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Level Zero **G**-Types ⋆

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Abstract. Let **G** be a connected reductive group defined over a local non-Archimedean field F with residue field $\mathbb{F}_q$; let P be a parahoric subgroup with associated reductive quotient $\mathbb{M}$. If σ is an irreducible cuspidal representation of $\mathbb{M}(\mathbb{F}_q)$ it provides an irreducible representation of P by inflation. We show that the pair (P, σ) is an $\mathfrak{S}$-type as defined by Bushnell and Kutzko. The cardinality of $\mathfrak{S}$ can be bigger than one; we show that if one replaces P by the full centraliser $\hat{P}$ of the associated facet in the enlarged affine building of G, and σ by any irreducible smooth representation $\hat{\sigma}$ of $\hat{P}$ which contains σ on restriction then $(\hat{P}, \hat{\sigma})$ is an $\mathfrak{s}$-type for a singleton set $\mathfrak{s}$. Our methods employ invertible elements in the associated Hecke algebra $\mathcal{H}(\sigma)$ and they imply that the appropriate parabolic induction functor and its left adjoint can be realised algebraically via pullbacks from ring homomorphisms.

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Introduction

Let $G = \mathbf{G}(F)$ be the group of rational points of a connected reductive group defined over a local non-Archimedean field F. Let $\mathcal{B}(G)$ be the set of classes of irreducible supercuspidal representations of rational Levi components of rational parabolic subgroups of G under the equivalence relation arising from G-conjugation and twisting by unramified quasicharacters of Levi components. If π is an irreducible representation of G, then it determines a unique element of $\mathcal{B}(G)$ which we denote by $\mathcal{I}(\pi)$ and call the *inertial equivalence class of* π. (This notation and definition is taken from [BK2].)

Now let $\mathfrak{S}$ be a subset of $\mathcal{B}(G)$. In [BK2] the authors define the notion of an $\mathfrak{S}$-type. This is an ordered pair (K, ρ) where K is a compact open subgroup of G and ρ is an irreducible smooth representation of K with the following property: an irreducible smooth representation π of G contains ρ if and only if the inertial equivalence class $\mathcal{I}(\pi)$ of π belongs to $\mathfrak{S}$. The authors show that $\mathfrak{S}$-types have many remarkable properties. In particular, if V denotes the space of ρ and $(\pi, \mathcal{V})$

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is a smooth representation of G let $\mathcal{V}_\rho = \mathrm{Hom}_K(V, \mathcal{V})$. Then the functor $\mathcal{V} \mapsto \mathcal{V}_\rho$ induces an equivalence of categories between the category $\mathfrak{S}\mathfrak{R}_\rho(G)$ of smooth representations of G generated by their ρ-isotypic component and the category of (unital) $\mathcal{H}(\rho)$-modules. Here $\mathcal{H}(\rho) = \mathcal{H}(G, \rho)$ denotes the algebra of $\check{\rho}$-spherical functions on G with compact support, where $\check{\rho}$ is the contragredient of ρ. The prototype (due to Borel [B]) of all $\mathfrak{S}$-types is the pair $(B, 1)$ where B is the centraliser of an alcove of the enlarged building for G, and 1 denotes the trivial representation of B. (In general the full centraliser of an alcove may be larger than the Iwahori subgroup that it contains.) Bushnell and Kutzko provide many other non trivial examples of $\mathfrak{S}$-types in [BK2] arising from their work on $\mathbb{GL}_n$ and $\mathbb{SL}_n$.

The prototype in the preceding paragraph can be generalised substantially in the following manner. Let P be a parahoric subgroup of G with reductive quotient M, and let σ be an irreducible cuspidal representation of M. One can view σ as a representation of P by inflation. Theorem 4.8 of this paper asserts that the pair (P, σ) is an $\mathfrak{S}$-type where $\mathfrak{S}$ is a finite set; in fact $\mathfrak{S} = \{[L, \rho_1], \dots, [L, \rho_n]\}$ where $L = \mathbb{L}(F)$ is the group of rational points of a canonically chosen Levi component. We remark that the number n can be larger than 1. The proof proceeds by associating to P a Levi component $\mathbb{L}$ in a canonical way; in fact if $\mathbb{P}$ is any parabolic subgroup containing $\mathbb{L}$ with a Levi decomposition $\mathbb{P} = \mathbb{L} \cdot \mathbb{U}$ there is an Iwahori decomposition $(P \cap \mathbb{U}(F)) \times (P \cap L) \times (P \cap \mathbb{U}^-(F)) \to P$. Further, $L \cap P$ is a maximal parahoric subgroup of L. The proof of Theorem 4.8 then depends on the following facts:

(i) Any irreducible smooth representation of G which contains a cuspidal representation of a maximal parahoric subgroup must be supercuspidal, and induced from an open, compact mod centre subgroup of G. (See Proposition 4.1.)

(ii) The intertwining algebra $\mathcal{H}(G, \sigma)$ contains invertible elements which are supported on double cosets PdP where d is *strongly* $(P, \mathbb{P})$ *-positive*. This is pointed out in Sections 2.4 and 3.3.

(iii) There is an isomorphism of isotypic components $\mathcal{V}^\sigma \to (\mathcal{V}_{\mathbb{U}})^{\sigma_{\mathbb{U}}}$, for any smooth representation $(\pi, \mathcal{V})$ which contains the type σ. Here as usual $\mathcal{V}_{\mathbb{U}}$ denotes the (unnormalised) Jacquet module. This is pointed out in Lemma 3.6; to prove it one uses property (ii) above. We remark that results of this sort go back to Jacquet; see [Cs].

As a variation, let $\hat{P}$ be the full centraliser in G of the facet associated to P, and $\hat{\sigma}$ be an irreducible representation of $\hat{P}/U$ which contains σ. We show that $(\hat{P}, \hat{\sigma})$ is an $\mathfrak{s}$-type for a singleton set $\mathfrak{s}$; see 4.7, 4.9. Lemma 3.9 is the vehicle we use to prove this; it is of independent interest. (See also the remark following Theorem 4.9.)

Property (iii) above has another consequence. We have the algebras $\mathcal{H}(G, \sigma)$, $\mathcal{H}(L, \sigma_{\mathbb{U}})$, and their respective categories of unital modules. On the other hand we have the categories $\mathfrak{S}\mathfrak{R}_\sigma(G)$, and $\mathfrak{S}\mathfrak{R}_{\sigma_\mathbb{U}}(L)$, and the categorical equivalences mentioned above. Theorem 4.8 implies that the (unnormalised) induction functor and (the projection of) the Jacquet functor provide adjoint functors between $\mathfrak{S}\mathfrak{R}_\sigma(G)$, and $\mathfrak{S}\mathfrak{R}_{\sigma_\mathbb{U}}(L)$. (See Theorem 4.10.) Theorem 4.12 says that these functors can be realised algebraically via (pullbacks of) a ring homomorphism from $\mathcal{H}(L, \sigma_\mathbb{U})$ to $\mathcal{H}(G, \sigma)$. This amounts to showing that one can apply Corollary 8.4 of [BK2].

Corrigenda

(i) The group denoted by H in [M]3.15 and elsewhere in that paper, should be replaced by the group $H = \ker \nu'$ in 1.6 below.

(ii) Contrary to what is asserted in *op. cit.* 3.15 the group $\mathfrak{M}' \cap P$ need not be special in $\mathfrak{M}'$; see 1.7 below. This does not affect the proofs. In particular, in *op. cit.* 4.14 the subgroup $\mathcal{M}_J$ need not be special.

Notation and Conventions

F: complete non-Archimedean field;
$\mathfrak{o}$: ring of integers of F;
$\mathfrak{p}$: maximal ideal of $\mathfrak{o}$;
$\mathbb{F}_q$: residue field $\mathfrak{o}/\mathfrak{p}$ ($q = p^n$, where p is some prime number);
$\mathbf{G}$: connected reductive F-group;
$\mathbf{Z}$: maximal F-split torus in the centre of $\mathbf{G}$;
$\mathbf{T}$: maximal F-split torus in $\mathbf{G}$;
$\mathbf{Z_G}(X)$ (resp. $\mathbf{N_G}(X)$): centraliser (resp. normaliser) in $\mathbf{G}$ of X.

In general if $\mathbf{V}$ is an algebraic F-variety we shall write V for the set $\mathbf{V}(F)$; *we make an exception for parabolic subgroups and their unipotent radicals*:

Remark. In this paper, the expression ‘parabolic subgroup’ will always mean ‘F-parabolic subgroup’. If $\mathbb{P}$ is such a group with unipotent radical $\mathbb{U}$ we shall write $\mathbb{P}(F)$, $\mathbb{U}(F)$ respectively for their F-rational points. If $\mathbb{L}$ is a Levi component for $\mathbb{P}$, we shall write $L = \mathbb{L}(F)$. We remark that all Levi decompositions will be assumed to be defined over F.

In fact, we shall write P, Q, etc., for parahoric subgroups of $G = \mathbf{G}(F)$.

Other notation is explained as needed.

1. Preliminaries

1.1. We begin with a quick review of the relevant aspects of the theory of reductive groups. Thus let $\mathbf{G}$ denote a connected reductive group defined over F and let Φ be the set of relative roots with respect to some maximal F-split torus $\mathbf{T}$; when

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necessary we shall write Δ for the set of simple roots corresponding to the choice of a minimal parabolic subgroup $\mathbb{P}_0$.

1.2. Let $\mathbb{P}$ be a parabolic subgroup with a Levi decomposition $\mathbb{P} = \mathbb{L} \cdot \mathbb{U}$.

THEOREM. (i) *There is a unique parabolic subgroup* $\mathbb{P}^-$ *containing* $\mathbb{L}$ *with Levi decomposition* $\mathbb{P}^- = \mathbb{L} \cdot \mathbb{U}^-$ *with the property that* $\mathbb{U} \cap \mathbb{U}^- = \{1\}$.

(ii) *Let* $\mathbb{P}^-$ *be as in* (i). *There is an* F*-isomorphism of varieties* $\mathbb{U}^- \times \mathbb{L} \times \mathbb{U} \to \mathbb{P}^- \cdot \mathbb{P}$ *induced by the multiplication map*; *the image is a Zariski open subset in* $\mathbf{G}$.

Proof. Except for the F-statements, (i) and (ii) are contained in [Bo] 14.21. If $\mathbb{P}$ is defined over F so is $\mathbb{P}^-$ ([Bo] 20.5). The multiplication map is defined over F, so the rest of (ii) follows since the image of an F-morphism is an F-variety ([Bo] AG14.3). $\square$

DEFINITION 1.3. We shall call the group $\mathbb{P}^-$ *the opposite parabolic subgroup to* $\mathbb{P}$ (with respect to $\mathbb{L}$).

PROPOSITION 1.4. *If* $\mathbf{S}$ *is an* F*-split subtorus of* $\mathbf{T}$ *then* $\mathbf{Z_G}(\mathbf{S})$ *is the Levi component of a parabolic subgroup of* $\mathbf{G}$.

Proof. This is Proposition 20.4 of [Bo]. $\square$

1.5. We take $\mathbf{T}$, Φ, $\mathbb{P}_0$, Δ just as above, and we write ${}^vW$ for the spherical Weyl group of the root system Φ. Let Σ be the set of affine roots associated to a reduced root system ${}^v\Sigma$ in the same real vector space as the root system Φ with affine Weyl group W'. We assume that ${}^v\Sigma$ and Φ have the same Weyl group. This is equivalent to assuming that if $\alpha \in \Phi$ then $\lambda(\alpha)\alpha \in {}^v\Sigma$ for a positive real number λ and that the map $\alpha \to \lambda(\alpha)\alpha$ is onto. A typical element a of Σ can be written as $Da + k$ where $Da \in {}^v\Sigma$ and k is an integer; we refer to Da as the *gradient* of a. There is also a homomorphism $D: W' \to {}^vW$.

