\}\Big) = \emptyset, \quad \forall k \in \mathbb{N}. \tag{2}$$

Let us take

$$y^*_k \in [U^* \cap \psi^{-1}\Big(\{e^*_{n_k}\}\Big) - W].$$

Since (U^*, w^*) is compact there must exist an accumulation point y^*_0 of $\{y^*_k\}$ and it is easily seen that $y^*_0(x) = 0$, $\forall x \in c_0$. Then

$$(y^*_k) - y^*_0 \in V^* \cap \psi^{-1}\Big(\{e^*_{n_k}\}\Big),$$

and 0 is an accumulation point of this sequence, which contradicts (1).

It is obvious that (V^*, w^*) is homeomorphic to (U^*, w^*), so (V^*, w^*) is cofinitely sequential. Therefore there exists a sequence $x^*_n \in F_n$ such that $\{x^*_n\}$ converges to 0. Then $P\colon X \to c_0$, $P(x) = \{x^*_n(x)\}$ is a continuous projection and $\|P\| \leqslant 2$. ∎

Since every metric space is cofinitely sequential, every separable Banach space satisfies the hypothesis of Theorem 3, so we obtain the result of Sobczyk [**10, 12**]. (In fact the proof of Theorem 3 is an adaptation of the proof of Veech of that result [**12**].) There are cofinitely sequential spaces that are not metrisable. In fact, we will see that every Corson-compact space is cofinitely sequential. Let us recall some notation and results. For any set I, we denote by $\sum(I)$ the subset of $[0, 1]^I$ consisting of functions $x(i)$ which are zero except on a countable subset of I. A compact space is said to be a Corson-compact space if it is homeomorphic to a subset of $\sum(I)$ for some set I.

LEMMA 4. [**1**]. *Let K be a compact subset of $[0, 1]^I$ such that $K \cap \sum(I)$ is dense in K and $J_0 \subset I$. There exists a subset J_1 of I, containing J_0, such that $|J_1| = |J_0|$ and $R_{J_1}(K) \subset K$, where, if J is a subset of I, $R_J\colon [0, 1]^I \to [0, 1]^I$ is defined by declaring $R_J(x)(i)$ to be $x(i)$ if $i \in J$ and 0 otherwise.*

PROPOSITION 5. *Every Corson-compact space is cofinitely sequential.*

PROOF: Let $\{F_n\}$ be a disjoint sequence of closed sets in a Corson-compact space K and y_0 a cofinitely near point to $\{F_n\}$. We can assume that K is included in $\sum(I)$ for some set I. If

$$J_0 = \{i \in I\colon y_0(i) \neq 0\},$$

Lemma 4 enables us to construct inductively a sequence $\{J_n\}$ of countable subsets of I such that

(i) $J_0 \subset J_n \subset J_{n+1}$, $\forall n \in \mathbb{N}$.
(ii) $R_{J_n}(F_n) \subset F_n$, $\forall n \in \mathbb{N}$.

Let us take $J = U\{J_n\colon n \in \mathbb{N}\}$. Since J is countable, $[0, 1]^J$ is metrisable, so there exists $x_n \in F_n$ such that $\{x_n(i)\}$ converges to $y_0(i)$ for every $i \in J$. Now it is enough to show that $\{R_{J_n}(x_n)\}$ converges to y_0 since from (ii), $R_{J_n}(x_n) \in F_n$. Indeed, let I_0 be a finite subset of I. Then

$$(1) \qquad y_0(i) = R_{J_n}(x_n)(i) = 0, \ \forall i \in I_0 - J.$$

Moreover, if $i \in I_0 \cap J$ there exists an n_0 such that $i \in J_n$, $n \geqslant n_0$, so $x_n(i) = R_{J_n}(x_n)(i)$ and

$$(2) \qquad y_0(i) = \lim x_n(i) = \lim R_{J_n}(x_n)(i).$$

From (1) and (2), $\lim R_{J_n}(x_n)(i) = y_0(i)$, $\forall i \in I_0$. ∎

COROLLARY 6. *Let X be a Banach space that contains c_0. If (U^*, w^*) is a Corson-compact space then there is a continuous projection P of X onto c_0 with $\|P\| \leqslant 2$.*

