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SOLVABILITY OF SOME SINGULAR BOUNDARY VALUE PROBLEMS ON THE SEMI-INFINITE INTERVAL

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ABSTRACT. Existence of solutions to the nonlinear boundary value problem on the semi-infinite interval $\frac{1}{p}(py')' = qf(t, y, py')$, $0 < t < \infty$, $y(0) = 0$, $y(t)$ bounded on $[0, \infty)$, are established. In the process we obtain new existence results for boundary value problems on compact intervals.

1. **Introduction.** This paper examines the existence of solutions to singular and nonsingular second order nonlinear differential equations

$$(1.1)\qquad \begin{cases} \frac{1}{p(t)}\big(p(t)y'(t)\big)' = q(t)f\big(t, y(t), p(t)y'(t)\big), & 0 < t < \infty \\ y(0) = 0,\ y(t) \text{ bounded on } [0, \infty). \end{cases}$$

Throughout $f\colon [0, \infty) \times (-\infty, \infty) \times (-\infty, \infty) \longrightarrow (-\infty, \infty)$ and $p, \frac{1}{q}\colon [0, \infty) \longrightarrow [0, \infty)$ are assumed to be continuous.

Boundary value problems on the semi-infinite interval have been examined extensively over the last ten years or so with most of the results obtained for the nonsingular problem ($p = q = 1$); see [1, 2, 6, 7] and their references. However recently [4, 11, 12] some results for nonsingular problems have been obtained. These papers were motivated by the Thomas-Fermi equation

$$y'' = t^{-\frac{1}{2}} y^{\frac{3}{2}}, \quad 0 < t < \infty$$

subject to the boundary condition corresponding to the isolated neutral atom

$$y(0) = 1, \quad \lim_{t \to \infty} y(t) = 0.$$

The technique, in establishing existence of solution to (1.1), in this paper involves obtaining results on the finite interval $[0, n]$, $n \in N^+ = \{1, 2, \ldots\}$ and then extending these results (using the Arzela-Ascoli theorem) to the semi-infinite interval. This technique was initiated in the papers [7, 12, 14].

The discussion will be in three parts. Firstly we examine the following boundary value problem on the finite interval

$$(1.2)\qquad \begin{cases} \frac{1}{p}(py')' = qf(t, y, py'), & 0 < t < n \\ y(0) = y(n) = 0 \end{cases}$$

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for each $n \in N^+$. Various existence results for problems of the form (1.2) will be obtained by exploiting the properties of the zero set of f. Partial results of this type may be found in [3, 5, 8, 9, 13]; however the results of this section are new and complement the existing theory. The following existence principle will be needed in Section 2; see [3, 10] for details.

THEOREM 1.1. *Let $n \in N^+$ be fixed. Assume*

(1.3) $\quad f\colon [0,\infty) \times \mathbf{R}^2 \longrightarrow \mathbf{R}$ *is continuous,* $q \in C(0,\infty)$ *with* $q > 0$ *on* $(0,\infty)$

(1.4) $\quad p \in C[0,\infty) \cap C^1(0,\infty)$ *together with* $p > 0$ *on* $(0,\infty)$

and

$$(1.5)\quad \begin{cases} \int_0^b \frac{ds}{p(s)} < \infty;\ \text{assume } \int_0^b \frac{1}{p(s)} \int_u^b p(s)q(s)\,ds\,du < \infty \text{ if } f(t,u,v) \equiv f(t,u) \text{ and} \\ \int_0^b p(s)q(s)\,ds < \infty \text{ otherwise, for any } b > 0.. \end{cases}$$

Now suppose there is a constant $M_0 > 0$, independent of λ, with

$$\max\Big\{\sup_{[0,n]} |y(t)|, \sup_{(0,n]} |p(t)y'(t)|\Big\} \le M_0$$

for any solution y to

$$(1.6)_\lambda \qquad \begin{cases} \frac{1}{p}(py')' = \lambda q f(t,y,py'), & 0 < t < n \\ y(0) = y(n) = 0 \end{cases}$$

for each $\lambda \in (0,1)$. Then (1.2) has at least one solution $y \in C[0,n] \cap C^2(0,n]$ with $py' \in C[0,n]$.