DEFINITION. An *échelonnage* $\mathcal{E} \subset \Phi \times \Sigma$ of Φ by Σ is a subset which satisfies the following properties:

(E1) if $(\alpha, a) \in \mathcal{E}$ then α is a positive multiple of Da;
(E2) if $w \in W'$ and $(\alpha, a) \in \mathcal{E}$ then $(Dw(\alpha), wa) \in \mathcal{E}$;
(E3) the projection maps from $\mathcal{E}$ to Φ, Σ are onto.

Remarks. (i) If $(\alpha, a) \in \mathcal{E}$ we say that α, Da are *associated*.

(ii) Let Φ_{nd} denote the set of non-divisible roots in Φ. Then (E1) and (E3) imply that there is a bijection $\rho: {}^v\Sigma \to \Phi_{nd}$ such that $\alpha = \mu_\alpha \rho(\alpha)$ with $\mu_\alpha > 0$.

1.6. Now we quickly review some aspects of Bruhat–Tits theory; as a general reference we suggest [T]. The group $G = \mathbf{G}(F)$ is naturally furnished with the structure of a second countable locally compact Hausdorff totally disconnected group ($=$ $t.d.$ *group*, in brief). The work of Bruhat and Tits associates to $(\mathbf{G}, \mathbf{T})$ an échelonnage $\mathcal{E} \subset \Phi \times \Sigma$. We remind the reader that the ambient vector space

V on which the roots in either Φ or ${}^{v}\Sigma$ act as functions is the real dual of the subspace of $X_F(\mathbf{T})\otimes\mathbb{R}$ generated by Φ, where $X_F(\mathbf{T})$ denotes the lattice of rational characters. In turn, from this and a choice of simple affine roots in Σ one obtains a normal subgroup G' in G, a compact open subgroup B in G' and a subgroup $N' = N'_{G'}(T)$ in G', and a set of reflections S in W' such that (G', B, N', S) is an affine Tits system with respect to the system Σ. (For the definition of G' see [BT2]5.2.11.) In particular there is a surjection $\nu'\colon N' \to W'$. We denote the kernel of ν' by H; it is a compact normal subgroup of the group $\mathbf{Z_G}(\mathbf{T})(F)$. We note that $N' \subseteq N = N_G(T)$, and the triple (G, B, N) is a *generalised* affine BN-pair in the sense of [M]3.2. The generalised affine Weyl group here is the quotient $W = N/H$; we write $\nu_W\colon N \to W$ for the natural projection.

Any subgroup conjugate in G (or G') to B is called an *Iwahori* subgroup of G. The affine Tits system (G', B, N', S) gives rise to a polysimplicial complex on which G and G' act, preserving the simplicial structure. The geometric realisation of this complex is called the *affine building associated to* (G', B, N', S); we denote it by $\mathcal{I}$. In fact $\mathcal{I}$ is obtained by pasting together copies (called *apartments*) of an affine Euclidean space A whose underlying space of translations is V above. The points of A correspond to *valuations* of $(G, Z_G(T), (U_\alpha)_{\alpha\in\Phi})$. For more on this see [BT1]6.2. In particular, A embeds into $\mathcal{I}$. The group N' acts on A as a group of affine automorphisms with kernel H. Furthermore, the affine root system Σ partitions A in the usual way into *facets*; it is this partition which gives rise to the underlying simplicial structure of $\mathcal{I}$. Thus the facets of A are facets of $\mathcal{I}$ and any facet of $\mathcal{I}$ is a translate by an element of G' of a facet of A. We remark that the choice of a different apartment amounts to choosing a different $\mathbf{T}$; the resulting $\mathcal{E}$ is the same. The G'-centralisers of facets in $\mathcal{I}$ are called *parahoric subgroups*; in particular the centralisers of chambers (facets of maximal dimension) in $\mathcal{I}$ are conjugates of B. Any parahoric subgroup is a compact open subgroup of G. See [BT1] 6.2, 6.5, and Section 2. Finally we have $H = B \cap N'$.

WARNING. *The subgroup H that we employ here is* not *the H employed in* [BT1, BT2]. *The subgroup that we denote by H is denoted by H^0 in* [BT2]4.6.3(4), *or by* $\mathfrak{Z}^o(\mathcal{O}^\natural)$ *in ibid.* 5.2.10.

For many purposes the *enlarged building* $\mathcal{I}^1$ is a more convenient object; in particular it guarantees that the centralisers in G of facets will be compact open subgroups of G. It is defined as follows. Let V^1 denote the dual of $X_F(\mathbf{G}) \otimes \mathbb{R}$ where as usual $X_F(\mathbf{G})$ is the group of rational characters of $\mathbf{G}$. Then $\mathcal{I}^1 = \mathcal{I} \times V^1$ and the action of G on $\mathcal{I}$ (which we have not explicitly defined) is extended to one on $\mathcal{I}^1$ by defining $\theta\colon G \times V^1 \to V^1$ by $\theta(g)(\chi) = -\omega(\chi(g))$, for all $\chi \in X_F(\mathbf{G})$.

We identify $\mathcal{I}$ with $\mathcal{I} \times \{0\}$, and we write G^1 for the stabiliser in G of this set. A facet $\mathcal{F}$ in $\mathcal{I}$ corresponds to a facet $\mathcal{F}^1 = \mathcal{F} \times V^1$ in $\mathcal{I}^1$. We write $\hat{P}_{\mathcal{F}}$ for the centraliser in G of the facet $\mathcal{F}^1 \subseteq \mathcal{I}^1$. It is also the centraliser in G^1 of $\mathcal{F}^1$, and it is the centraliser in G of the 'facet' $\mathcal{F} \times \{0\}$ in $\mathcal{I}^1$. We always have $G' \subseteq G^1$; if G is semisimple we have $G = G^1$, and $\mathcal{I} = \mathcal{I}^1$. We note that the group

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$G' \subseteq G = \mathbf{G}(F)$ is the subgroup of G generated by the connected centralisers (= parahoric subgroups) of facets of the enlarged building $\mathcal{I}^1$.

We have $G/G' = N/N'$. We set $\Omega = N/N'$. Let $W = \mathbf{N_G}(\mathbf{T})(F)/H$ be the full affine Weyl group associated to $(\mathbf{G}, \mathbf{T})$. It is a semidirect product $W' \rtimes \tilde{\Omega}$ where $\tilde{\Omega}$ is the subgroup of elements stabilising some specified alcove in $\mathcal{I}^1$; in particular under the obvious projection map $W \to W/W'$ the group $\tilde{\Omega}$ maps isomorphically to Ω. The group N acts on A by affine transformations; this defines a map $\nu: N \to \mathrm{Aff}(A)$ (as in [BT1]), which factors through ν_W. Indeed the generalised affine Weyl group W is an extension of ${}^{v}W$ by the group $D = \mathbf{Z_G}(\mathbf{T})(F)/H$; here D is an extension of the lattice $\mathbf{Z_G}(\mathbf{T})(F)/\ker \nu_W$ by the finite Abelian group $\ker \nu_W/H$.

1.7. The choice of B amounts to choosing a set of simple affine roots Π in Σ, and one can attach a *local Dynkin diagram* to this in a way similar to the usual case of ordinary root systems. For example if G is split this diagram is just the usual completed Dynkin diagram. If $\mathcal{F}$ is a facet in $A \subset \mathcal{I}$ we take the set $\Sigma_{\mathcal{F}}$ of affine roots vanishing on $\mathcal{F}$. The set of roots $\Phi_{\mathcal{F}} \in \Phi$ associated to this set is a not necessarily closed subroot system of Φ: for example, if $\alpha \in \Phi_{\mathcal{F}}$ it need not be the case that $2\alpha \in \Phi_{\mathcal{F}}$; we denote its closure by ${}^{c}\Phi_{\mathcal{F}}$. We remark that it can happen that ${}^{c}\Phi_{\mathcal{F}} = \Phi$ if $\mathcal{F}$ is a nonspecial vertex, even if Φ is reduced; if $\mathbf{G}$ is split this does not occur. In particular, let $\mathcal{F}$ be a facet in the closure of the chamber (alcove) corresponding to B. Then $\mathcal{F}$ also corresponds to a subset $J = J_{\mathcal{F}} \subseteq \Pi$ giving rise to a *finite* reflection group W_J and a subset of Π; the group W_J is generated by the fundamental reflections associated to the elements of J. Then W_J is the Weyl group for $\Phi_{\mathcal{F}}$ (but not necessarily ${}^{c}\Phi_{\mathcal{F}}$), and the Dynkin diagram for $\Phi_{\mathcal{F}}$ is obtained from the local Dynkin diagram by striking out all the nodes not corresponding to elements of J and all edges meeting such a node. Each of these objects only depend on $\mathcal{F}$; we sometimes write Φ_J instead. See [T] Section 1 and [BT1] 6.2,6.4.

1.8. The root system Φ_J has the following interpretation. Let P be the parahoric subgroup centralising the facet $\mathcal{F}$. There is a short exact sequence $0 \to U \to P \to M \to 0$ where U is an open compact pro-p subgroup of G and M is the group of $\mathbb{F}_q$-rational points of a connected $\mathbb{F}_q$-reductive group $\mathbb{M}$. There is an obvious $\mathfrak{o}$-split torus scheme $\mathcal{T}$ whose generic fibre is $\mathbf{T}$ and whose reduction mod $\mathbb{F}_q$ gives a maximal $\mathbb{F}_q$-split torus $\mathbb{T}$ in $\mathbb{M}$. The root system for $\mathbb{M}$ with respect to $\mathbb{T}$ is then just Φ_J. See [T] 3.5.1.

1.9. The structure of P can be described more precisely as follows. First, for any element α of Φ_{nd} let $a(\alpha, \mathcal{F})$ be the smallest affine root which is nonnegative on $\mathcal{F}$ and which corresponds to α by the map in 1.5: i.e. $\rho(Da(\alpha, \mathcal{F})) = \alpha$. For each affine root a with $\rho(Da) = \alpha$ there is a compact open subgroup U_a of $U_\alpha = \mathbb{U}_\alpha(F)$. Let $U^+(\mathcal{F})$ be the group generated by all the $U_{a(\alpha,\mathcal{F})}$ for $\alpha \in \Phi^+_{nd} = \Phi_{nd} \cap \Phi^+$ and define $U^-(\mathcal{F})$ in a similar way. Here Φ^+ denotes the set of positive roots with respect to Δ. Finally let $N'(\mathcal{F})$ be the subgroup in N' which fixes $\mathcal{F}$ pointwise.

THEOREM. (i) *The product map* $\prod_{\alpha\in\Phi^+_{nd}} U_{a(\alpha,\mathcal{F})} \to U^+(\mathcal{F})$ *is bijective for every ordering of the factors, and similarly for* $U^-(\mathcal{F})$.

(ii) $P_{\mathcal{F}} = U^-(\mathcal{F})\cdot U^+(\mathcal{F})\cdot N'(\mathcal{F})$.

