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POROSITY OF CERTAIN SUBSETS OF LEBESGUE SPACES ON LOCALLY COMPACT GROUPS

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Abstract

Let G be a locally compact group. In this paper, we show that if G is a nondiscrete locally compact group, $p \in (0, 1)$ and $q \in (0, +\infty]$, then $\{(f, g) \in L^p(G) \times L^q(G) : f * g \text{ is finite } \lambda\text{-a.e.}\}$ is a set of first category in $L^p(G) \times L^q(G)$. We also show that if G is a nondiscrete locally compact group and $p, q, r \in [1, +\infty]$ such that $1/p + 1/q > 1 + 1/r$, then $\{(f, g) \in L^p(G) \times L^q(G) : f * g \in L^r(G)\}$, is a set of first category in $L^p(G) \times L^q(G)$. Consequently, for $p, q \in [1 + \infty)$ and $r \in [1, +\infty]$ with $1/p + 1/q > 1 + 1/r$, G is discrete if and only if $L^p(G) * L^q(G) \subseteq L^r(G)$; this answers a question raised by Saeki ['The L^p-conjecture and Young's inequality', *Illinois J. Math.* **34** (1990), 615–627].

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1. Introduction and preliminaries

Throughout this work, let G denote a locally compact group with a fixed left Haar measure λ. The modular function on the locally compact group G is denoted by Δ. It is well known that Δ is a continuous homomorphism on G. Moreover, for every measurable subset A of G,

$$\lambda(A^{-1}) = \int_A \Delta(x^{-1})\, d\lambda(x);$$

for more details see [2] or [5].

For $1 \le p < \infty$, the Lebesgue space $L^p(G)$ with respect to λ is defined as the Banach space of all (equivalence classes of) Borel measurable functions f on G with

$$\|f\|_p = \left(\int_G |f(x)|^p\, d\lambda(x)\right)^{1/p} < \infty.$$

If $0 < p < 1$, it is known that $L^p(G)$ is a complete metric space with the metric

$$d(f, g) = \int_G |f - g|^p\, d\lambda \quad (f, g \in L^p(G)).$$

In this case, for convenience, we put

$$|f|_p = d(f, 0) = \int_G |f|^p \, d\lambda.$$

If $p = \infty$, $L^p(G)$ is the Banach space of all (equivalence classes of) essentially bounded measurable functions f on G with the norm $\|f\|_\infty = \text{esssup}|f|$.

For measurable functions f and g on G, the *convolution*

$$(f * g)(x) = \int_G f(y)g(y^{-1}x) \, d\lambda(y)$$

is defined at each point $x \in G$ for which the function $y \mapsto f(y)g(y^{-1}x)$ is Haar integrable. For $r \in [1, \infty]$, we write $f * g \in L^r(G)$ to mean that $|f * g(x)| < \infty$ for λ-almost every $x \in G$, $f * g$ is λ measurable on the set of all such x, and $\|f * g\|_r < \infty$.

Quek and Yap [6] proved the following interesting theorem.

Theorem 1.1. *Let G be an infinite locally compact abelian group. Let $p, q > 1$ be real numbers such that $1 < p < \infty$, $1 < q < \infty$ and $1/p + 1/q > 1$, and let r be defined by $1/r = 1/p + 1/q - 1$. Then:*

(i) *if G is compact, then*

$$L^p(G) * L^q(G) \nsubseteq \bigcup \{L^s(G) : r < s\};$$

(ii) *if G is discrete, then*

$$L^p(G) * L^q(G) \nsubseteq \bigcup \{L^s(G) : s < r\};$$

(iii) *if G is neither compact nor discrete, then*

$$L^p(G) * L^q(G) \nsubseteq \bigcup \{L^s(G) : s \neq r\}.$$

Motivated by this result, Saeki [8] posed the following question.