Section 3 will now use the results of Section 2 to establish existence of solutions to (1.1). Again here the results are new and extend and complement the existing theory found in [4, 11, 12]. Finally it is easy to see that the linear problem

$$\begin{cases} y'' = y + M, & 0 < t < \infty \\ y(0) = 0 \end{cases}$$

with $M > 0$ a constant, has exactly one bounded solution $y(t) = Me^{-t} - M$. In this case $\lim_{t \to \infty} y(t) = -M$. Now in Section 4, results in Section 3 are used to deduce possible values of $\lim_{t\to\infty} y(t)$ if such a limit exists.

For notational purpose let $BC^2[0,\infty)$ denote the space of functions u with u, pu' bounded and continuous on $[0,\infty)$ and $(pu')'$ continuous on $(0,\infty)$.

2. **Finite interval problems.** This section obtains existence results for problems of the form (1.2). However our eventual goal to discuss (1.1) and the technique involves obtaining the M_0, in Theorem 1.1, independent of λ and n; so in hindsight we will obtain M_0 independent of λ and n. If we were just interested in the finite interval problem we need only obtain M_0 independent of λ and so it will be obvious from our analysis that some of the assumptions given below can be relaxed. We just remark on this and will not discuss it further.

We begin with a generalization of Theorem 2.1 of [7].

THEOREM 2.1. *Suppose (1.3), (1.4) and (1.5) are satisfied. In addition assume either*

(2.1) *there is a constant* $M > 0$ *with* $uf(t,u,0) > 0$ *for* $|u| > M$ *and* $t \in [0,\infty)$

or

$$(2.2)\quad \begin{cases} \text{there are constants } M > 0,\ \sigma > 0 \text{ with } uf(t,u,z) > 0 \text{ for } |u| > M, \\ t \in [0,\infty),\ z \in (-\sigma,\sigma) \text{ and } u \neq c_i,\ i = 1,\dots,m. \text{ Here } f(t,c_i,0) = 0, \\ t \in [0 \in \infty) \text{ and } i = 1,\dots,m \end{cases}$$

holds. Also suppose

$$(2.3)\quad \begin{cases} \text{there are continuous functions } \psi,\ \phi\colon [0,\infty) \to [0,\infty) \text{ with } |f(t,u,z)| \leq \\ \phi(t)\psi(|z|) \text{ for } u \in [-M,M] \end{cases}$$

and

$$(2.4)\qquad p^2 q\phi \text{ is bounded on } [0,\infty).$$

Define $H(z) = \int_0^z \frac{u}{\psi(u)}\,du$, $z > 0$ *which is strictly increasing and suppose*

$$(2.5)\qquad 2M \sup_{[0,\infty)} p^2 q\phi \in \mathrm{dom}(H^{-1}).$$

Then (1.2) has a solution $y \in C[0,n] \cap C^2(0,n]$ *with* $py' \in C[0,n]$. *Moreover we have*

$$\sup_{[0,n]} |y(t)| \leq M, \quad \sup_{(0,n]} |p(t)y'(t)| \leq H^{-1}\big(2M \sup_{[0,\infty)} p^2 q\phi\big) \equiv M_1$$

and $\big|\big(p(t)y'(t)\big)'\big| \leq M_2 p(t)q(t)\phi(t)$, $t \in (0,n)$ *where* $M_2 = \sup_{[0,M_1]} \psi(v)$.

REMARK. Let $K_1 = M$ in (2.1) and $K_2 = M$ in (2.2). Then (2.2) implies (2.1) with $K_1 = \max\{K_2, |c_i|\}$. However for the semi-infinite problem (2.1), with $K_1 = \max\{K_2, |c_i|\}$, may be too restrictive in some situations; see example (i) in Section 3.

PROOF. Let y be a solution to $(1.6)_\lambda$. We first show that

$$(2.6)\qquad \sup_{[0,n]} |y(t)| \leq M.$$

To begin with suppose (2.1) is satisfied. Suppose $|y(t)|$ achieves a maximum at $t_0 \in (0,n)$. Then $y'(t_0) = 0$ and $y(t_0)y''(t_0) \leq 0$. Assume $|y(t_0)| > M$. Then

$$y(t_0)\big(p(t_0)y'(t_0)\big)' = \lambda y(t_0)p(t_0)q(t_0)f\big(t_0, y(t_0), 0\big) > 0,$$

i.e. $y(t_0)p(t_0)y''(t_0) > 0$, a contradiction. Consequently $|y(t_0)| \leq M$ and (2.6) is proven in this case.