(iii) *If* $\mathcal{F}$ *is a chamber, the product map* $\prod_{\alpha\in\Phi_{nd}} U_{a(\alpha,\mathcal{F})}\times H \to B$ *is a bijection for every ordering of the product.*

(iv) *Let* U *be as in* 1.8. *For each* $U_{a(\alpha,\mathcal{F})}$ *as above let* $U^*_{a(\alpha,\mathcal{F})} = U_{a(\alpha,\mathcal{F})}\cap U$ *and let* $H^* = H\cap U$. *Then the product map* $\prod_{\alpha\in\Phi_{nd}} U^*_{a(\alpha,\mathcal{F})}\times H^* \to U$ *is a bijection for every ordering of the product.*

Proof. Statements (i) and (ii) are proved in 6.4.9 and 7.1.8 of [BT1]. Statement (iii) is proved in 6.4.48 of *op. cit.* Statement (iv) also follows from that result, on using the concave function $f^*_{\mathcal{F}}$ of 6.4.23 of *op. cit.* □

1.10. Let $P = P_J$ be as in 1.8. There are then the subroot systems $\Phi_J \subset {}^c\Phi_J \subset \Phi$. Since ${}^c\Phi_J$ is closed there is a connected reductive F-subgroup $\mathfrak{M}\subset \mathbf{G}$ containing $\mathbf{T}$ and which has the relative root system ${}^c\Phi_J$ with respect to $\mathbf{T}$. Indeed this group is generated by those root groups $\mathbb{U}_\alpha$ with $\alpha\in {}^c\Phi_J$, and by $\mathbf{T}$. (One may have $\mathfrak{M} = \mathbf{G}$ when P is maximal but not special.) We let $\mathfrak{M}'\subseteq \mathfrak{M}(F)$ be the subgroup generated by the $\mathbb{U}_\alpha(F)$ with $\alpha\in {}^c\Phi_J$, and by H.

PROPOSITION. (i) *If* $P = P_{\mathcal{F}}$ *for a facet* $\mathcal{F}$ *in the apartment* A *with respect to* $\mathbf{T}$ *then* $P\cap\mathfrak{M}'$ *is a parahoric subgroup of* $\mathfrak{M}$, *and there is a short exact sequence* $0\to U\cap\mathfrak{M}'\to P_{\mathcal{F}}\cap\mathfrak{M}'\to M\to 0$.

(ii) *Similarly, if* $\hat{P} = \hat{P}_{\mathcal{F}}$ *for a facet* $\mathcal{F}$ *in the apartment* A *with respect to* $\mathbf{T}$ *then* $\hat{P}\cap\mathfrak{M}'$ *centralises a facet for* $\mathfrak{M}$, *and there is a short exact sequence*

$$0\to U\cap\mathfrak{M}'\to \hat{P}_{\mathcal{F}}\cap\mathfrak{M}'\to \hat{M}\to 0.$$

Proof. Let $\mathfrak{M}'_0$ be the group generated by the U_α with $\alpha\in{}^c\Phi_J$. Then $\mathfrak{M}' = H\cdot\mathfrak{M}'_0$. Taking the valuated root system $(\varphi_\alpha)_{\alpha\in\Phi}$ that gives rise to the affine Tits system (G', B, N', S) and applying [BT1] 7.6.3 (see also 1.12 below) to the groups $G_1 = \mathfrak{M}'$ and $G^0_1 = \mathfrak{M}'_0$ we see that we obtain a valuated root system on $\mathfrak{M}'$. Now observe that the group T_1 of *loc. cit.* is just the group $H\cdot(\mathbf{Z}_{\mathbf{G}}(\mathbf{T})(F)\cap\mathfrak{M}'_0)$. In particular this enables us to apply Corollary 7.6.5 of *op. cit.*, which implies (i), and the proof of (ii) is similar. □

1.11. There is a bijective correspondence between parahoric subgroups contained in P and ($\mathbb{F}_q$-rational points of) parabolic subgroups of the group $\mathbb{M}$. This correspondence is realised by $Q\mapsto U\backslash Q$. This is part (i) of Proposition 5.1.32 of [BT2].

1.12. We conclude this section by comparing parahoric subgroups of a Levi component $L = \mathbb{L}(F)$ (as in 1.2) with parahoric subgroups of G. Let $\mathbb{L}$ be a Levi component of $\mathbf{G}$ defined by some subset Θ of the set of simple roots Δ as in 1.3. Thus $\mathbb{L} = \mathbf{Z}_{\mathbf{G}}(\mathbf{S})$ where $\mathbf{S} = \bigcap_{\alpha\in\Theta}\ker\alpha$. The set Θ is a basis for a closed subroot system $\Phi_{\mathbb{L}}$; indeed this last is the root system for $(\mathbb{L},\mathbf{T})$. Let L', L^1 be the analogues for L of G', G^1, and let L^0 be the subgroup of $\mathbb{L}$ generated by the root groups $\mathbb{U}_\alpha(F), \alpha \in \Phi_{\mathbb{L}}$. Then $L' = H \cdot L^0$ and $L^1 = (Z_G(T) \cap L^1)L^0$. For the time being let L_1 be any subgroup of L which is generated by L^0 and a subgroup $Z_G(T)^1 \subseteq Z_G(T)$ which contains $Z_G(T) \cap L^0$. According to [BT1]7.6.3 if $\varphi = (\varphi_\alpha)_{\alpha\in\Phi}$ is a valuation for $(G, Z_G(T), (U_\alpha)_{\alpha\in\Phi})$ then $\varphi_{L_1} = (\varphi_\alpha)_{\alpha\in\Phi_{\mathbb{L}}}$ is a valuation for $(L_1, Z_G(T)^1, (U_\alpha)_{\alpha\in\Phi_{\mathbb{L}}})$.

We write $V(L), A(L), \mathcal{I}(L), \dots$, etc. to denote the corresponding objects for L that have been defined previously for G. We also let $V^1(L) = \bigcap_{\alpha\in\Theta} \ker \alpha$, where now the intersection is taken in the vector space V of 1.6, and we define $V_1(L) = V/V^1(L)$; this last space can be identified with $V(L)$. In particular there is a natural map ${}^v\pi\colon V \to V(L)$. If we suppose that a ${}^vW$-invariant inner product has been chosen on V with orthogonal projection $p\colon V \to V^1(L)$ then $V(L)$ can be identified with $\ker p$. As before, we can form the buildings $\mathcal{I}(L), \mathcal{I}(L_1)$; as complexes these are the same, with the action of L extending that of L_1. We then have the following facts.

(i) Let π be the map $A \to A(L)$ defined by $\varphi + v \to \varphi_{L_1} +^v \pi(v)$. Proposition 7.6.4 in [BT1]) says that

(a) there is a unique L_1-equivariant map $\tilde{\pi}\colon L_1 \cdot A \to \mathcal{I}(L_1)$ extending π; the inverse image of an apartment, half-apartment, wall, is an apartment, half-apartment, wall in $\mathcal{I}$;

(b) there is a unique action $V^1(L) \times L_1 \cdot A \to L_1 \cdot A$ extending the action $V^1(L) \times A \to A$; this action factors through $\tilde{\pi}$ and the quotient map defines a bijection $(L_1 \cdot A)/V^1(L) \to \mathcal{I}(L_1)$.

Note that $L_1 \cdot A$ has the structure of a polysimplicial complex, inherited from that of $\mathcal{I}$. The definition of affine roots for $\mathcal{I}(L_1)$ implies that if $\mathcal{F}$ is a facet in $L_1 \cdot A$ then $\tilde{\pi}(\mathcal{F})$ lies in a unique facet, but this image is not necessarily a facet.

(ii) Let $\Omega \subseteq L_1 \cdot A \subseteq \mathcal{I}$; write $\hat{P}_\Omega$ for the pointwise centraliser of Ω. Then [BT1]7.6.5 says in particular that

(a) $\hat{P}_\Omega \cap L_1 \subseteq \hat{P}_{\tilde{\pi}(\Omega)}$ (the pointwise centraliser in L_1 of $\tilde{\pi}(\Omega)$), and

(b) if the subgroup $Z_G(T)^1$ is contained in $\ker(p \circ \nu)$ where $\nu\colon N_G(T) \to \mathrm{Aff}(A)$ then $\hat{P}_\Omega \cap L_1 = \hat{P}_{\tilde{\pi}(\Omega)}$.

(iii) Now choose a point $\varphi \in A$ and consider the affine subspace $\varphi + \ker p$. We can then form $\mathcal{I}' = L^0 \cdot (\varphi + \ker p) \subseteq \mathcal{I}$ since $L^0 \subseteq G$. According to [BT2]4.2.17,

(a) the restriction of $\tilde{\pi}$ in (i)(a) provides an L^0-equivariant isometry $\pi'\colon \mathcal{I}' \to \mathcal{I}(L)$ extending the map $\varphi + \ker p \to A(L)$;

(b) the inverse j of π' provides a bijection $(y, v) \mapsto j(y) + v$ from $\mathcal{I}(L) \times V^1(L)$ to $L \cdot A$;

(c) there is a homomorphism $\theta(L_1)\colon L_1 \to V^1(L)$ such that for any $\ell \in L_1$, $y \in \mathcal{I}(L)$, $v \in V^1(L)$, $\ell \cdot (j(y) + v) = j(\ell \cdot y) + v + \theta(L_1)(\ell)$, and $\theta(L_1)|Z_G(T) = p \circ \nu$; $\theta(L_1)|L^0 = 0$.

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The affine subspace $\varphi + \ker p$ inherits a polysimplicial decomposition from A. We note that the isometries π', j take facets to facets.

(iv) Taking $L_1 = L$ in (iii) and applying the definitions of $\mathcal{I}^1$, $\mathcal{I}(L)^1$ one deduces *op. cit.* 4.2.18 that $\mathcal{I}(L)^1$ can be isometrically identified with the smallest subset of $\mathcal{I}^1$ stable by L and containing the 'enlarged' apartment $A \times V^1$. Under this identification the map θ_L for the enlarged building $\mathcal{I}(L)^1$ which corresponds to the map θ in 1.6 for G, is given by $\theta(L) + \theta$; thus $L^1 = \ker\theta(L) \cap \ker\theta$.

LEMMA 1.13. *Let $\mathcal{F}$ be a facet in $L \cdot A$, $\hat{P} = \hat{P}_{\mathcal{F}}$ and $P \subseteq \hat{P}$ the corresponding parahoric subgroup. Then*

(i) *$\hat{P} \cap L^1 = \hat{P} \cap L$ is the centraliser of a facet in $\mathcal{I}(L)^1$;*
(ii) *$P \cap L' = P \cap L$ is a parahoric subgroup in L.*

Proof. From 1.12(i) we see that $\tilde{\pi}(\mathcal{F})$ lies in a facet $\mathcal{F}_L$ in $\mathcal{I}(L)$ and $\mathcal{F}_L$ identifies with a facet $j(\mathcal{F}_L)$ in the complex $\mathcal{I}'$ of 1.12(iii). Applying 1.12(ii) to the group L' we see that if $P = P_{\mathcal{F}}$ is a parahoric subgroup in G' then $P \cap L'$ is the parahoric subgroup in L' for the facet $j(\mathcal{F}_L)$. Now, $P \cap L \subseteq \ker\theta(L) \cap \ker\theta$ by 1.12(iii)(c), hence $P \cap L \subseteq \hat{P}_{j(\mathcal{F}_L)\times\{0\}\times\{0\}}$. But this last group only differs from its connected component by elements of $(Z_G(T) \cap L^1) - H \subseteq Z_G(T) - H$ and these cannot lie in P in any case. This proves (ii).

Applying 1.12(i)–(iii) in a similar way to the group L^1 we see that if $\hat{P} \subset G^1$ fixes pointwise a facet in $L \cdot A$ then $\hat{P} \cap L^1$ is the full centraliser in L^1 of a facet in $\mathcal{I}(L^1) = \mathcal{I}(L)$, hence the centraliser in L^1 of a facet in $\mathcal{I}(L^1) = \mathcal{I}(L)$. Thus it is the centraliser in L of a facet in $\mathcal{I}(L)^1$. $\square$

2. Parabolics and Parahorics

2.1. We now fix a facet $\mathcal{F} \subseteq A$ and let $P = P_{\mathcal{F}}$ be the associated parahoric subgroup, with corresponding short exact sequence $0 \to U \to P \to M \to 0$ as in 1.8, and associated root system $\Phi_J = \Phi_{\mathcal{F}}$. As in 1.6 we write $\hat{P}$ for the full centraliser in G^1 of $\mathcal{F}$. We remark that P is the group of integral points of the connected component of a smooth $\mathfrak{o}$-group scheme $\hat{\mathcal{P}}$ such that $\hat{\mathcal{P}}(\mathfrak{o}) = \hat{P}$. There is an exact sequence $0 \to U \to \hat{P} \to \hat{M} \to 0$, where $\hat{M}$ is the group of rational points of a reductive $\mathbb{F}_q$-group $\hat{\mathbb{M}}$, and the group denoted $\mathbb{M}$ in 1.8 is the identity component of $\hat{\mathbb{M}}$.