Question. Let G be a locally compact group and let $p, q, r \in [1, \infty]$. If

$$\frac{1}{r} < \frac{1}{p} + \frac{1}{q} - 1 \quad \text{and} \quad L^p(G) * L^q(G) \subseteq L^r(G),$$

does it follow that G is discrete?

Recently, Głąb and Strobin [4], using the notion of porosity, generalised and considerably extended some interesting results on the convolution of functions essentially due to Rickert [7] and Żelazko [10]. See also [1, 3, 8] for some related results.

Let us recall the notion of porosity. Let X be a metric space. The open ball with centre $x \in X$ and radius $r > 0$ is denoted by $B(x, r)$. For a given number $0 < c \leq 1$, a subset M of X is called *c-lower porous* if

$$\liminf_{R \to 0+} \frac{\gamma(x, M, R)}{R} \geq \frac{c}{2}$$

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for all $x \in M$, where

$$\gamma(x, M, R) = \sup\{r \geq 0 : \exists\, z \in X, B(z, r) \subseteq B(x, R) \setminus M\}.$$

It is clear that M is c-lower porous if and only if

$$\forall x \in M, \forall \alpha \in (0, c/2), \exists\, R_0 > 0, \forall R \in (0, R_0), \exists\, z \in X, B(z, \alpha R) \subseteq B(x, R) \setminus M.$$

A set is called *σ-c-lower porous* if it is a countable union of c-lower porous sets with the same constant $c > 0$. It is easy to see that a σ-c-lower porous set is meagre, and the notion of σ-porosity is stronger than that of meagreness. For more details, see [4, 9].

In this work, we present a generalisation of the interesting result due to Żelazko in [11]. We also answer the question asked by Saeki.

2. Results

Let us remark that we equip here the space $L^p(G) \times L^q(G)$ with the complete metric $\mathbf{d}$ defined by

$$\mathbf{d}((f_1, g_1), (f_2, g_2)) := \begin{cases} \max\{d(f_1, f_2), d(g_1, g_2)\} & \text{for } p \in (0, 1), q \in (0, 1), \\ \max\{d(f_1, f_2), \|g_1 - g_2\|_q\} & \text{for } p \in (0, 1), q \in [1, +\infty], \end{cases}$$

for all $(f_i, g_i) \in L^p(G) \times L^q(G)$ and $i = 1, 2$.

We begin with the following theorem in which we use a technique from [4].

Theorem 2.1. *Let G be a nondiscrete locally compact group and let $p \in (0, 1)$ and $q \in (0, +\infty]$. Then for any symmetric compact neighbourhood V of the identity element of G, the set*

$$E_V = \{(f, g) \in L^p(G) \times L^q(G) : \forall x \in V \text{ with } |f * g(x)| < \infty \text{ a.e.}\}$$

is a σ-c-lower porous set for some $c > 0$.

Proof. Let V be a symmetric compact neighbourhood of the identity element of G. For a natural number n, put

$$E_n = \left\{(f, g) \in L^p(G) \times L^q(G) : \forall x \in V \text{ with } \int_G |f(y)||g(y^{-1}x)|\, d\lambda(y) \leq n \text{ a.e.}\right\}.$$

So, $E_V = \bigcup_{n\in\mathbb{N}} E_n$. Hence we only need to show that for each $n \in \mathbb{N}$, E_n is c-lower porous for some $c > 0$. For this end we consider three cases.

Case 1. $p \in (0, 1)$ and $q \in [1, +\infty)$.