Now suppose (2.2) is satisfied. Suppose $|y(t)|$ achieves a maximum at $t_0 \in (0,n)$, so $y'(t_0) = 0$ and $y(t_0)y''(t_0) \leq 0$. Assume $|y(t_0)| > M$. If $y(t_0) \neq c_i$, $i = 1,\dots,m$, we have a contradiction as before, so $|y(t_0)| \leq M$. On the other hand suppose $y(t_0) = c_i$ for some $i = 1,\dots,m$, say c_1. There exists by (2.2), $t_1, t_2 \in (0,n)$ with $y(t) = c_1$ for $t \in [t_1,t_2]$,

$t_1 \leq t_0 \leq t_2$, and $y(t) \neq c_1$ for $t > t_2$ and close to t_2 and $t < t_1$ and close to t_1. Then (2.2) implies that there exists intervals (t_2, δ) and (τ, t_1) with

$$y(t)f\big(t, y(t), p(t)y'(t)\big) > 0$$

for $t \in (t_2, \delta)$ and $t \in (\tau, t_1)$. Consequently $y(t)\big(p(t)y'(t)\big)' > 0$ for $t \in (t_2, \delta)$ and $t \in (\tau, t_1)$. Suppose $y(t_0) > 0$. Then $(py')' > 0$ for $t > t_2$ and close to t_2 and $t < t_1$ and close to t_1. This together with $y'(t) = 0$, $t_1 \leq t_0 \leq t_2$ implies $y' < 0$ for $t < t_1$ and close to t_1 and $y' > 0$ for $t > t_2$ and close to t_2, which contradicts the maximality of $y(t_0) = |y(t_0)|$. A similar contradiction occurs if $y(t_0) < 0$. So (2.6) is also true in this case.

Now the boundary conditions imply that y' has at least one zero in $(0, n)$. Consequently, if for some $t \in [0, n]$ with $p(t)y'(t) \neq 0$, then there is an interval $[\mu, \nu]$ containing t on which py' maintains a constant sign and py' vanishes at one of the endpoints. To be definite assume $py' > 0$ on (μ, ν) and $p(\mu)y'(\mu) = 0$. Then on (μ, ν), $(py')' \leq pq\phi\psi(py')$ so

$$\frac{py'(py')'}{\psi(py')} \leq p^2 q\phi y'$$

and integration from μ to t yields

$$H\big(p(t)y'(t)\big) \leq [y(t) - y(\mu)] \sup_{[0,\infty)} p^2 q\phi.$$

Thus

$$|p(t)y'(t)| \leq H^{-1}\big(2M \sup_{[0,\infty)} p^2 q\phi\big) \equiv M_1. \tag{2.7}$$

The same bound M_1 is obtained if $py' < 0$ on (μ, ν) and/or py' vanishes at ν.

Now Theorem 1.1 implies that (1.2) has a solution y. In addition the properties of y given in the statement of the theorem follow from (2.6), (2.7), (2.3) together with the differential equation. ∎

Next two results are presented which rely on the "zero set" of the nonlinearity f. The first establishes the existence of a nonpositive solution. We remark that an analogue result could be obtained for nonnegative solutions.

THEOREM 2.2. *Suppose (1.3), (1.4), (1.5) and (2.3) with $u \in [-M, 0]$ are satisfied. In addition assume*

(2.8) *there is a constant $M > 0$ with $f(t, u, 0) < 0$ for $u < -M$ and $t \in [0, \infty)$*

or

(2.9) $\begin{cases} \text{there are constants } M > 0,\ \sigma > 0 \text{ with } f(t,u,z) < 0 \text{ for } u < -M, \\ t \in [0,\infty),\ z \in (-\sigma,\sigma) \text{ and } u \neq c_i,\ i = 1,\dots,m. \text{ Here } f(t,c_i,0) = 0, \\ t \in [0,\infty) \text{ and } i = 1,\dots,m \end{cases}$

hold.