Recall the group $\mathfrak{M}$ in 1.10; it has a centre containing an F-split component $\mathbf{S}$. The centraliser of $\mathbf{S}$ is a (connected) reductive F-group $\mathbb{L}$. Note that $\mathbf{S}$ is the F-split component of $\mathbf{Z}_{\mathbb{L}}$: we have $\mathbb{L} \supset \mathfrak{M}$ so the F-split component of $\mathbf{Z}_{\mathbb{L}}$ centralises $\mathfrak{M}$ and contains $\mathbf{S}$, hence it must be $\mathbf{S}$. Moreover, $\mathbf{S} = \mathbf{Z}(\mathbf{G})$ if and only if P is maximal.

THEOREM. *The group $\mathbb{L}$ is the Levi component of a parabolic subgroup $\mathbb{P} = \mathbb{L}\cdot\mathbb{U}$; set $L = \mathbb{L}(F)$. Furthermore, the following properties hold.*

(i) $\hat{P} \cap L = \hat{Q}$ *is the centraliser in* L *of a vertex of* $\mathcal{I}(L)^1$, *and* $Q = P \cap L$ *is a maximal parahoric subgroup of* L, *which is contained in* $\hat{Q}$. *There are short exact sequences* $0 \to U \cap L \to Q \to M \to 0, 0 \to U \cap L \to \hat{Q} \to \hat{M} \to 0$.

(ii) *Let* $\mathbb{P}$ *be a parabolic subgroup containing* $\mathbb{L}$ *with Levi decomposition* $P = \mathbb{L} \cdot \mathbb{U}$. *There is a homeomorphism in the* p*-adic topology*

$$U \cap \mathbb{U}^- \times \hat{P} \cap L \times U \cap \mathbb{U} \to \hat{P},$$

and there is a similar decomposition for the group P.

Proof. The first assertion follows from 1.4 and the first exact sequence follows from the observation that Q contains the subgroup $P \cap \mathfrak{M}'$ and this group projects onto M as in 1.10. Now let $\mathbb{P} = \mathbb{L} \cdot \mathbb{U}$ be an F- parabolic for which $\mathbb{L}$ is a Levi component. (Note that if P is maximal then $\mathbb{P} = \mathbf{G}$ trivially satisfies (i), (ii) and (iii).)

Applying 1.13 we see that $\hat{P} \cap L$ is the centraliser of a facet in the enlarged building for L, and $Q = P \cap L$ is the corresponding parahoric subgroup. The remark above implies that $\mathfrak{M}$ is the connected reductive subgroup of $\mathbb{L}$ associated to Q as in 1.10. The definition of affine roots for $\mathcal{I}(L)$ and the identifications of 1.12 imply immediately that $\tilde{\pi}(\mathcal{F})$ is a point; in fact it is not difficult to see by unravelling the definitions in 1.12–13 that it must be a vertex. Alternatively, if Q were not maximal in L we could repeat the argument above in L itself to produce a proper Levi component $\mathbb{K}$ within $\mathbb{L}$ with the same properties (with respect to $\mathbb{L}$ and Q). Since $\mathfrak{M}$ is the connected reductive subgroup of $\mathbb{L}$ associated to Q we have $\mathbb{K} = \mathbf{Z}_{\mathbb{L}}(\mathbf{S}) = \mathbb{L}$. It follows that Q is a maximal parahoric subgroup in L as claimed. For the last part of (i) observe that $P \cap L$ contains $P \cap \mathfrak{M}'$ which projects onto M as in 1.10; similarly $\hat{P} \cap L$ contains $\hat{P} \cap \mathfrak{M}'$ which projects onto $\hat{M}$.

To prove (ii) recall from 1.2 that given any parabolic subgroup $\mathbb{P} = \mathbb{L} \cdot \mathbb{U}$ with opposite parabolic subgroup $\mathbb{P}^- = \mathbb{L} \cdot \mathbb{U}^-$ there is an isomorphism of varieties induced by multiplication: $\mathbb{U} \times \mathbb{L} \times \mathbb{U}^- \to \mathbb{P} \cdot \mathbb{P}^-$ and the image is an open set in $\mathbf{G}$. In particular, if $\mathbb{P}$ is defined over F we can take F-valued points to get a homeomorphism in the p-adic topology on G. Now consider the restriction

$$\hat{P} \cap \mathbb{U} \times \hat{P} \cap L \times \hat{P} \cap \mathbb{U}^- \to \hat{P} \cap (\mathbb{P} \cdot \mathbb{P}^-)(F).$$

To finish we need only show that the image is all of $\hat{P}$. Let $x \in \hat{P}$; by (i) we can find $l \in \hat{P} \cap L$ with $y = l^{-1}x \in U$. If I is any Iwahori subgroup contained in P then $U \subset I$; this follows immediately from 1.11. Invoking [BT1] 6.4.9, 6.4.48 we see that (ii) is true if we replace $\hat{P}$ by I, hence it is true if we replace $\hat{P}$ by U. (See 1.9.) Write $y = u_1 m u_2$ with $m \in U \cap L$, $u_1 \in U \cap \mathbb{U}(F)$, $u_2 \in U \cap \mathbb{U}^-(F)$. Thus $x = l u_1 m u_2$. Since $P \cap L$ normalises $U \cap \mathbb{U}$ we can rewrite this as $x = v l m u_2$ with $v \in U \cap \mathbb{U}$. The argument for P is the same. □

Remark 2.2. Although $(P \cap L)^0$ is maximal in L, it is not usually special in L (as easy examples show), even if $P \cap \mathfrak{M}(F)$ is special in $\mathfrak{M}(F)$. (Observe that $P \cap \mathfrak{M}(F) = P \cap \mathfrak{M}'$ because $\mathfrak{M}'$ is to $\mathfrak{M}(F)$ as G' is to G.)

2.3. We assume that $\mathbb{L}$ is standard with respect to the basis Δ of 1.3. Thus $\mathbb{L} = \mathbb{L}_\Theta$ for some $\Theta \subseteq \Delta$, and we write $\mathbf{S}$ for its split centre. Write $L = \mathbb{L}(F)$ as usual; observe that $\mathbf{T} \subseteq \mathbb{L}$. The generalised affine Weyl group $W_L = W_{L,aff}$ for L is an extension of D (see 1.6) by $W_\Theta = {}^vW_{\rho(\Theta)} \subseteq {}^vW$.

Let $X_*(\mathbf{S})$ denote the group of rational cocharacters of $\mathbf{S}$. Recall that there is a homomorphism $H_S: S \to X_*(\mathbf{S}) \otimes \mathbb{R}$ defined by $H_S(s)(\chi) = -\mathrm{ord}_F(\chi(s))$ if $\chi \in X(\mathbf{S}) = X_F(\mathbf{S})$. Let $D_S = \mathrm{im}(H_S)$.

2.4. Now let $P, \mathbb{L}, Q = P \cap L$ be the particular subgroups of Section 2.1. The F-split torus $\mathbf{S}$ acts on $\mathbb{U}$ by conjugation; from this one obtains a set of weights which we denote by $\Phi(\mathbb{P}, \mathbf{S})^+$. The elements of this set can be obtained by considering the nontrivial restrictions to $\mathbf{S}$ of the roots in Φ^+; if we write $\Delta(\mathbb{P}, \mathbf{S})$ for the set of nontrivial restrictions of the elements of the basis Δ then each element of $\Phi(\mathbb{P}, \mathbf{S})^+$ can be expressed as a linear combination of elements of $\Delta(\mathbb{P}, \mathbf{S})$ with nonnegative coefficients. (As usual, we are assuming $\mathbb{L}$ is standard.)

Since the elements of $\Phi(\mathbb{P}, \mathbf{S})$ are rational characters for $\mathbf{S}$, obtained by restriction from the elements of Φ we can write $S^+ = \{s \in \mathbf{S}(F) = S | H_S(s)(\alpha) \geqslant 0\}, \alpha \in \Delta(\mathbb{P}, \mathbf{S})$. We define S^{++} by replacing inequality by strict inequality in the definition above.

LEMMA. (i) *Let* $s \in S^+$. *Then*

$$s(\mathbb{U}(F) \cap U)s^{-1} \subseteq \mathbb{U}(F) \cap U; \; s^{-1}(\mathbb{U}^-(F) \cap U)s \subseteq \mathbb{U}^-(F) \cap U.$$

(ii) *If* $s \in S^{++}$ *then*

(a) *For any pair of compact open subgroups* H_1 *and* H_2 *of* $\mathbb{U}(F)$ *there is a non-negative integer* n *such that* $s^n H_1 s^{-n} \subseteq H_2$.
(b) *For any pair of compact open subgroups* K_1 *and* K_2 *of* $\mathbb{U}^-(F)$ *there is a non-positive integer* n *such that* $s^n K_1 s^{-n} \subseteq K_2$.

Proof. In (i) we shall only prove the second assertion. We suppose that the parabolic subgroup $\mathbb{P}$ corresponds to the subset $\Theta \subseteq \Delta$. The group $\mathbb{U}^-$ is directly spanned by root groups $\mathbb{U}_\gamma$ where $\gamma \in \Phi_{nd}$ and $\gamma = \sum_{\alpha\in\Delta} m_\alpha \alpha$ with at least one $\alpha \notin \Theta$ with $m_\alpha < 0$. It suffices to show that $s^{-1}U_{\gamma,r}s \subseteq U_{\gamma,r}$ if $U_{\gamma,r} \subseteq \mathbb{U}_\gamma(F)$ is a valuation group. Write $\gamma = \sum_{\alpha\notin\Theta} m_\alpha\alpha + \sum_{\beta\in\Theta} m_\beta\beta$. If $s \in S$ then [BT2]5.1.22(2) implies that $s^{-1}U_{\gamma,r}s = U_{\gamma,r-\sum_{\alpha\notin\Theta}(H_S(s),\alpha)m_\alpha}$; the assertion for $s \in S^+$ follows immediately.

For (ii) it is enough to show (*c.f.* [BK2] 6.14) that if $s \in S^{++}$ then

$$\bigcap_{n\geqslant 0} s^n(\mathbb{U}(F) \cap U)s^{-n} = \{1\}, \qquad \bigcup_{n\leqslant 0} s^n(\mathbb{U}(F) \cap U)s^{-n} = \mathbb{U}(F),$$

or again that $sU_{\gamma,r}s^{-1} \subsetneq U_{\gamma,r}$, $s^{-1}U_{\gamma,r}s \supsetneq U_{\gamma,r}$, if $U_{\gamma,r} \subseteq \mathbb{U}_\gamma(F) \subset \mathbb{U}(F)$ is a valuation group. This follows from the argument for (i). □

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2.5. In the language of [BK2]6.16, Lemma 2.4 says that the elements of S which lie in S^{++} are *strongly* $(\mathbb{P}, P)$*-positive.*

3. Invertible Elements in $\mathcal{H}(G, \sigma)$

3.1. We retain the notation of the previous sections. In particular, $P = P_J$ and we have the short exact sequence of 1.8: $0 \to U \to P \to M \to 0$. Let (σ, V) be an irreducible cuspidal representation of the group $M = \mathbb{M}(\mathbb{F}_q)$ with contragredient representation $(\check{\sigma}, V^{\vee})$ This inflates to a representation of the group P, and we can form the compactly induced representation $c\text{-Ind}_P^G(\check{\sigma})$. The intertwining algebra $\mathrm{End}_G(c\text{-Ind}_P^G(\check{\sigma}))$ is isomorphic to the algebra $\mathcal{H}(G, \sigma) = \mathcal{H}(\sigma)$ of $\check{\sigma}$-spherical functions on G with compact support where the multiplication in the latter is given by the standard convolution product (see [M] Section 4, or [BK2] Section 2.6). In [M] this algebra was analysed and described by generators and relations. Roughly speaking it is an affine Hecke algebra twisted by a group algebra (with a 2-cocycle).