Let $\sup_{x\in V} \Delta(x) = \eta$ and $c \in (0, 1)$ be such that

$$\frac{c}{1-c} + \frac{\eta\lambda(V^2)}{\lambda(V)}\Big(\frac{c}{1-c}\Big)^q = 1.$$

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Then, clearly, for $0 < \alpha < c$,

$$\frac{\alpha}{1-\alpha} + \frac{\eta\lambda(V^2)}{\lambda(V)}\Big(\frac{\alpha}{1-\alpha}\Big)^q < 1.$$

By continuity of the map $x \mapsto \alpha/x + (\eta\lambda(V^2)/\lambda(V))(\alpha/x)^q$ on $(0, 1)$, we infer that there exist $0 < \beta < 1 - \alpha$ and $d > 1$ such that

$$\rho = 1 - \frac{\alpha}{\beta}\Big(\frac{d}{d-1}\Big)^p - \frac{\eta\lambda(V^2)}{\lambda(V)}\Big(\frac{\alpha d}{\beta(d-1)}\Big)^q > 0.$$

Fix a natural number n and suppose that $(f, g) \in E_n$. Since G is not discrete, $\inf\{\lambda(U) : \lambda(U) > 0\} = 0$, and for $R > 0$, we can choose compact symmetric neighbourhoods L and K contained in V such that $K \subseteq L$, $\lambda(LK)\lambda(V) \le \lambda(L)\lambda(V^2)$,

$$\int_L |f|^p \, d\lambda < (1 - \alpha - \beta)R$$

and

$$\lambda(L)^{1-1/p} > n\Big(d^{-2}\beta^{1+1/p}R^{1+1/p}\eta^{1-1/p}\Big(\frac{1}{\lambda(V^2)}\Big)^{1/q}\rho\Big)^{-1}. \tag{2.1}$$

Let s, t be such that

$$s\lambda(L) = \beta R \quad \text{and} \quad t(\lambda(LK))^{1/q} = \beta R. \tag{2.2}$$

Define functions $\widetilde{f}$ and $\widetilde{g}$ on G by setting

$$\widetilde{f}(x) := \begin{cases} (s\Delta(x^{-1}))^{1/p} & \text{if } x \in L, \\ f(x) & \text{otherwise} \end{cases}$$

and

$$\widetilde{g}(x) := \begin{cases} g(x) & \text{if } x \notin LK, \\ g(x) + t & \text{if } x \in LK,\ \mathrm{Re}(g(x)) \ge 0, \\ g(x) - t & \text{if } x \in LK,\ \mathrm{Re}(g(x)) < 0. \end{cases}$$

Hence

$$\begin{aligned} |\widetilde{f} - f|_p &= \int_L |s^{1/p}\Delta(x^{-1})^{1/p} - f(x)|^p \, d\lambda(x) \\ &\le \int_L s\Delta(x^{-1}) \, d\lambda(x) + \int_L |f|^p \, d\lambda \\ &\le s\int_L \Delta(x^{-1}) \, d\lambda(x) + (1 - \alpha - \beta)R \\ &\le \beta R + (1 - \alpha - \beta)R \\ &= R - \alpha R. \end{aligned}$$

Moreover,

$$\|\widetilde{g} - g\|_q = \|t\chi_{LK}\|_q = t(\lambda(LK))^{1/q} = \beta R \le R - \alpha R.$$

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Hence $B((\widetilde{f},\widetilde{g}),\alpha R)\subseteq B((f,g),R)$. It remains only to prove that $B((\widetilde{f},\widetilde{g}),\alpha R)\cap E_n=\varnothing$. Fix any $(h,k)\in B((\widetilde{f},\widetilde{g}),\alpha R)$ and let

$$A_1=\{x\in L:|h(x)|<d^{-1}(s\Delta(x^{-1}))^{1/p}\},\quad A_2=L\setminus A_1,\tag{2.3}$$

and

$$B_1=\{x\in LK:|k(x)|<d^{-1}t\},\quad B_2=LK\setminus B_1.\tag{2.4}$$

Then

$$\begin{aligned}\alpha R\geq \|h-\widetilde{f}\|_p &\geq \int_{A_1}|h(x)-s^{1/p}\Delta(x^{-1})^{1/p}|^p\,d\lambda(x)\\ &\geq \int_{A_1}s\Big(\frac{d-1}{d}\Big)^p\Delta(x^{-1})\,d\lambda(x)\\ &= s\Big(\frac{d-1}{d}\Big)^p\int_{A_1}\Delta(x^{-1})\,d\lambda(x)\\ &= s\Big(\frac{d-1}{d}\Big)^p\lambda(A_1^{-1}),\end{aligned}$$