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(i) Suppose there exists s_1, r_1 *with* $s_1 < 0 < r_1$ *and*

$$(2.10)\begin{cases} f(t,u,r_1) \leq 0,\ t \in [0,\infty) \text{ and } -M \leq u \leq 0 \text{ and } f(t,u,s_1) \leq 0,\ t \in [0,\infty) \\ \text{and } -M \leq u \leq 0 \end{cases}$$

and

$$(2.11) \qquad f(t,0,0) \geq 0, \quad t \in (0,\infty)$$

hold. Then (1.2) has a solution y with

$$s_1 \leq p(t)y'(t) \leq r_1 \text{ and } -M \leq y(t) \leq 0 \quad \text{for } t \in [0,n],$$

and

$$\left|\big(p(t)y'(t)\big)'\right| \leq p(t)q(t)\phi(t) \sup_{[s_1,r_1]} \psi(|z|), \quad t \in (0,n).$$

In addition if

$$(2.12) \qquad f(t,u,z) \geq 0, \quad t \in [0,\infty),\ u \in [-M,0] \text{ and } z \in (s_1,r_1)$$

then $(py')' \geq 0$ *for* $t \in (0,n)$.

(ii) Suppose there exists $r_1 > 0$ *with (2.11) and*

$$(2.13) \qquad f(t,u,r_1) \leq 0, \quad t \in [0,\infty) \text{ and } -M \leq u \leq 0$$

holding. Define $J(z) = \int_0^z \frac{u}{\psi(u)}\,du$, $z > 0$ *which is strictly increasing and suppose (2.4) and*

$$(2.14) \qquad M \sup_{[0,\infty)} p^2 q\phi \equiv N \in \operatorname{dom}(J^{-1})$$

hold. Then (1.2) has a solution y with

$$-J^{-1}(N) \leq p(t)y'(t) \leq r_1 \text{ and } -M \leq y(t) \leq 0 \quad \text{for } t \in [0,n],$$

and

$$\left|\big(p(t)y'(t)\big)'\right| \leq p(t)q(t)\phi(t) \sup_{[-J^{-1}(N),r_1]} \psi(|z|), \quad t \in (0,n).$$

In addition if

$$(2.15) \qquad f(t,u,z) \geq 0, \quad t \in [0,\infty),\ u \in [-M,0] \text{ and } z \in (-\infty,r_1)$$

then $(py')' \geq 0$ *for* $t \in (0,n)$.

(iii) Suppose there exists $s_1 < 0$ *with (2.4), (2.11) and (2.14) holding. Also suppose*

$$(2.16) \qquad f(t,u,s_1) \leq 0, \quad t \in [0,\infty) \text{ and } -M \leq u \leq 0$$

is satisfied. Then (1.2) has a solution y with

$$s_1 \leq p(t)y'(t) \leq J^{-1}(N) \text{ and } -M \leq y(t) \leq 0 \quad \text{for } t \in [0,n],$$

and

$$\left|\big(p(t)y'(t)\big)'\right| \le p(t)q(t)\phi(t) \sup_{[s_1,J^{-1}(N)]} \psi(|z|), \quad t \in (0,n).$$

In addition if

$$(2.17) \qquad f(t,u,z) \ge 0, \quad t \in [0,\infty),\ u \in [-M,0] \text{ and } z \in (s_1,\infty)$$

then $(py')' \ge 0$ *for* $t \in (0,n)$.

(iv) Suppose (2.4), (2.11) and (2.14) hold. Then (1.2) has a solution y with

$$-J^{-1}(N) \le p(t)y'(t) \le J^{-1}(N) \text{ and } -M \le y(t) \le 0 \quad \text{for } t \in [0,n],$$

and

$$\left|\big(p(t)y'(t)\big)'\right| \le p(t)q(t)\phi(t) \sup_{[0,J^{-1}(N)]} \psi(|z|), \quad t \in (0,n).$$

In addition if

$$(2.18) \qquad f(t,u,z) \ge 0, \quad t \in [0,\infty),\ u \in [-M,0] \text{ and } z \in (-\infty,\infty)$$

then $(py')' \ge 0$ *for* $t \in (0,n)$.