Indeed, let $S_J = \{w \in W \mid wJ = J\}$ and put $N_J = P \cap N$; then S_J is a complement in $N_W(W_J)$ to the finite group W_J and, moreover, one has

$$\frac{N_N(P \cap \mathfrak{M}')}{N_J} \simeq \frac{N_W(W_J)}{W_J} \simeq S_J.$$

For this see [M] 4.16, 6.1. It then follows that

$$W(J, \sigma) = W(\sigma) = \{w \in S_J \mid w\sigma \simeq \sigma\}$$

is well defined. (Note that σ can be viewed as a representation on $P \cap \mathfrak{M}'$.)

PROPOSITION. *There is a* (*canonically defined*) *affine Coxeter subgroup* $R(\sigma) \subset W(\sigma)$ *together with a* (*canonically defined*) *complement* $C(\sigma)$ $W(\sigma) = R(\sigma) \rtimes C(\sigma)$. *Moreover, there is a canonical choice for a set of simple roots in the affine root system associated to* $R(\sigma)$, *once a set of set of positive roots has been chosen in* Σ.

This is proved in [M] 7.3. Henceforth we suppose that a set of positive affine roots for the affine system Σ has been chosen, as well as the matching affine basis in the root system associated with $R(\sigma)$.

3.2. The definition of $W(\sigma)$ implies the existence of a 2-cocycle $\mu\colon W(\sigma) \times W(\sigma) \to \mathbb{C}^{\times}$, which is nontrivial only on $C(\sigma) \times C(\sigma)$. (See [M] 6.2, 7.11.)

THEOREM. *The algebra* $\mathcal{H}(\sigma)$ *is generated by elements* T_w, $w \in W(\sigma)$ *subject to the following relations. Let* $w \in W(\sigma), t \in C(\sigma)$ *and let* v *be a reflection in* $R(\sigma)$ *corresponding to a simple root* a (*chosen as above in* 3.1).

(i) $T_w T_t = \mu(w, t) T_{wt}$;
(ii) $T_t T_w = \mu(t, w) T_{tw}$;

(iii) $T_vT_w = \begin{cases} T_{vw}, & \textit{if } w^{-1}a > 0; \\ p_aT_{vw} + (p_a - 1)T_w, & \textit{if not}; \end{cases}$

(iv) $T_wT_v = \begin{cases} T_{wv}, & \textit{if } wa > 0; \\ p_aT_{wv} + (p_a - 1)T_w, & \textit{if not}. \end{cases}$

Here $p_a \neq 1$ is a nonnegative power of p (the residue characteristic), and the element T_w is supported on one double coset of the form $P\dot{w}P$ where $\dot{w}$ is an element in $N(T)$ such that $\nu_W(\dot{w}) = w$.

This is Theorem 7.12 in [M]. We remark that $R(\sigma)$ can be trivial.

3.3. Now consider the translations $T(J)$ in $W(\sigma)$ provided by the group of rational points of the split centre of $\mathfrak{M}'$. They always provide a lattice in $W(\sigma)$ of rank at least as large as the lattice of all translations in the group denoted R_J in [M]7.3. (See also [M]2.6-2.7.) Further, their definition and that of the 2-cocycle μ, ensure that μ restricted to $T(J)$ is always trivial. (See remark (b) following *loc. cit.*). If we take $w = v$ in 3.2(iii) and (iv) we see that T_v is invertible when v is a fundamental reflection in the 'quotient' affine root system. Then, by writing an arbitrary $w \in R(\sigma)$ as a minimal product of such reflections, we see that T_w is invertible for any such w, again using 3.2(iii) and (iv). In general we can express $w = rc$ where $r \in R(\sigma)$ and $c \in C(\sigma)$; since T_c is invertible by 3.2(i), it follows that T_w is invertible by 3.2(i) or (ii) once again. In particular we have the following result.

LEMMA. *The elements $T_d, d \in T(J)$, are invertible.*

3.4. Let $\mathcal{H}(G) = \{f: G \to \mathbb{C} \mid f$ locally constant, compact support$\}$. This is an associative algebra with multiplication defined by convolution $f * h(x) = \int_G f(xg^{-1})h(g)\,\mathrm{d}g$.

With σ as above define $e_\sigma \in \mathcal{H}(G)$ by

$$e_\sigma(x) = \begin{cases} (1/\mathrm{vol}\, P)\dim(\sigma)\,\mathrm{trace}\,(x^{-1}), & \text{if } x \in P; \\ 0, & \text{if not.} \end{cases}$$

This is an idempotent in $\mathcal{H}(G)$; we then have the algebra $e_\sigma * \mathcal{H}(G) * e_\sigma$ which has as an identity the element e_σ. From Proposition 4.2.4 of [BK1] there is a canonical isomorphism $\Upsilon: \mathcal{H}(\sigma) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(V) \to e_\sigma * \mathcal{H}(G) * e_\sigma$.

It is realised in one direction in the following manner. We identify the left side with $\mathcal{H}(\sigma) \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}} V^\vee$ where we denote the dual of V by $V^\vee$. Then $\Upsilon(\Phi \otimes v \otimes \check{v})$ is the function $\phi(g) = \dim(\sigma)\langle v, \Phi(g)\check{v}\rangle$, where $\langle\ ,\ \rangle$ denotes the canonical pairing on $V \times V^\vee$. The isomorphism Υ implies that the algebras $\mathcal{H}(\sigma)$, $e_\sigma * \mathcal{H}(G) * e_\sigma$ are Morita equivalent, hence their module categories are equivalent.This is realised as follows. If M is an $\mathcal{H}(\sigma)$-module then $M \otimes_{\mathbb{C}} V$ is the corresponding $e_\sigma * \mathcal{H}(G) * e_\sigma (\simeq \mathcal{H}(\sigma) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(V))$-module. Conversely, if N is an $e_\sigma * \mathcal{H}(G) * e_\sigma$-module,

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we view $V^\vee$ as a right $\mathrm{End}_{\mathbb{C}}(V)$-module and form $V^\vee \otimes_{\mathrm{End}_{\mathbb{C}}(V)} N$. We then get a $\mathcal{H}(\sigma)$-module via the right factor since there is an embedding $\mathcal{H}(\sigma) \to e_\sigma * \mathcal{H}(G) * e_\sigma$. For more details we refer the reader to [BK1] Ch.4. We shall denote the equivalence between the module categories by Υ^*.

3.5. Now we take $\mathbb{P}$ as in Section 2 with respect to P. We denote by $\sigma_{\mathbb{U}}$ the representation of $\mathbb{L} \cap P$ on V by restriction of σ: it is also the inflation of σ on $M = \mathbb{M}(\mathbb{F}_q)$ (notation of 1.8) hence is irreducible. Let $(\pi, \mathcal{V})$ be a smooth representation of G. We denote by $\mathcal{V}^\sigma$ the σ-isotypic part of $\mathcal{V}$. Recall that there is a representation of $\mathcal{H}(G)$ on $\mathcal{V}$ defined by $\pi(f)v = \int_G f(x)\pi(x)v\,\mathrm{d}x$.

Given (σ, V) as above, and $(\pi, \mathcal{V})$ a smooth representation of G we define $\mathcal{V}_\sigma = \mathrm{Hom}_P(V, \mathcal{V}) \simeq \mathrm{Hom}_G(c - \mathrm{Ind}_P^G(\sigma), \mathcal{V})$, the isomorphism following from Frobenius reciprocity for compact induction. The algebra $\mathcal{H}(\check{\sigma})$ acts on the left on $c - \mathrm{Ind}_P^G(\sigma)$ via convolution $\phi * f(x) = \int_G \phi(y) f(y^{-1}x)\,\mathrm{d}y$, if $\phi \in \mathcal{H}(\check{\sigma})$, and $f \in c - \mathrm{Ind}_P^G(\sigma)$. On the other hand there is a canonical anti-isomorphism of algebras with identity provided by the map $\phi \mapsto \check{\phi}$ where $\check{\phi}(x) = (\phi(x^{-1}))^{\vee}$.

This means that $\mathcal{V}_\sigma$ is canonically a *left* $\mathcal{H}(\sigma)$-module.

There is an obvious evaluation map $\mathcal{V}_\sigma \otimes V \to \mathcal{V}^\sigma$; in terms of the canonical isomorphism Υ of 3.4 one deduces that there is a natural isomorphism of $e_\sigma * \mathcal{H}(G) * e_\sigma$-modules $\mathcal{V}^\sigma \simeq \Upsilon^*(\mathcal{V}_\sigma \otimes V)$ provided by this evaluation map. See [BK2]2.13 for more details on this.

3.6. From Theorem 2.1 we have

(i) $(P \cap \mathbb{U}(F)) \cdot (P \cap L) \cdot (P \cap \mathbb{U}^-(F)) = P$;

(ii) σ is trivial on $P \cap \mathbb{U}(F)$, $P \cap \mathbb{U}^-(F)$, since it factors through $L \cap P$.

In the terminology of [BK2]6.1, (i) and (ii) say that the pair (P, σ) is *decomposed* with respect to $(\mathbb{L}, \mathbb{P})$. Indeed 2.1 says that it is decomposed with respect to $(\mathbb{L}, \mathbb{P}')$ where $\mathbb{P}'$ is *any* parabolic which contains $\mathbb{L}$ as Levi component.

Let $s \in S$. Recall from Section 2, that s lies in the split centre of $\mathbb{L}$ by construction. We have already seen that the elements $T_{\nu(s)}$ (ν as in 1.12(ii)(b)) are invertible, hence any non zero element of $\mathcal{H}(G, \sigma)$ which is supported on PsP is invertible. Lemma 2.4 says that an abundance of such s are strongly $(\mathbb{P}, P)$ positive. The above observations tell us that Theorem 7.9 of [BK2] is applicable in this situation. We immediately deduce the following lemma.

LEMMA. *Let $(\pi, \mathcal{V})$ be a smooth representation of G. Write $(\pi_{\mathbb{U}}, \mathcal{V}_{\mathbb{U}})$ for the Jacquet module of $(\pi, \mathcal{V})$ with respect to $\mathbb{P}$. Then there is a canonical isomorphism $\mathcal{V}^\sigma \to (\mathcal{V}_{\mathbb{U}})^{\sigma_{\mathbb{U}}}$.*

This isomorphism can be described as follows. Let $r: \mathcal{V} \to \mathcal{V}_{\mathbb{U}}$ denote the quotient map. We then obtain a map $q: \mathrm{Hom}_P(V, \mathcal{V}) = \mathcal{V}_\sigma \to \mathrm{Hom}_Q(V, \mathcal{V}_{\mathbb{U}}) = (\mathcal{V}_{\mathbb{U}})_{\sigma_{\mathbb{U}}}$ by composing with r; here $Q = P \cap L$ as in Section 2. The map q then induces the isomorphism in Lemma 3.6.

Remark. If P is not maximal then the Levi component $\mathbb{L}$ of 2.1 is proper. Suppose that $(\pi, \mathcal{V})$ is irreducible admissible containing (σ, V). Then 4.1 implies that the Jacquet module $\mathcal{V}_{\mathbb{U}}$ cannot be zero. In particular, $(\pi, \mathcal{V})$ cannot be supercuspidal. This gives an alternative proof of [M1]3.5. We point that each of these proofs requires some knowledge of the structure of the Hecke algebra.