whence

$$\lambda(A_1^{-1})\leq\frac{\alpha R}{s}\Big(\frac{d}{d-1}\Big)^p\overset{(2.2)}{=}\lambda(L)\Big(\frac{\alpha}{\beta}\Big)\Big(\frac{d}{d-1}\Big)^p.\tag{2.5}$$

In the same way, by noting that $|\widetilde{g}(x)|\geq t$, for $x\in LK$,

$$\alpha R\geq\|k-\widetilde{g}\|_q\geq\|(k-\widetilde{g})\chi_{B_1}\|_q=t\Big\|\Big(\frac{k}{t}-\frac{\widetilde{g}}{t}\Big)\chi_{B_1}\Big\|_q\geq t\Big(\frac{d-1}{d}\Big)(\lambda(B_1))^{1/q}.$$

Noting that $L\subseteq V$, it follows that

$$\lambda(B_1)\leq\Big(\frac{\alpha dR}{t(d-1)}\Big)^q\overset{(2.2)}{=}\lambda(LK)\Big(\frac{\alpha d}{\beta(d-1)}\Big)^q\leq\lambda(L)\frac{\lambda(V^2)}{\lambda(V)}\Big(\frac{\alpha d}{\beta(d-1)}\Big)^q.\tag{2.6}$$

The above inequalities show that A_2 and B_2 are of positive measure and so nonempty. Now let $z\in K$ be an arbitrary element, and define the set $F=(A_2^{-1}z)\cap B_2$ and $H=zF^{-1}$. Since $Lz\subseteq LK$, $A_2^{-1}z\subseteq LK$. Hence

$$\begin{aligned}\lambda(H^{-1})=\lambda(Fz^{-1})&=\lambda(A_2^{-1})-\lambda(A_2^{-1}\setminus(B_2z^{-1}))\\ &\geq\lambda(A_2^{-1})-\lambda((LK\setminus B_2)z^{-1})=\lambda(A_2^{-1})-\lambda(B_1z^{-1})\\ &=\lambda(L)-\lambda(A_1^{-1})-\Delta(z^{-1})\lambda(B_1)\\ &\overset{(2.5),(2.6)}{\geq}\lambda(L)\rho.\end{aligned}\tag{2.7}$$

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Also, $H \subseteq A_2$, $F \subseteq B_2$ and $H^{-1}z = F$. Finally, we conclude that

$$\begin{aligned}
\int_H |h(y)||k(y^{-1}z)|\,d\lambda(y) &\overset{(2.3),(2.4)}{\geq} d^{-2}s^{1/p}t\int_H \Delta(y^{-1})^{1/p}\,d\lambda(y)\\
&= d^{-2}s^{1/p}t\int_H \Delta(y^{-1})^{1/p-1}\Delta(y^{-1})\,d\lambda(y)\\
&\geq d^{-2}s^{1/p}t\int_H \eta^{1-1/p}\Delta(y^{-1})\,d\lambda(y)\\
&= d^{-2}s^{1/p}t\eta^{1-1/p}\lambda(H^{-1}) \overset{(2.7)}{\geq} d^{-2}s^{1/p}t\eta^{1-1/p}\lambda(L)\rho\\
&\overset{(2.2)}{=} d^{-2}\beta^{1+1/p}R^{1+1/p}\rho\Big(\frac{1}{\lambda(LK)}\Big)^{1/q}\eta^{1-1/p}\lambda(L)^{1-1/p}\\
&\geq d^{-2}\beta^{1+1/p}R^{1+1/p}\rho\Big(\frac{1}{\lambda(V^2)}\Big)^{1/q}\eta^{1-1/p}\lambda(L)^{1-1/p}\\
&\overset{(2.1)}{>} n.
\end{aligned}$$

Case 2. $p \in (0, 1)$ and $0 < q < 1$.