REMARK 7. In [**11**] a result is proved that improves Corollary 6 since its applications are not restricted to copies of c_0. A compact space K is said to be a Valdivia-compact space if there exists a set I such that K is homeomorphic to a closed subset F of $[0, 1]^I$ such that $F \cap \sum(I)$ is dense in F [**3**]. Then in [**11**] it is shown that if K is a Valdivia-compact space then every separable subspace of $C(K)$ is contained in a complemented separable subspace S of $C(K)$. In fact, there is a projection P from $C(K)$ onto S with $\|P\| \leqslant 1$.

Therefore if X is a Banach space containing c_0, such that (U^*, w^*) is a Valdivia-compact space, there is a projection P of X onto c_0 with $\|P\| \leqslant 2$. Indeed it is enough to consider $c_0 \subset X \subset C(U^*)$ and apply the previous observation and the result of Sobczyk.

If Corson-compact is changed for Rosenthal-compact in Corollary 6, the assertion becomes false; this and other facts will be derived from the following example.

EXAMPLE 8. Let us take $L = \cup\{\{0, 1\}^\alpha : 0 \leqslant \alpha \leqslant \omega\}$. An element of L is a function whose domain is α with $0 \leqslant \alpha \leqslant \omega$, where ω is the first infinite ordinal. When $\alpha = 0$ there is exactly one element of $\{0, 1\}^\alpha$, namely the empty mapping from $\emptyset$ to $\{0, 1\}$; we shall write 0 for this trivial object. We will define an order $\preccurlyeq$ on L (the usual order in the real numbers is denoted by $\leqslant$).

$$[s \preccurlyeq t] \leftrightarrow [\operatorname{dom} s \leqslant \operatorname{dom} t \text{ and } t|_{\operatorname{dom} s} = s].$$

We equip L with a topology (the order-topology) by declaring the element 0 to be an isolated point while taking basic neighbourhoods of points $t \neq 0$ to be intervals $(s, t]$ with $s \prec t$. Thus L is scattered and locally compact. Let K be the Alexandroff compactification of L; $K = L \cup \{\infty\}$.

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We will show that K is a Rosenthal-compact space such that there exists a non-complemented copy of c_0 in $C(K)$. Let E be the closed linear subspace of $C(K)$ spanned by $\{\mathbb{1}_{\{t\}}: t \in L_0\}$, where $\mathbb{1}_A$ stands for the indicator function of the set A and $L_0 = L - \{0, 1\}^\omega$. Then if we write $L_0 = \{x_n: n \in \mathbb{N}\}$ it is easy to check that $\psi: c_0 \to E$, $\psi(\{t_n\}) = \sum t_n \mathbb{1}_{\{x_n\}}$ is a linear isometry. Moreover E is not complemented in $C(K)$. Indeed, otherwise there would exist a projection $R: C(K) \to E$; then if $P = \psi^{-1} \circ R$, by considering $P^*(e_i^*)$ we would obtain measures $\{\mu_t: t \in L_0\}$ such that

(i) $\langle \mathbb{1}_{\{t\}}, \mu_t \rangle = 1$, $\forall t \in L_0$.

(ii) $\langle \mathbb{1}_{\{t\}}, \mu_s \rangle = 0$, $\forall s, t \in L_0$, $s \neq t$.

(iii) $\{\mu_t: t \in L_0\}$ weak*-converges to zero.

Let us take

$$B_t = \{p \in K: \mu_t(\{p\}) \neq 0\}.$$

According to (i) and (ii) we have $B_t \cap L_0 = \{t\}$, $\forall t \in L_0$. Moreover each B_t must be countable so $\bigcup\{B_t: t \in L_0\}$ is countable; then

(iv) $H = \{0, 1\}^\omega - (\bigcup\{B_t: t \in L_0\}) \neq \emptyset$.