3.7. The fact that the elements T_d are invertible has a further consequence. Note that in addition to $\mathcal{H}(G, \sigma)$ there is also the intertwining algebra $\mathcal{H}(L, \sigma_{\mathbb{U}})$ for the pair $(Q, \sigma_{\mathbb{U}})$. Let $\varphi \in \mathcal{H}(L, \sigma_{\mathbb{U}})$ have support $Q\ell Q$ for some $\ell \in L$. Because (P, σ) is decomposed relative to $(\mathbb{L}, P)$ there is a unique element $T\varphi = \Phi$ in $\mathcal{H}(G, \sigma)$ with support in $P\ell P$; see [BK2]6.3. Let $\mathcal{H}^+(L, \sigma_{\mathbb{U}}) \subset \mathcal{H}(L, \sigma_{\mathbb{U}})$ denote the collection of functions whose support is contained in a union of double cosets of the form $Q\ell Q$ where ℓ is positive relative to $(P, \mathbb{P})$. Corollary 6.12, and Theorem 7.2 of *op. cit.* then tell us in particular the following.

THEOREM. (i) *$\mathcal{H}^+(L, \sigma_{\mathbb{U}})$ is a subalgebra of $\mathcal{H}(L, \sigma_{\mathbb{U}})$ with the same identity element.*

(ii) *The map T induces an injective homomorphism of algebras with identity*

$$T\colon \mathcal{H}^+(L, \sigma_{\mathbb{U}}) \to \mathcal{H}(G, \sigma).$$

(iii) *The map T in* (ii) *extends uniquely to an injective homomorphism of algebras with identity*

$$t\colon \mathcal{H}(L, \sigma_{\mathbb{U}}) \to \mathcal{H}(G, \sigma).$$

We remark that the proof of (i) and (ii) does not require the existence of an invertible element T_d, but that of (iii) does.

3.8. We now have accumulated the following results concerning the pair (P, σ) and its relation with *any* parabolic subgroup $\mathbb{P}$ containing the Levi component $\mathbb{L}$:

(i) the pair (P, σ) is decomposed with respect to $(\mathbb{L}, \mathbb{P})$;

(ii) the representation $\sigma_{\mathbb{U}}$ is smooth irreducible for the (maximal) parahoric subgroup $Q = P \cap L$ in L;

(iii) there is a strongly $(\mathbb{P}, P)$-positive element $s \in S \subset \mathbf{Z}(\mathbb{L})(F)$ such that PsP supports an invertible element of $\mathcal{H}(\sigma)$.

In the language of [BK2]8.1 the pair (P, σ) is a *cover* for the pair $(Q, \sigma_{\mathbb{U}})$.

3.9. The following lemma will be used in 3.10 below; it is of independent interest. We start with a Levi component L in the group G. Suppose that $\hat{J} \supset J$ are compact open subgroups in G. Now let $\hat{\tau}$ be a smooth irreducible representation of $\hat{J}$ whose restriction $\hat{\tau} \mid J$ contains τ.

LEMMA. *Suppose* (i) (J, τ) *is a cover for* (J_L, τ_L) *in the sense of* [BK2]8.1.

(ii) *if $\mathbb{P}$ is any parabolic subgroup containing $\mathbb{L}$ with Levi decomposition $\mathbb{P} = \mathbb{L}{\cdot}\mathbb{U}$ and opposite parabolic $\mathbb{P}^- = \mathbb{L}{\cdot}\mathbb{U}^-$ then* $\hat{J} = (J \cap \mathbb{U}^-(F))(\hat{J} \cap L)(J \cap \mathbb{U}(F))$.

(iii) $(\hat{J} \cap L)/\ker(\hat{\tau}|(\hat{J} \cap L)) \cong \hat{J}/\ker \hat{\tau}$.
Then $(\hat{J}, \hat{\tau})$ *is a cover for the pair* $(\hat{J} \cap L, \hat{\tau} \mid (\hat{J} \cap L))$.

Proof. Assumption (ii) guarantees an Iwahori decomposition for $\hat{J}$ with respect to $(\mathbb{L}, \mathbb{P})$, and assumption (iii) ensures that $(\hat{J}, \hat{\tau})$ is decomposed with respect to $(\mathbb{L}, \mathbb{P})$ for any $\mathbb{P}$ containing $\mathbb{L}$ as Levi component. Thus our pair $(\hat{J}, \hat{\tau})$ satisfies condition (i) of *loc. cit.*, and condition (ii) is trivially satisfied by construction. We must verify condition (iii).

Now define $\tau_* = \mathrm{Ind}_J^{\hat{J}}(\tau)$; then $\hat{\tau}$ occurs in τ_*. Just as before we can define the algebras $\mathcal{H}(G, \hat{\tau})$, $\mathcal{H}(G, \tau_*)$. According to [BK1]4.1.3, there is a canonical isomorphism of algebras $\Gamma: \mathcal{H}(G, \tau) \to \mathcal{H}(G, \tau_*)$ with the property that if $\phi \in \mathcal{H}(G, \tau)$ has support JxJ then $\Gamma(\phi)$ has support $\hat{J}x\hat{J}$, and if $\Phi \in \mathcal{H}(G, \tau_*)$ has support $\hat{J}x\hat{J}$ then $\Gamma^{-1}(\Phi)$ has support JxJ. On the other hand, the algebra $\mathcal{H}(G, \hat{\tau})$ can be identified (non canonically) with a subalgebra of $\mathcal{H}(G, \tau_*)$. To see this it is enough to replace the representations in the algebras in question by their contragredients, since taking contragredients commutes with induction. Denoting contragredients by '$\vee$' we see that $\mathcal{H}(G, (\hat{\tau})^{\vee})$ can be identified with some τ_*-spherical functions which transform via $\hat{\tau}$. Indeed, let V_* denote the space of τ_*; then $V_* = \oplus_{i=1}^{n} V_i$ where V_i runs through the not necessarily distinct irreducible constituents of τ_*. We can then identify $\hat{\tau}$ with (at least) one of these, and the assertion follows from this. Moreover we see that the identity in $\mathcal{H}(G, (\tau_*)^{\vee})$ can be written as a sum of the identities of the algebras $\mathrm{End}_{\mathbb{C}}(V_i)$ corresponding to the irreducible constituents V_i counted according to multiplicity. We conclude that indeed $\mathcal{H}(G, \hat{\tau})$ can be identified with a subalgebra of $\mathcal{H}(G, \tau)$; furthermore the identity of $\mathcal{H}(G, \hat{\tau})$ occurs as a nonzero direct summand of the identity of $\mathcal{H}(G, \tau)$.

Let s be an element of the split centre S of L. It fixes L pointwise under conjugation, hence does the same to any subgroup of L; in particular it fixes pointwise the subgroups $\hat{J}_L = \hat{J} \cap L$, $J_L = P \cap L$. It follows that s fixes $\tau_*, \hat{\tau}, \tau$ (not merely up to isomorphism); hence there are nonzero spherical functions $\phi_s^*, \hat{\phi}_s, \phi_s$ in $\mathcal{H}(G, \tau_*)$, $\mathcal{H}(G, \hat{\tau})$, $\mathcal{H}(G, \tau)$ respectively. Furthermore the isomorphism $\mathcal{H}(G, \tau) \simeq \mathcal{H}(G, \tau_*)$ identifies ϕ_s with a non zero multiple of ϕ_s^*. Since (J, τ) is a cover for (J_L, τ_L) condition (iii) of Definition 8.1 in [BK2] says that there is an s such that ϕ_s is invertible. It follows that ϕ_s^* is invertible in $\mathcal{H}(G, \tau_*)$. Now ϕ_s^* is a direct sum of operators $\phi_s^{(1)}, \phi_s^{(2)}, \dots, \phi_s^{(r)}$ corresponding to the irreducible constituents of τ_*, since s acts trivially on each constituent. Since ϕ_s^* is invertible so is each $\phi_s^{(1)}, \phi_s^{(2)}, \dots, \phi_s^{(r)}$. But $\hat{\phi}_s$ is a non zero multiple of one of these, hence it is invertible in the subalgebra $\mathcal{H}(G, \hat{\tau})$. It follows that Condition 8.1(iii) holds for the pair $(\hat{J}, \hat{\tau})$ as well. □

VARIANT 3.10. We resume the notation and conventions of 1.6, 1.12 and 2.1. In particular if $\mathcal{F}$ is a facet in $\mathcal{I}$ we write $P = P_{\mathcal{F}}$ for the corresponding parahoric subgroup and we write $\hat{P} = \hat{P}_{\mathcal{F}} \subseteq G^1$ for the full centraliser of $\mathcal{F}$. We then write $\hat{M} = \hat{P}/U$; it is the group of $\mathbb{F}_q$-rational points of a disconnected $\mathbb{F}_q$-reductive group whose connected component is the group $\mathbb{M}$ of 1.8. We suppose that we are given an irreducible representation $\hat{\sigma}$ of $\hat{M}$ which contains σ; as usual we view it also as a representation of $\hat{P}$. We also write $\hat{Q} = \hat{P} \cap L$.

We now let $\hat{\sigma}_{\mathbb{U}}$ be the restriction to $\hat{Q}$ of $\hat{\sigma}$. It is immediate from 2.1 that the hypotheses (ii) and (iii) of 3.9 hold for the pair $(\hat{P}, \hat{\sigma})$, and we have already seen in 3.8 that hypothesis (i) holds. We immediately deduce the following.

COROLLARY. *The pair* $(\hat{P}, \hat{\sigma})$ *is a cover for the pair* $(\hat{Q}, \hat{\sigma}_{\mathbb{U}})$.

4. *G*-types

4.1. We continue with the notation of Section 3. We begin by recalling a result from [M1]; see also the remark following 3.7. Namely, let $\mathbb{L}$ be a connected reductive F-group with $L = \mathbb{L}(F)$; let Q be a maximal parahoric subgroup of L with short exact sequence $0 \to U \to Q \to M \to 0$ and suppose that (σ, V) is an irreducible cuspidal representation of M. We regard (σ, V) as a representation of Q by inflation.

PROPOSITION. *Let* $(\tau, \mathcal{V})$ *be an irreducible smooth representation of* L *containing* (σ, V). *Then* $(\tau, \mathcal{V})$ *is supercuspidal, and there is an irreducible smooth representation* (ρ, W) *of* $Q^+ = N_L(Q)$ *containing* (σ, V) *such that* $(\tau, \mathcal{V}) = c - \mathrm{Ind}_{Q^+}^{L}(\rho)$.

Proof. This is proved in [M1] Sections 1–2. □

4.2. Next we recall some ideas and results from [BK2] Sections 3–4.

First, we consider pairs (L, ρ) where $\mathbb{L}$ is a (rational) Levi subgroup, $L = \mathbb{L}(F)$, and ρ is an irreducible supercuspidal representation of L. As usual if $g \in G$ we write ${}^{g}\rho$ for the (supercuspidal) representation on gLg^{-1} defined by ${}^{g}\rho(\ell) = \rho(g^{-1}\ell g)$. Finally, we let $X_u(G)$ denote the group of unramified quasicharacters of the (rational points of the) reductive group G: the elements of $X_u(G)$ are finite products of quasicharacters of the form $g \mapsto |\phi(g)|^s$ for some $s \in \mathbb{C}$ and some $\phi \in X_F(G)$, where $X_F(G)$ denotes the rational character group of $\mathbf{G}$.

DEFINITION. The pairs (L, ρ), (L', ρ') are *inertially equivalent* if there is a $g \in G$ and $\xi \in X_u(L')$ such that $L' = gLg^{-1}$ and ${}^{g}\rho \simeq \rho' \otimes \xi$. We denote the equivalence class containing (L, ρ) by $[L, \rho]$.

We write $\mathcal{B}(G)$ for the set of equivalence classes arising from the relation in the definition above.