PROOF. (i) Let y be a solution to

$$(2.19)_\lambda \qquad \begin{cases} \frac{1}{p}(py')' = \lambda q f_1(t,y,py'), & 0 < t < n \\ y(0) = y(n) = 0 \end{cases}$$

where $0 < \lambda < 1$ and

$$f_1(t,u,v) = \begin{cases} f(t,0,v) + u, & u \ge 0 \\ f(t,u,v), & u \le 0,\ s_1 \le v \le r_1 \\ f(t,u,r_1), & u \le 0,\ v \ge r_1 \\ f(t,u,s_1), & u \le 0,\ v \le s_1. \end{cases}$$

We will show that any solution y of $(2.19)_\lambda$ is a solution of $(1.6)_\lambda$. We first show that

$$(2.20) \qquad -M \le y(t) \le 0, \quad t \in [0,n].$$

Suppose y has a positive maximum at $t_0 \in (0,n)$. Then

$$(py')'(t_0) = \lambda p(t_0)q(t_0)[f(t_0,0,0) + y(t_0)] > 0,$$

a contradiction. Thus $y \le 0$ on $[0,n]$. Now $y \ge -M$ follows as in Theorem 2.1 since $s_1 < 0 < r_1$. Thus (2.20) is true.

REMARK. (2.20) is also true if $\lambda = 1$.

We now show

$$(2.21) \qquad s_1 \le p(t)y'(t) \le r_1, \quad t \in [0,n].$$

If $p(t)y'(t) \not\leq r_1$ then there exists $t_1 < t_2 \in [0,n]$ such that $y(t) \leq 0$, $p(t)y'(t) > r_1$ for $t \in (t_1,t_2)$ with $p(t_1)y'(t_1) = r_1$ and $p(t_2)y'(t_2) > r_1$. Consequently

$$0 < p(t_2)y'(t_2) - p(t_1)y'(t_1) = \int_{t_1}^{t_2} (py')'\,ds = \lambda \int_{t_1}^{t_2} p(s)q(s)f\big(s,y(s),r_1\big)\,ds \leq 0,$$

a contradiction. Thus $p(t)y'(t) \leq r_1$ for $t \in [0,n]$. Similarly $p(t)y'(t) \geq s_1$ for $t \in [0,n]$ and (2.21) follows.

REMARK. (2.21) is also true if $\lambda = 1$.

Now Theorem 1.1 guarantees that $(2.19)_1$ has a solution y. Hence y is a solution of (1.2) and all the properties hold.

(ii) Let y be a solution to

$$(2.22)_\lambda \qquad \begin{cases} \frac{1}{p}(py')' = \lambda q f_2(t,y,py'), & 0 < t < n \\ y(0) = y(n) = 0 \end{cases}$$

where $0 < \lambda < 1$ and

$$f_2(t,u,v) = \begin{cases} f(t,0,v) + u, & u \geq 0 \\ f(t,u,v), & u \leq 0, v \leq r_1 \\ f(t,u,r_1), & u \leq 0, v \geq r_1. \end{cases}$$

If y is a solution to $(2.22)_\lambda$ then (2.20) holds. In addition, as in part (i), we have

$$(2.23) \qquad p(t)y'(t) \leq r_1, \quad t \in [0,n].$$

The fact that

$$(2.24) \qquad -J^{-1}(N) \leq p(t)y'(t), \quad t \in [0,n]$$

follows as in Theorem 2.1. Now Theorem 1.1 guarantees that $(2.22)_1$ has a solution y. Hence y is a solution (1.2) and all the properties hold.

(iii) and (iv). The proof follows from a slight modification of the above arguments. ∎

Of course we may obtain analogue results for nonnegative solutions and solutions with no fixed sign. We illustrate with one example.