Let s_0 be an element of H and let us consider the clopen set

$$C = \{s \in L: s \leqslant s_0\}.$$

Since $s_0 \in H$, according to (iv) we have

$$\langle \mathbb{1}_C, \mu_t \rangle = 1, \ \forall t \in C \cap L_0,$$

which contradicts (iii).

In order to see that K is a Rosenthal-compact space we will define a function $\varphi: K \to C(\Delta)$, where Δ stands for $\{0, 1\}^\omega$ endowed with the pointwise topology. If $\alpha \in L$ we take

$$\varphi(\alpha) = \mathbb{1}_{U(\alpha)} \text{ where } U(\alpha) = \{\beta \in \{0, 1\}^\omega: \alpha \leqslant \beta\}, \text{ and}$$
$$\varphi(\infty) = 0, \text{ the null function.}$$

If $\alpha \in L_0$, $\varphi(\alpha)$ is the characteristic function of a clopen set so $\varphi(\alpha)$ is continuous in $\{0, 1\}^\omega$ and $\varphi(\alpha) = \lim \varphi(\alpha |_n)$, for $\alpha \in \{0, 1\}^\omega$. Therefore every element of $\varphi(K)$ is a function of the first Baire class on Δ. Moreover, if $\varphi(K)$ is endowed with the topology of the pointwise convergence it is easy to check that φ is continuous. Since φ is injective and K is compact we have that K is homeomorphic to $\varphi(K)$ which shows that K is a Rosenthal-compact space.

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REMARK 9. This example gives a simple proof of the fact that c_0 is not complemented in ℓ_∞. Indeed, let us write $L_0 = \{x_n : n \in \mathbb{N}\}$. Since L_0 is dense in K the mapping $\varphi : C(K) \to \ell_\infty$ defined by $\varphi(f) = \{f(x_n)\}$ is an isometric embedding. Moreover we have that $\varphi(E) = c_0$. Then c_0 is not complemented in ℓ_∞ since E is not complemented in $C(K)$. (For other simple proofs see [8] and [13].) Since every ℓ_∞-valued continuous linear mapping defined in a subspace of a Banach space can be extended to a linear continuous mapping in the whole space, it is easy to deduce that there is no complemented copy of c_0 in ℓ_∞.

REMARK 10. In [10] p.945 it is asked if there exists a closed linear subspace S of ℓ_∞, $c_0 \subset S$, such that there is no projection of ℓ_∞ onto S, and no projection of S onto c_0. By means of the previous example it is easy to construct a subspace with these properties. Indeed, let φ be the mapping defined in Remark 9 and $\varphi(C(K)) = S$; it has been shown that $c_0 \subset S$ and c_0 is not complemented in S, so we have only to show that S is not complemented in ℓ_∞. By construction there are infinite convergent sequences in K so $C(K)$ is not a Grothendieck space [6]. Since φ is a linear isometry, S is not a Grothendieck space, therefore S is not complemented in ℓ_∞.

REMARK 11. If K is the Rosenthal-compact space constructed in Example 8, we have that (U^*, w^*), the unit ball of the dual of $C(K)$, is a Rosenthal-compact space [5], so it is a Fréchet topological space [2]. Therefore, according to Theorem 3, we have an example of a Fréchet compact space that is not cofinitely sequential. Moreover $C(K)$ is a Banach space with a non-complemented copy of c_0, such that the unit ball of its dual (U^*, w^*) is a Rosenthal-compact space; this fact shows that if Corson-compact is changed for Rosenthal-compact in Corollary 6, the assertion becomes false.

Let us recall that a projectional resolution of identity on a Banach space X is a set of projections $\{P_\alpha : \omega \leqslant \alpha \leqslant \mu\}$ where μ is the first ordinal whose cardinality equals the density character $\operatorname{dens}(X)$ of X, which satisfies:

(i) $\|P_\alpha\| = 1$, $\forall \alpha$.
(ii) $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$ if $\omega \leqslant \alpha \leqslant \beta \leqslant \mu$.
(iii) $\operatorname{dens}(P_\alpha(X)) \leqslant |\alpha|$, $\forall \alpha$.
(iv) $\bigcup\{P_{\beta+1}(X) : \beta < \alpha\}$ is dense in $P_\alpha(X)$.
(v) $P_\mu = \operatorname{Id}_X$.

