ON RADICAL EXTENSIONS OF RINGS

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A ring K is a *radical extension* of a subring B if for each $x \in K$ there is an integer $n = n(x) > 0$ such that $x^n \in B$. In [2] and [3], C. Faith considered radical extensions in connection with commutativity questions, as well as the generation of rings. In this paper additional commutativity theorems are established, and rings with right minimum condition are examined. The main tool is Theorem 1.1 which relates the Jacobson radical of K to that of B, and is of independent interest in itself. The author is indebted to the referee for his helpful suggestions, in particular for the strengthening of Theorem 2.1.

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In [3] it was established that if the primitive ring K is a radical extension of the subring B then B is primitive. In what follows $J(R)$ denotes the Jacobson radical of the ring R. Recall that $J(R)$ can be characterized as the intersection of all primitive ideals of R (if any). Also $J(R) = \{x \in R : xR$ is a right quasi-regular right ideal of $R\}$ [5].

THEOREM 1.1. *If K is a radical extension of a subring B then* $J(B) = J(K) \cap B$.

PROOF. Let $x \in J(K) \cap B$ and consider $xB \subseteqq B$. Since $xB \subseteqq xK$ and $x \in J(K)$, every element of xB has a right quasi-inverse in K. It will be shown that this quasi-inverse is actually in B and hence $x \in J(B)$. Thus let $y \in xB$ and $a \in K$ such that

$$(*) \qquad y+a-ya = 0 \text{ and } a^n \in B \text{ for some integer } n \geqq 1.$$

From (*) $ya^{n-1} = ya^n - a^n \in B$. In (*) left multiply by y and right multiply by a^{n-2} to get $y^2a^{n-2} = y^2a^{n-1} - ya^{n-1} \in B$. Again, left multiply by y^2 and right multiply by a^{n-3} in (*) to get $y^3 a^{n-3} \in B$. Repetition of this procedure yields $y^{n-1}a \in B$. Then from (*), $a = ya - y = y(ya-y) - y = \cdots$, and so $a = y^{n-1}a - y^{n-1} - \cdots - y \in B$, as was to be shown. Thus $J(K) \cap B \subseteqq J(B)$.

For the opposite inclusion, let $\mathscr{P}$ denote the set of primitive ideals of K and $\mathscr{M}$ denote the set of primitive ideals of B. If $J(K) = K$ then $J(B) \subseteqq K \cap B = J(K) \cap B$. If $J(K) \neq K$ then as in [3, page 281]

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$$J(B) = \bigcap_{Q \in \mathcal{M}} Q \subset (\bigcap_{I \in \mathcal{P}} I) \bigcap B = J(K) \bigcap B.$$

COROLLARY. *If K is semisimple and a radical extension of a subring B then B is semisimple; if K is a radical extension of a semisimple ring B then $J(K)$ is a nil ideal.*

It was shown in [3] that a semisimple radical extension of a commutative subring is commutative; the question was raised as to whether or not semisimplicity could be replaced by the weaker condition of having no non-zero nil ideals, with the possibility of the subring being semisimple.

THEOREM 1.2. *Let K be a ring with no nil ideals $\neq (0)$ such that K is a radical extension of a commutative subring B.*

(i) *If $J(B) = (0)$, then K is commutative.*

(ii) *If B is a left (right) ideal of K, then K is commutative.*

PROOF. (i) Since $J(B) = (0)$, $J(K)$ is a nil ideal by the corollary, hence $J(K) = (0)$ so that K is commutative.

(ii) We first show that in any ring without nilpotent ideals $\neq (0)$ any commutative one-sided ideal is contained in the center. Let $a(B) = \{x \in K : xB = (0)\}$. Since B is a left ideal $a(B)$ is an ideal and $B^* = a(B) \cap B$ is a left ideal of K. Now $B^{*2} \subseteqq a(B)B = (0)$ so B^* gives rise to a nilpotent ideal of K containing B^*. Hence $B^* = (0)$. Let $x \in B$, $r \in K$; for any $y \in B$, $(xr - rx)y = x(ry) - r(xy) = 0$. Thus $xr - rx \in B^* = (0)$ and so B is contained in the center of K. If K is a radical extension of B having no nil ideals $\neq (0)$, then K is a radical extension of its center, so from [4] K is commutative.