THEOREM 2.3. *Suppose (1.3), (1.4), (1.5), (2.1) or (2.2), and (2.3) are satisfied. Suppose there exists s_1, r_1 with $s_1 < 0 < r_1$ and*

$$(2.25) \begin{cases} uf(t,u,r_1) \geq 0,\ t \in [0,\infty) \text{ and } -M \leq u \leq M \text{ and } uf(t,u,s_1) \geq 0, \\ t \in [0,\infty) \text{ and } -M \leq u \leq M. \end{cases}$$

Then (1.2) has a solution y with

$$s_1 \leq p(t)y'(t) \leq r_1 \text{ and } -M \leq y(t) \leq M \quad \text{for } t \in [0,n],$$

and

$$\left|\big(p(t)y'(t)\big)'\right| \leq p(t)q(t)\phi(t) \sup_{[s_1,r_1]} \psi(|z|), \quad t \in (0,n).$$

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PROOF. Let y be a solution to

$$(2.26)_\lambda \qquad \begin{cases} \frac{1}{p}(py')' = \lambda q f_3(t,y,py'), & 0<t<n \\ y(0)=y(n)=0 \end{cases}$$

where $0<\lambda<1$ and

$$f_3(t,u,v) = \begin{cases} f(t,u,r_1), & v \ge r_1 \\ f(t,u,v), & s_1 \le v \le r_1 \\ f(t,u,s_1), & v \le s_1. \end{cases}$$

We will show that any solution y of $(2.26)_\lambda$ is a solution $(1.6)_\lambda$. As in Theorem 2.1 we have (2.6) holding (since $s_1<0<r_1$), *i.e.*,

$$-M \le y(t) \le M, \quad t\in[0,n].$$

We now show

$$(2.27) \qquad s_1 \le p(t)y'(t) \le r_1, \quad t\in[0,n].$$

If $p(t)y'(t) \not\le r_1$ then one of the following conditions occur:

(i) there exists $t_1<t_2\in[0,n]$ such that $y(t)\ge 0$, $p(t)y'(t)>r_1$ for $t\in(t_1,t_2)$ with $p(t_1)y'(t_1)>r_1$ and $p(t_2)y'(t_2)=r_1$

or

(ii) there exists $t_1<t_2\in[0,n]$ such that $y(t)\le 0$, $p(t)y'(t)>r_1$ for $t\in(t_1,t_2)$ with $p(t_1)y'(t_1)=r_1$ and $p(t_2)y'(t_2)>r_1$.

If (i) holds then

$$0>p(t_2)y'(t_2)-p(t_1)y'(t_1)=\int_{t_1}^{t_2}(py')'\,ds=\lambda\int_{t_1}^{t_2}p(s)q(s)f\big(s,y(s),r_1\big)\,ds\ge 0,$$

a contradiction. If (ii) holds then

$$0<p(t_2)y'(t_2)-p(t_1)y'(t_1)=\int_{t_1}^{t_2}(py')'\,ds=\lambda\int_{t_1}^{t_2}p(s)q(s)f\big(s,y(s),r_1\big)\,ds\le 0,$$

a contradiction. Thus $p(t)y'(t)\le r_1$ for $t\in[0,n]$. Similarly $p(t)y'(t)\ge s_1$ for $t\in[0,n]$ and (2.27) follows. Now Theorem 1.1 guarantees that $(2.26)_1$ has a solution y. Hence y is a solution of (1.2) and all the properties hold. ∎

3. **Global solvability.** The results of Section 2 together with Arzela-Ascoli theorem will now imply the solvability of (1.1).

THEOREM 3.1. *Suppose (1.3), (1.4), (1.5), (2.1) or (2.2), (2.3), (2.4) and (2.5) are satisfied. In addition suppose*

$$(3.1) \qquad \int_0^b p(s)q(s)\phi(s)\,ds<\infty \quad \textit{for any } b>0.$$

Then (1.1) has at least one solution $y\in BC^2[0,\infty)$ *with*

$$|y(t)|\le M,\ |p(t)y'(t)|\le H^{-1}\big(2M\sup_{[0,\infty)}p^2q\phi\big) \quad \textit{for } t\in[0,\infty).$$