THEOREM 12. *Assuming the Continuum Hypothesis, there exists a Rosenthal-compact space such that there is no projectional resolution of the identity in $C(K)$.*

PROOF: Let K be the Rosenthal-compact space constructed in Example 8. We will suppose there is a projectional resolution of the identity on $C(K)$ and obtain a contradiction. The density character in $C(K)$ is $\mathbf{c}$, the continuum. Therefore, assuming the continuum hypothesis, there exists a set of projections $\{P_\alpha : \omega \leqslant \alpha \leqslant \omega_1\}$ which

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satisfies the above conditions. In Example 8 a non-complemented copy E of c_0 was obtained. Let $\{e_n\}$ be a basis of this copy and $\{f_n\}$ linear functionals on E satisfying

$$x = \sum f_n(x_n)e_n, \ \forall x \in E.$$

According to (iv) and (v) there exist α_n, $\omega \leqslant \alpha_n \leqslant \omega_1$ and $a_n \in P_{\alpha_n}(C(K))$ such that

$$\|e_n - a_n\| < 2^{-(n+1)} \|f_n\|^{-1}.$$

So
$$\sum \|f_n\| \, \|e_n - a_n\| < 1.$$

Then $\{a_n\}$ is a basis equivalent to $\{e_n\}$ and its closed linear span $[a_n]$ is not complemented in $C(K)$ [7].

On the other hand, the supremum α of the sequence α_n must satisfy $\omega \leqslant \alpha < \omega_1$, so $a_n \in P_{\alpha_n}(C(K)) \subset P_\alpha(C(K))$ and $[a_n] \subset P_\alpha(C(K))$. Then $[a_n]$ is a copy of c_0 in the separable space $P_\alpha(C(K))$ so $[a_n]$ is complemented in $P_\alpha(C(K))$ [10]. Since $P_\alpha(C(K))$ is complemented in $C(K)$, $[a_n]$ must be complemented in $C(K)$, a contradiction. □

According to the result of Sobczyk, whenever c_0 is a closed linear subspace of a Banach space X with countable density character (that is, separable), c_0 is complemented in X [10]. On the other hand, c_0 is not complemented in ℓ_∞ [9] and $\text{dens}(\ell_\infty) = \mathbf{c}$, the continuum. Then the following question arises: Which are the cardinal numbers α such that c_0 is complemented in every Banach space X, containing c_0 as a closed subspace and $\text{dens}(X) = \alpha$? According to the previous remarks such a cardinal α satisfies $\omega \leqslant \alpha < \mathbf{c}$, but this does not give a complete answer unless we assume the continuum hypothesis. In the following example we will show that there exists no cardinal α with this property such that $\omega < \alpha < \mathbf{c}$.

EXAMPLE 13. Let ρ be a cardinal number such that $\omega < \rho \leqslant \mathbf{c}$. Let A be a subset of $\{0, 1\}^\omega$, $|A| = \rho$, and let us consider

$$M = [\bigcup\{D(\alpha) \colon 0 \leqslant \alpha < \omega\}] \cup A, \ \text{ where } D(\alpha) = \{0, 1\}^\alpha.$$

Thus M is included in L of Example 8, and we will consider M endowed with the subspace topology, for which M is scattered and locally compact. Let K_0 be the Alexandroff compactification of M. Then $\text{dens}(C(K_0)) = \rho$ and we can construct a non-complemented copy of c_0 in $C(K_0)$. Indeed, a copy of c_0 is spanned by $\{\mathbb{1}_{\{t\}} : t \in L_0\}$ where L_0 was defined in Example 8. Reasoning as in Example 8, we obtain that this copy is not complemented.

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