4.3. If $\mathbb{P}$ is a parabolic subgroup with Levi decomposition $\mathbb{P} = \mathbb{L} \cdot \mathbb{U}$ we let $\delta_{\mathbb{P}}$ denote the associated modulus quasicharacter; it provides an unramified quasicharacter of L. We write $\mathrm{Ind}_{\mathbb{P}}^{G}$ to denote unnormalised induction from $\mathbb{P}$ to G and $\boldsymbol{\iota}_{\mathbb{P}}^{G}$ to denote normalised induction. These are related by $\boldsymbol{\iota}_{\mathbb{P}}^{G}(\tau) = \mathrm{Ind}_{\mathbb{P}}^{G}(\tau \otimes \delta_{\mathbb{P}}^{-1/2})$. The left adjoint for $\boldsymbol{\iota}_{\mathbb{P}}^{G}$ is denoted by $\mathbf{r}_{\mathbb{P}}^{G}$; it is simply the unnormalised Jacquet functor

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(of 3.6) tensored by $\delta_{\mathbb{P}}^{1/2}$. If $(\pi, \mathcal{V})$ is an irreducible smooth representation of G, there is always a parabolic subgroup $\mathbb{P}$ with Levi decomposition $\mathbb{P} = \mathbb{L} \cdot \mathbb{U}$ such that π is equivalent to a subquotient of $\iota_{\mathbb{P}}^{G}(\rho)$ for some irreducible supercuspidal representation of L; see [Cs]. The resulting inertial class $\mathcal{I}(\pi) = [L, \rho]$ is determined uniquely by π, and is called the *inertial support* of π. Note that since $\delta_{\mathbb{P}}$ is an unramified quasicharacter of L, the remarks above imply that the inertial class could have been defined by replacing ι by Ind.

Let $\mathfrak{S}\mathfrak{R}(G)$ denote the category of smooth representations of G. If $\mathfrak{S} \subset \mathcal{B}(G)$ we write $\mathfrak{S}\mathfrak{R}^{\mathfrak{S}}(G)$ for the full subcategory of $\mathfrak{S}\mathfrak{R}(G)$ whose objects are those objects $(\pi, \mathcal{V})$ of $\mathfrak{S}\mathfrak{R}(G)$ for which every irreducible subquotient has inertial support in $\mathfrak{S}$. If $\mathfrak{S} = \{\mathfrak{s}\}$ we shall simply write $\mathfrak{S}\mathfrak{R}^{\mathfrak{s}}(G)$ rather than $\mathfrak{S}\mathfrak{R}^{\mathfrak{S}}(G)$. According to Proposition 2.10 of [BD] the category $\mathfrak{S}\mathfrak{R}(G)$ is the *direct product* of the categories $\mathfrak{S}\mathfrak{R}^{\mathfrak{s}}(G)$ as $\mathfrak{s}$ runs through $\mathcal{B}(G)$. This means that

(a) for each smooth representation $\mathcal{V}$, and for each $\mathfrak{s} \in \mathcal{B}(G)$ there is a unique G-subspace $\mathcal{V}^{\mathfrak{s}}$ which is an object in $\mathfrak{S}\mathfrak{R}^{\mathfrak{s}}(G)$, maximal with respect to this property, and $\mathcal{V}$ is the direct sum of the $\mathcal{V}^{\mathfrak{s}}$ as $\mathfrak{s}$ runs through $\mathcal{B}(G)$;

(b) if $\mathcal{V}$, $\mathcal{W}$ are objects in $\mathfrak{S}\mathfrak{R}(G)$ then $\mathrm{Hom}_G(\mathcal{V}, \mathcal{W})$ is the direct product of the various $\mathrm{Hom}_G(\mathcal{V}^{\mathfrak{s}}, \mathcal{W}^{\mathfrak{s}})$.

DEFINITION. Let $\mathfrak{S}$ be a subset of $\mathcal{B}(G)$. An $\mathfrak{S}$-*type in G* is a pair (K, σ) where K is a compact open subgroup of G, and σ is an irreducible smooth representation of K with the following property: an irreducible smooth representation $(\pi, \mathcal{V})$ of G contains σ if and only if $\mathcal{I}(\pi) \in \mathfrak{S}$.

If $\mathfrak{S} = \{\mathfrak{s}\}$ is a singleton, we shall abuse notation and write '$\mathfrak{s}$-type'.

4.4. Definition 4.3 has significant consequences, some of which we shall list below. In what follows, (K, σ) always denotes an $\mathfrak{S}$-type. If $(\pi, \mathcal{V})$ is a smooth representation we shall write $\mathcal{V}[\sigma]$ for the G-module generated by the σ-isotypic vectors. Recall that one can form $e_\sigma * \mathcal{V}$ which provides an $e_\sigma * \mathcal{H}(G) * e_\sigma$-module. Composing this with the Morita equivalence of 3.4 then provides a functor $M_\rho \colon \mathfrak{S}\mathfrak{R}_\sigma(G) \to \mathcal{H}(\sigma)$-$\mathfrak{Mod}$.

We then have the following result.

THEOREM ([BK2] Theorem 4.3). (i) *There is a uniquely determined G-space $\mathcal{U}$ such that $\mathcal{V} = \mathcal{V}[\sigma] \oplus \mathcal{U}$.*

(ii) *If $\mathcal{V} = \mathcal{V}[\sigma]$ then any irreducible G-subquotient of $\mathcal{V}$ contains σ.*

(iii) *The functor M_ρ above provides an equivalence of categories $\mathfrak{S}\mathfrak{R}_\sigma(G) \to \mathcal{H}(\sigma) - \mathfrak{Mod}$.*

(iv) $\mathfrak{S}\mathfrak{R}^{\mathfrak{S}}(G) = \mathfrak{S}\mathfrak{R}_\sigma(G)$

4.5. In [BK2] the authors provide many examples of $\mathfrak{s}$-types drawn from their work on linear and special linear groups. The prototype of all $\mathfrak{s}$-types is the pair $(B, 1)$ where B is the centraliser of an alcove in the 'enlarged' building for G and 1 is the trivial representation of B. The full centraliser is typically larger than the corresponding Iwahori subgroup (connected centraliser). The admissible form of 4.4(iv) in this case is due to Borel [B]; see [BK2] for a simple proof of the more general situation, based on ideas in [MW].

THEOREM. *Let (σ, V) be an irreducible cuspidal representation as above and suppose that P is a maximal parahoric subgroup. Then (σ, V) is an $\mathfrak{S}$-type, for a finite set $\mathfrak{S}$.*

Proof. Let $(\pi, \mathcal{V})$ be an irreducible smooth representation containing (σ, V). From 4.1 we can write $\pi = c - \mathrm{Ind}_{P^+}^G(\rho)$ where ρ is an irreducible smooth representation for P^+ which contains σ. Let χ denote the central quasicharacter for π, and let π' be another such representation which also contains σ and which also has central quasicharacter χ. We suppose that $\pi' = c - \mathrm{Ind}_{P^+}^G(\rho')$. The representations ρ', ρ are determined on ZU, where U is the prounipotent radical of P hence we can write $\rho' = \rho \otimes \tau$ where τ is an irreducible representation of the finite group P^+/ZU. In particular if we fix a central quasicharacter there are only finitely many choices for the representation ρ and hence there are only finitely many such π containing σ with prescribed central quasicharacter.

Now suppose that we consider π and π' as above but with possibly different central quasicharacters χ, χ'. We have $\chi |\ Z \cap P = \chi' |\ Z \cap P$ in any case. Let Z_c be the kernel of the map H_Z defined in 2.3 for the Levi component G. From 2.3 there is an exact sequence $0 \to Z_c \to Z \to \Lambda \to 0$ where Λ is a lattice of finite rank and $\mathrm{rank}\Lambda = \text{split rank } Z$. On the other hand if H is the group in 1.6, then $Z_c \subset H \subset P$ for any parahoric subgroup P centralising a facet in A, since $\mathbf{Z} \subset \mathbf{T}$. In particular $\chi^{-1} \cdot \chi'$ is trivial on Z_c hence comes from a quasicharacter on Λ. Now Λ is a lattice of the same rank as the dual of the rational character group $X_F(\mathbf{G})$ of $\mathbf{G}$. Indeed $X_F(\mathbf{G})$ is a subgroup of finite index in $X_F(\mathbf{Z})$ as one sees from the isogeny $\mathbf{Z} \times \mathbf{G}_{\mathrm{der}} \to \mathbf{G}$. Practically by definition, any quasicharacter of Λ is a (finite) product of ones of the form $z (\mathrm{mod}\, Z_c) \mapsto |\psi(z)|^s$ for some $s \in \mathbb{C}$ and some $\psi \in X_F(\mathbf{Z})$.

It follows immediately that any quasicharacter of Z which is trivial on Z_c is the restriction of an *unramified* quasicharacter of G (i.e. one which is a product of ones of the form $g \mapsto |\phi(g)|^s$ for some $s \in \mathbb{C}$ and some $\phi \in X_F(\mathbf{G})$). In particular $\chi^{-1} \cdot \chi'$ is such a quasicharacter. Thus replacing π by $\pi \otimes \phi$ for a suitable unramified quasicharacter ϕ of G we see that $\pi \otimes \phi$ and π' have the same central quasicharacter and we are in the situation of the previous paragraph. □

Remark 4.6. One can easily produce examples (σ, V) for which the set $\mathfrak{S}$ is not a singleton, by considering the case where σ is unipotent cuspidal. In fact, many of the cases considered in [M1] provide such examples.

VARIANT 4.7. By modifying the pair (P, σ) slightly the set $\mathfrak{S}$ can be reduced to a singleton. Indeed we know from 4.1 that any irreducible smooth representation $(\pi, \mathcal{V})$ containing (σ, V) has the form $\pi = c - \mathrm{Ind}_{P^+}^G(\rho)$, where ρ is an irreducible smooth representation for P^+ which contains σ. Since P is maximal it fixes an 'enlarged' vertex $v \times V^1$ in $\mathcal{I}^1$, and P^+ is the stabiliser in G of $v \times V^1$. It follows that $G^1 \cap P^+ = \hat{P}$ is the centraliser in G^1 of $v \times V^1$. Let $\hat{\sigma}$ be any irreducible component of $\rho \mid \hat{P}$. The group $\hat{P}$ is open compact in G; in fact it is the maximal compact subgroup of P^+.

THEOREM. *$(\hat{P}, \hat{\sigma})$ is a $[G, \pi]$-type.*

Proof. To say that π', π are inertially equivalent means that $\pi' \simeq \pi \otimes \chi$ where χ is an unramified quasicharacter of G. But then $\pi' \simeq c - \mathrm{Ind}_{P^+}^G(\rho \otimes (\chi|P^+))$. Since χ is trivial on G^1 hence $\hat{P}$, it follows that π' contains $\hat{\sigma}$. On the other hand if π' contains $\hat{\sigma}$ then $\pi' = c - \mathrm{Ind}_{P^+}^G(\rho')$ where ρ' is an irreducible constituent of $c - Ind_{\hat{P}}^{P^+}(\hat{\sigma})$. Now $P^+/\hat{P}$ can be identified with a subgroup of the lattice G/G^1, and it contains the group denoted Λ in the proof of 4.5 because P^+ contains Z. It follows that $P^+/\hat{P}$ is a sublattice of G/G^1 of the same rank, hence any quasicharacter of it extends to a quasicharacter of G/G^1. Now observe that if ρ, ρ' both contain $\hat{\sigma}$ then they are determined on $Z \cap \hat{P}$ by the central character of $\hat{\sigma}$; since Z is a split F-torus this means that the representation $\hat{\sigma}$ can be extended to $Z\hat{P}$ (by an unramified quasicharacter of Z). Clifford–Mackey theory then implies that $\rho' = \rho \otimes \chi$ for some quasicharacter χ of $P^+\hat{P}$, and then that $\pi' = \pi \otimes \chi'$ for some extension χ' to G of χ. □

Remark. Note that this result says that each irreducible constituent $\hat{\sigma}$ of $\rho|\hat{P}$ is an $\mathfrak{s}$-type for the same singleton $\mathfrak{s}$.