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In this section the structure of rings which are radical extensions of rings with right minimum condition, subject to the condition that the overring has no nil ideals $\neq (0)$, is determined. Note that something similar to having no nil ideals is necessary since radical extensions of arbitrary rings can be obtained by forming direct sums with nil rings.

THEOREM 2.1. *Let K be a ring with no nil ideals $\neq (0)$ which is a radical extension of a subring B and assume B satisfies the right minimum condition. Then B is semisimple, $B = M_1 \oplus \cdots \oplus M_r$, where M_i is a full ring of $n_i \times n_i$ matrices over a division ring D_i, $i = 1, \cdots, r$, and K is semisimple, $K = K_1 \oplus \cdots \oplus K_r$, where K_i is a simple ring with identity radical over M_i, $i = 1, \cdots, r$. Moreover, if char $K_i \neq 2$, then either $K_i = M_i$ or else $n_i = 1$ and K_i is a (commutative) field radical over D_i.*

PROOF. From the right minimum condition on B we infer that $J(B)$ is a nilpotent ideal of B. By Theorem 1.1, $J(B) = J(K) \cap B$ and this implies $J(K)$ is a nil ideal of K. Hence $J(K) = (0)$ and so $J(B) = (0)$. By the Wedderburn-Artin Theorem, $B = M_1 \oplus \cdots \oplus M_r$, where each M_i is a complete matrix ring over a division ring, $i = 1, \cdots, r$.

If P is a primitive ideal of K, then $P \cap B$ is a primitive ideal of B. If $P \cap B = (0)$ then P is a nil ideal of K and hence $P = (0)$. Thus K is a primitive ring and so B is primitive. Now a primitive ring is a prime ring, hence can have no proper ideal direct summands and so $B = M_i$ for some i. Then B is a simple ring with identity and therefore K is a simple ring with identity by [2; Corollary, Thm. 3.1]. Now suppose $P \cap B \neq (0)$ for all primitive ideals P of K. Let

$$T_i = M_1 \oplus \cdots \oplus M_{i-1} \oplus M_{i+1} \oplus \cdots \oplus M_r \text{ for } i = 1, \cdots, r.$$

Then T_i is a maximal ideal of B for $i = 1, \cdots, r$ and it is easy to see that these are the only ones since B has an identity and each M_i is simple. Now $P \cap B$ is a direct sum of some non-null subset of the M_i's, say $P \cap B = M_{i1} \oplus \cdots \oplus M_{it}$. Then $B/(P \cap B)$ is isomorphic to $M_{it+1} \oplus \cdots \oplus M_{ir}$ and the primeness of $B/(P \cap B)$ implies that there is only one summand, which is a simple ring. Hence $P \cap B$ is a maximal ideal of B so that $P \cap B = T_j$ for some j. If for some j, $P \cap B \neq T_j$ for all primitive ideals of K, then $M_j \subseteqq T_i$ for all $i \neq j$ and consequently $M_j \subseteqq \cap (P \cap B) \subseteqq \cap P = J(K) = (0)$. We have shown that for each T_j there is a primitive ideal P_j of K such that $P_j \cap B = T_j$, $j = 1, \cdots, r$.

Suppose $x \in \bigcap_{j=1}^r P_j$; then for some integer $n \geqq 1$, $x^n \in P_j \cap B$ for $j = 1, \cdots, r$ so that $x^n \in \bigcap_{j=1}^r T_j = (0)$. Thus $\bigcap_{j=1}^r P_j = (0)$, being a nil ideal of K. Consequently K is isomorphic to a subdirect sum of the primitive rings $K_j = K/P_j$, $j = 1, \cdots, r$, each of which is a radical extension of a corresponding simple subring with identity which is isomorphic to M_j, and hence each K_j is itself a simple ring with identity, $j = 1, \cdots, r$. Finally, if any P_j is omitted from $\bigcap_{j=1}^r P_j$ the resulting intersection contains M_j, hence is non-zero. By [5; p. 59] K is then a direct sum of the K_j's. If char $K_i \neq 2$ and $K_i \neq B_i$ then [1, Corollary 5] implies that $n_i = 1$. But then by [2, Theorem A], K_i is a field.

COROLLARY. *If K and B are as in Theorem* 2.1 *and B is commutative, then K is commutative and each K_j is a field.*

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