The proof is similar to the proof of Case 1 with $q = 1$.

Case 3. $p \in (0, 1)$ and $q = +\infty$.

Let $c = \frac{1}{2}$, so that $c/(1 - c) = 1$. Fix $0 < \alpha < 1/2$, so that $\alpha/(1 - \alpha) < 1$. By continuity of the map $x \mapsto \alpha/x$ on $(0, 1)$, there exist $0 < \beta < 1 - \alpha$ and $d > 1$ such that $\alpha/\beta(d/(d - 1))^p < 1$. Similarly to Case 1, we can choose a compact symmetric neighbourhood L contained in V such that

$$\int_L |f(x)|^p\,d\lambda(x) < 1 - \alpha - \beta \quad \text{and} \quad \lambda(L)^{1-1/p} > n(R^{1+1/p}(1 - 2\alpha)d^{-1}\rho\beta^{1/p}\eta^{1-1/p})^{-1},$$

where $\sup_{x\in V}\Delta(x) = \eta$ and $\rho = 1 - (\alpha/\beta)(d/(d - 1))^p$. Define

$$\widetilde{f}(x) = \begin{cases} (s\Delta(x^{-1}))^{1/p} & \text{if } x \in L,\\ f(x) & \text{otherwise,}\end{cases}$$

and

$$\widetilde{g}(x) = \begin{cases} g(x) + R(1 - \alpha) & \text{if } \operatorname{Re}(g(x)) \geq 0,\\ g(x) - R(1 - \alpha) & \text{if } \operatorname{Re}(g(x)) < 0,\end{cases}$$

in which $s\lambda(L) = \beta R$. By these definitions,

$$|\widetilde{f} - f|_p < R - \alpha R \quad \text{and} \quad \|\widetilde{g} - g\|_\infty = R - \alpha R.$$

Hence $B((\widetilde{f}, \widetilde{g}), \alpha R) \subset B((f, g), R)$. But $B((\widetilde{f}, \widetilde{g}), \alpha R) \cap E_n = \varnothing$. To show this, take any $(h, k) \in B((\widetilde{f}, \widetilde{g}), \alpha R)$ and let

$$L_1 = \{x \in L : |h(x)| > d^{-1}(s\Delta(x^{-1}))^{1/p}\}, \qquad L_2 = L \setminus L_1.$$

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Then

$$\lambda(L_2^{-1}) \le \lambda(L)\Big(\frac{\alpha}{\beta}\Big)\Big(\frac{d}{d-1}\Big)^p \quad \text{and} \quad |k(x)| \ge R(1-2\alpha).$$

Now let $z \in L$ be an arbitrary element. Define the sets $F = L_1^{-1}z$ and $E = zF^{-1}$. It follows that

$$\lambda(E^{-1}) = \lambda(Fz^{-1}) = \lambda(L_1^{-1}) = \lambda(L) - \lambda(L_2^{-1}) \ge \rho\lambda(L).$$

Consequently,

$$\begin{aligned}\int_E |h(y)||k(y^{-1}z)|\, d\lambda(y) &\ge R(1-2\alpha)d^{-1}s^{1/p}\eta^{1-1/p}\rho\lambda(L)\\ &= R^{(1+1/p)}(1-2\alpha)d^{-1}\rho\beta^{1/p}\eta^{1-1/p}\lambda(L)^{1-1/p}\\ &> n.\end{aligned}$$

Thus $(h, k) \notin E_n$, as required. □

As an immediate consequence of this theorem we obtain a result that generalises a well-known theorem of Żelazko [11] which states that $L^p(G)$, $0 < p < 1$, is an algebra under convolution if and only if G is discrete.