4.8. We now combine 3.8, 4.5, and [BK2] Theorem 8.3, to deduce the following result.

THEOREM. *If (σ, V) is an irreducible cuspidal representation as above where P is not necessarily maximal, then (σ, V) is an $\mathfrak{S}$-type for a finite set $\mathfrak{S}$.*

Proof. Let $\mathbb{L}$ be as in 2.1. Applying Theorem 4.5 to $\mathbb{L}$ and the pair $(Q, \sigma_{\mathbb{U}})$ we see that $(Q, \sigma_{\mathbb{U}})$ is an $\mathfrak{S}_L$-type for some finite set $\mathfrak{S}_L$. Here $\mathfrak{S}_L$ consists of a finite set of inertial equivalence classes with respect to L of the form $[L, \tau]$ where τ is an irreducible supercuspidal representation of L. On the other hand 3.8 says that (P, σ) is a G-cover for the pair $(Q, \sigma_{\mathbb{U}})$. Theorem 8.3 of [BK2] then says that in this situation (P, σ) is an $\mathfrak{S}_G$-type where $\mathfrak{S}_G$ is the finite set formed from the inertial equivalence classes with respect to G of the elements in $\mathfrak{S}_L$.

Briefly, the argument goes as follows. First, let $(\pi, \mathcal{V})$ be an irreducible smooth representation of G containing (σ, V). There is always an irreducible supercuspidal representation τ of $\mathbb{L}$ containing $\sigma_{\mathbb{U}}$ such that π is isomorphic to a G-subspace of $\mathrm{Ind}_{\mathbb{P}}^G(\tau)$. Indeed 3.6 implies that the unnormalised Jacquet module $(\pi_{\mathbb{U}}, \mathcal{V}_{\mathbb{U}})$ contains $\sigma_{\mathbb{U}}$. Since $\sigma_{\mathbb{U}}$ is an $\mathfrak{S}_L$-type Proposition 2.10 of [BD] (described in 4.3 above) and part (iv) of Theorem 4.4 imply that some irreducible quotient τ has $\mathcal{I}(\tau) \in \mathfrak{S}_L$. Since $\delta_{\mathbb{P}}$ is unramified the same is true on replacing the unnormalised Jacquet module by the normalised version. Frobenius reciprocity then implies that $\mathcal{I}(\pi) \in \mathfrak{S}_G$.

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To go the other way, let $\mathfrak{S}_G$ be as in the preceding paragraph, and suppose that $\mathcal{I}(\pi) \in \mathfrak{S}_G$. This means that π occurs as a subrepresentation of $\iota_{\mathbb{P}}^G(\rho)$ for some (L, ρ) with $[L, \rho]$ in $\mathfrak{S}$, and by construction ρ contains $\sigma_{\mathbb{U}}$. One may now apply 3.6 to see that π contains σ. □

VARIANT 4.9. Again, by replacing the pair (P, σ) by the pair $(\hat{P}, \hat{\sigma})$ where $\hat{P}$ is the full centraliser of the appropriate facet and $\hat{\sigma}$ is an irreducible smooth representation of $\hat{P}$ which contains σ as in 3.10, we can deduce the following.

THEOREM. *$(\hat{P}, \hat{\sigma})$ is an $\mathfrak{s}$-type for a singleton set $\mathfrak{s}$.*

Proof. We know from Variant 3.10 that $(\hat{P}, \hat{\sigma})$ is a G-cover for the pair $(\hat{Q}, \hat{\sigma}_{\mathbb{U}})$. Here the $(\hat{P}, \hat{\sigma})$ is with respect to G, while $(\hat{Q}, \hat{\sigma}_{\mathbb{U}})$ is with respect to L. The result then follows immediately from Variant 4.7 and Theorem 8.3 of [BK2]. □

Remark. The technique above can be codified into a general principle. We revert to the notation of 3.9, and assume we have pairs $(\hat{J}, \hat{\tau})$, (J, τ) satisfying the conditions in Lemma 3.9. Assume further that $(\hat{J} \cap L, \hat{\tau} \mid (\hat{J} \cap L))$ is an $\mathfrak{s}$-type for a singleton $\mathfrak{s}$. The argument above implies *that $(\hat{J}, \hat{\tau})$ is an $\mathfrak{s}_G$-type for a singleton $\mathfrak{s}_G$.*

4.10. Now we recall some results in [BD]. First, let $\mathfrak{s} = [L, \tau] \in \mathcal{B}(G)$ and let (L, τ) be a representative for it. Then (L, τ) determines a class $\mathfrak{s}_L \in \mathcal{B}(L)$. If we change the representative then it must have the form $({}^gL, {}^g\tau \otimes \chi)$ for some $g \in G$ and $\chi \in X_u({}^gL)$. If we write L' for ${}^gL$ and $\mathfrak{s}_{L'}$ for the resulting class in $\mathcal{B}(L')$ then conjugation by g provides an equivalence of categories $\mathfrak{S}\mathfrak{R}^{\mathfrak{s}_L}(L) \simeq \mathfrak{S}\mathfrak{R}^{\mathfrak{s}_{L'}}(L)$.

Second, if we interpret [BD] 2.8 in the language above (*c.f.* [BK2] 2.3,6.1) we obtain the following statements.

THEOREM. (i) *Let $(\pi, \mathcal{V})$ be an object of $\mathfrak{S}\mathfrak{R}^{\mathfrak{s}}(G)$. Then $(\pi_{\mathbb{U}}, \mathcal{V}_{\mathbb{U}})$ is an object of the subcategory $\Pi_{\mathfrak{t}}\mathfrak{S}\mathfrak{R}^{\mathfrak{t}}(L)$ of $\mathfrak{S}\mathfrak{R}(L)$ where $\mathfrak{t}$ runs through the $N_G(L)$ orbit of $\mathfrak{s}_L$.*

(ii) *The representation $(\pi, \mathcal{V})$ is an object of $\mathfrak{S}\mathfrak{R}^{\mathfrak{s}}(G)$ if and only if there are parabolic subgroups $\mathbb{P}$ of G each of which has Levi component $\mathbb{L}$, and smooth representations $\tau_L \in \mathfrak{S}\mathfrak{R}^{\mathfrak{s}_L}(L)$ and a G-injection $\pi \to \coprod_{\mathbb{P}} \mathrm{Ind}_{\mathbb{P}}^G(\tau_L)$.*

4.11. The unnormalised Jacquet functor provides a functor $r_{\mathbb{U}}\colon \mathfrak{S}\mathfrak{R}^{\mathfrak{S}}(G) \to \mathfrak{S}\mathfrak{R}(L)$. Composing this with the projection functor $\mathbf{p}^{\mathfrak{S}_L}\colon \mathfrak{S}\mathfrak{R}(L) \to \mathfrak{S}\mathfrak{R}^{\mathfrak{S}_L}(L)$ guaranteed by Theorem 4.4(i) we obtain a functor $\mathbf{r}_{\mathbb{U}}\colon \mathfrak{S}\mathfrak{R}^{\mathfrak{S}}(G) \to \mathfrak{S}\mathfrak{R}^{\mathfrak{S}_L}(L)$, since this last category is also the category $\mathfrak{S}\mathfrak{R}_{\sigma_{\mathbb{U}}}(L)$ by 4.4(iv).

Going the other way, 4.10(ii) implies that the unnormalised induction functor Ind takes the category $\mathfrak{S}\mathfrak{R}^{\mathfrak{s}_L}(L)$ to the category $\mathfrak{S}\mathfrak{R}^{\mathfrak{s}}(G)$. Here $\mathfrak{s}$ is the class determined by $\mathfrak{s}_L$ as in the proof of Theorem 4.8. It follows that Ind takes $\mathfrak{S}\mathfrak{R}^{\mathfrak{S}_L}(L)$ to the category $\mathfrak{S}\mathfrak{R}^{\mathfrak{S}}(G)$.

If τ is an object in $\mathfrak{S}\mathfrak{R}^{\mathfrak{s}_L}(L)$ we then have $\mathrm{Hom}_G(\pi, \mathrm{Ind}_{\mathbb{P}}^G(\tau) \simeq \mathrm{Hom}_L(r_{\mathbb{U}}(\pi), \tau) \simeq \mathrm{Hom}_L(\mathbf{p}^{\mathfrak{S}_L} r_{\mathbb{U}}(\pi), \tau)$.

In other words, we have the following result.

PROPOSITION. *The unnormalised Jacquet functor* $r_{\mathbb{U}}$ *provides a functor*

$$\mathbf{r}_{\mathbb{U}}: \mathfrak{S}\mathfrak{R}^{\mathfrak{S}}(G) \to \mathfrak{S}\mathfrak{R}^{\mathfrak{S}_L}(L).$$

It has a right adjoint functor provided by the unnormalised induction functor Ind.

Remark. If we used normalised induction here we would have to (un)twist the Jacquet functor by $\delta_{\mathbb{P}}^{-1/2}$.

4.12. If $f: A \to B$ is a homomorphism of associative rings, and M is a B-module we write $f^*(M)$ for the A-module M induced by f. If N is an A-module we write $f_*(N)$ for the B-module $\mathrm{Hom}_A(B, N)$.

Theorem 4.8 guarantees equivalences of categories

$$\mathfrak{S}\mathfrak{R}_{\sigma}(G) \to \mathcal{H}(G, \sigma) - \mathfrak{Mod}, \qquad \mathfrak{S}\mathfrak{R}_{\sigma_{\mathbb{U}}}(L) \to \mathcal{H}(L, \sigma_{\mathbb{U}}) - \mathfrak{Mod}.$$

Furthermore, Proposition 4.11 implies that unnormalised induction provides a functor $\mathfrak{S}\mathfrak{R}_{\sigma_{\mathbb{U}}}(L) \to \mathfrak{S}\mathfrak{R}_{\sigma}(G)$, and that the Jacquet functor $r_{\mathbb{U}}$ provides a functor $\mathbf{r}_{\mathbb{U}}: \mathfrak{S}\mathfrak{R}_{\sigma}(G) \to \mathfrak{S}\mathfrak{R}_{\sigma_{\mathbb{U}}}(L)$. Recall the injective algebra homomorphism $t_{\mathbb{P}}: \mathcal{H}(L, \sigma_{\mathbb{U}}) \to \mathcal{H}(G, \sigma)$ of 3.7. Applying Corollary 8.4 of [BK2] to this we immediately obtain the following result.

THEOREM. *Each of the following diagrams is commutative*:

$$\begin{array}{ccc}
\mathfrak{S}\mathfrak{R}_{\sigma}(G) & \xrightarrow{M_{\sigma}} & \mathcal{H}(G, \sigma) - \mathfrak{Mod} \\
\downarrow{\scriptstyle \mathbf{r}_{\mathbb{U}}} & & \downarrow{\scriptstyle t_{\mathbb{P}}^*} \\
\mathfrak{S}\mathfrak{R}_{\sigma_{\mathbb{U}}}(L) & \xrightarrow{M_{\sigma_{\mathbb{U}}}} & \mathcal{H}(L, \sigma_{\mathbb{U}}) - \mathfrak{Mod};
\end{array}$$

$$\begin{array}{ccc}
\mathfrak{S}\mathfrak{R}_{\sigma}(G) & \xrightarrow{M_{\sigma}} & \mathcal{H}(G, \sigma) - \mathfrak{Mod} \\
\uparrow{\scriptstyle \mathrm{Ind}} & & \downarrow{\scriptstyle t_{\mathbb{P}*}} \\
\mathfrak{S}\mathfrak{R}_{\sigma_{\mathbb{U}}}(L) & \xrightarrow{M_{\sigma_{\mathbb{U}}}} & \mathcal{H}(L, \sigma_{\mathbb{U}}) - \mathfrak{Mod}.
\end{array}$$

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