Corollary 2.2. *Let G be a locally compact group and let $p \in (0, 1)$ and $q \in (0, +\infty]$. Then $f * g$ exists for all $f \in L^p(G)$ and $g \in L^q(G)$ if and only if G is discrete.*

Proof. Recall that $L^s(G) \subseteq L^t(G) \subseteq L^1(G)$ if $s \le t \le 1$ and G is discrete. This proves the 'if' part. For the converse we only need to note that a σ-c-lower porous set is of first category, and $(L^p(G) \times L^q(G), \mathbf{d})$ is a complete metric space. □

Theorem 2.3. *Let G be a locally compact group and let $p, q \in [1, +\infty)$ and $r \in [1, +\infty]$. If $1/p + 1/q > 1 + 1/r$ and G is nondiscrete, then for any symmetric compact neighbourhood V of the identity element of G, the set*

$$E_V = \{(f, g) \in L^p(G) \times L^q(G) : f * g \in L^r(V, \lambda_{|V})\}$$

is σ-c-lower porous for some $c > 0$.

Proof. Let V be a symmetric compact neighbourhood of the identity element of G, and let $p, q \in [1, +\infty)$ and $r \in [1, +\infty]$ be such that $1/p + 1/q > 1 + 1/r$. For a natural number $n > 0$, put

$$E_n = \Big\{(f, g) \in L^p(G) \times L^q(G) : \int_V \Big(\int_G |f(y)||g(y^{-1}x)|\, d\lambda(y)\Big)^r d\lambda(x) \le n\Big\};$$

if $r = \infty$ we instead consider the condition $\int_G |f(y)||g(y^{-1}x)|\, d\lambda(y) \le n$ for λ-almost every $x \in V$ in the above set. So, $E_V = \bigcup_{n\in\mathbb{N}} E_n$. Hence we only need to show that for each $n \in \mathbb{N}$, E_n is c-lower porous for some $c > 0$. To prove this, let $\sup_{x\in V} \Delta(x) = \eta$ and $c \in (0, 1)$ be such that

$$\Big(\frac{c}{1-c}\Big)^p + \eta\Big(\frac{c}{1-c}\Big)^q \frac{\lambda(V^2)}{\lambda(V)} = 1.$$

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Then, clearly, for $0 < \alpha < c$,

$$\Big(\frac{\alpha}{1-\alpha}\Big)^p + \eta\Big(\frac{\alpha}{1-\alpha}\Big)^q \frac{\lambda(V^2)}{\lambda(V)} < 1.$$

By continuity of the map $x \mapsto (\alpha/x)^p + \eta(\alpha/x)^q(\lambda(V^2)/\lambda(V))$ on $(0, 1)$, we infer that there exist $0 < \beta < 1 - \alpha$ and $d > 1$ such that

$$\rho = 1 - \Big(\frac{\alpha d}{\beta(d-1)}\Big)^p - \eta\Big(\frac{\alpha d}{\beta(d-1)}\Big)^q \frac{\lambda(V^2)}{\lambda(V)} > 0.$$

Fix a natural number n and suppose that $(f, g) \in E_n$. Since G is not discrete, $\inf\{\lambda(U) : \lambda(U) > 0\} = 0$, and for $R > 0$, we can choose compact symmetric neighbourhoods K and L contained in V such that $K \subseteq L$, $\lambda(L) < 1$, and $\lambda(LK)\lambda(V) \leq \lambda(L)\lambda(V^2)$ with

$$\Big(\int_L |f|^p \, d\lambda\Big)^{1/p} + \Big(\int_{LK} |g|^q \, d\lambda\Big)^{1/q} < (1 - \alpha - \beta)R$$

and

$$\lambda(L)^{1+1/r-1/p-1/q} > n\Big(d^{-2}\beta^2 R^2 \eta^{1/p-1}\Big(\frac{\lambda(V)}{\lambda(V^2)}\Big)^{1/q} \rho\Big)^{-1}.$$

Let s, t be such that

$$s(\lambda(L))^{1/p} = \beta R \quad \text{and} \quad t(\lambda(LK))^{1/q} = \beta R.$$

Define functions $\widetilde{f}$ and $\widetilde{g}$ on G by setting

$$\widetilde{f}(x) = \begin{cases} s\Delta(x^{-1})^{1/p} & \text{if } x \in L, \\ f(x) & \text{otherwise,} \end{cases} \quad \text{and} \quad \widetilde{g}(x) = \begin{cases} t & \text{if } x \in LK, \\ g(x) & \text{otherwise.} \end{cases}$$

Then $B((\widetilde{f}, \widetilde{g}), \alpha R) \subseteq B((f, g), R)$. It remains only to prove that $B((\widetilde{f}, \widetilde{g}), \alpha R) \cap E_n = \varnothing$. Fix any $(h, k) \in B((\widetilde{f}, \widetilde{g}), \alpha R)$ and let

$$L_1 = \{x \in L : |h(x)| < d^{-1} s\Delta(x^{-1})^{1/p}\}, \quad L_2 = L \setminus L_1,$$

and

$$B_1 = \{x \in LK : |k(x)| < d^{-1} t\}, \quad B_2 = LK \setminus B_1.$$

Then

$$\lambda(L_1^{-1}) \leq \Big(\frac{\alpha dR}{s(d-1)}\Big)^p = \lambda(L)\Big(\frac{\alpha d}{\beta(d-1)}\Big)^p,$$

and similarly

$$\lambda(B_1) \leq \Big(\frac{\alpha dR}{t(d-1)}\Big)^q = \lambda(LK)\Big(\frac{\alpha d}{\beta(d-1)}\Big)^q \leq \lambda(L)\frac{\lambda(V^2)}{\lambda(V)}\Big(\frac{\alpha d}{\beta(d-1)}\Big)^q.$$

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Now let $z \in K$ be an arbitrary element, and define the set $F = (L_2^{-1}z) \cap B_2$ and $H = zF^{-1}$. Since $Lz \subseteq LK$, $L_2^{-1}z \subseteq LK$. Hence $\lambda(H^{-1}) \geq \lambda(L)\rho$. Also, $H \subseteq L_2$, $F \subseteq B_2$ and $H^{-1}z = F$. Finally, we conclude that

$$\begin{aligned}\int_H |h(y)||k(y^{-1}z)|\,d\lambda(y) &\geq d^{-2}st\int_H \Delta(y^{-1})^{1/p}\,d\lambda(y)\\ &= d^{-2}st\int_H \Delta(y^{-1})^{1/p-1}\Delta(y^{-1})\,d\lambda(y)\\ &\geq d^{-2}st\int_H \eta^{1/p-1}\Delta(y^{-1})\,d\lambda(y)\\ &= d^{-2}st\eta^{1/p-1}\lambda(H^{-1}) \geq d^{-2}st\eta^{1/p-1}\lambda(L)\rho\\ &\geq d^{-2}\beta^2R^2\Big(\frac{\lambda(V)}{\lambda(V^2)}\Big)^{1/q}\eta^{1/p-1}\lambda(L)^{(1-1/p-1/q)}\rho\\ &> \frac{n}{\lambda(L)^{1/r}}.\end{aligned}$$

Thus $(h, k) \notin E_n$, as required. □

The fact that a σ-c-lower porous set is of first category, together with Theorem 2.3, gives an answer to the question raised by Saeki.

Corollary 2.4. *Let G be a locally compact group and let $p, q \in [1 + \infty)$ and $r \in [1, +\infty]$ be such that $1/p + 1/q > 1 + 1/r$. Then G is discrete if and only if $L^p(G) * L^q(G) \subseteq L^r(G)$.*

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