PROOF. By Theorem 2.1 there exists a solution y_n to (1.2) with

$$\sup_{[0,n]} |y_n(t)| \leq M, \quad \sup_{[0,n]} |p(t)y_n'(t)| \leq H^{-1}\big(2M \sup_{[0,\infty)} p^2 q\phi\big) \equiv M_1$$

and $\left|\big(p(t)y_n'(t)\big)'\right| \leq M_2 p(t)q(t)\phi(t)$, $t \in (0,n)$ where $M_2 = \sup_{[0,M_1]} \psi(v)$. Consequently for $t, s \in [0,n]$ we have

$$|y_n(t) - y_n(s)| = \left|\int_s^t \frac{1}{p(u)} p(u)y_n'(u)\,du\right| \leq M_1 \left|\int_s^t \frac{du}{p(u)}\right|$$

and

$$|p(t)y_n'(t) - p(s)y_n'(s)| = \left|\int_s^t \big(p(u)y_n'(u)\big)'\,du\right| \leq M_1 \left|\int_s^t p(u)q(u)\phi(u)\,du\right|.$$

Now define functions u_n on $[0,\infty)$ by $u_n(x) = y_n(x)$ for $x \in [0,n]$ and $u_n(x) = 0$ for $x > n$. Each u_n belongs to $C[0,\infty)$ and is twice continuously differentiable on $(0,\infty)$ except possibly at $x = n$. Let $S = \{u_n\}_{n=1}^{\infty}$. By the Arzela-Ascoli theorem there is a subsequence N_1^* of N^+ and functions $z_1, pz_1' \in C[0,1]$ with $u_n(x) \longrightarrow z_1(x)$, $p(x)u_n'(x) \longrightarrow p(x)z_1'(x)$ uniformly on $[0,1]$ as $n \longrightarrow \infty$ through N_1^*. Let $N_1 = N_1^*/\{1\}$. Then by the Arzela-Ascoli theorem there is a subsequence N_2^* of N_1 and functions $z_2, pz_2' \in C[0,2]$ with $u_n(x) \longrightarrow z_2(x)$, $p(x)u_n'(x) \longrightarrow p(x)z_2'(x)$ uniformly on $[0,2]$ as $n \longrightarrow \infty$ through N_2^*. Note since $N_2^* \subset N_1$ we have $z_2 = z_1$ on $[0,1]$. Let $N_2 = N_2^*/\{2\}$ and proceed inductively to obtain for $k = 1,2,\ldots$, a subsequence $N_k^* \subset N_{k-1}$ and functions $z_k, pz_k' \in C[0,k]$ with $u_n(x) \longrightarrow z_k(x)$, $p(x)u_n'(x) \longrightarrow p(x)z_k'(x)$ uniformly on $[0,k]$ as $n \longrightarrow \infty$ through N_k^*. Note since $N_k^* \subset N_{k-1}$ we have $z_k = z_{k-1}$ on $[0,k-1]$.

Define a function y as follows. Fix $x \in [0,\infty)$ and let $k \in N^+$ with $x \leq k$. Then define $y(x) = z_k(x)$. Now y is well defined with $y \in C[0,\infty)$ and $py' \in C[0,\infty)$. Fix x and choose and fix $k \geq x$, $k \in N^+$. Then

$$\begin{aligned} u_n(x) = {} & \int_0^k \frac{1}{p(v)} \int_v^k p(s)q(s)f\big(s, u_n(s), p(s)u_n'(s)\big)\,ds\,dv \left(\int_0^k \frac{ds}{p(s)}\right)^{-1} \int_0^x \frac{ds}{p(s)} \\ & - \int_0^x \frac{1}{p(v)} \int_v^k p(s)q(s)f\big(s, u_n(s), p(s)u_n'(s)\big)\,ds\,dv + u_n(k)\frac{x}{k} \end{aligned}$$

for $n \in N_k$. Let $n \longrightarrow \infty$ through N_k to obtain

$$\begin{aligned} z_k(x) = {} & \int_0^k \frac{1}{p(v)} \int_v^k p(s)q(s)f\big(s, z_k(s), p(s)z_k'(s)\big)\,ds\,dv \left(\int_0^k \frac{ds}{p(s)}\right)^{-1} \int_0^x \frac{ds}{p(s)} \\ & - \int_0^x \frac{1}{p(v)} \int_v^k p(s)q(s)f\big(s, z_k(s), p(s)z_k'(s)\big)\,ds\,dv + z_k(k)\frac{x}{k}. \end{aligned}$$

Thus

$$\begin{aligned} y(x) = {} & \int_0^k \frac{1}{p(v)} \int_v^k p(s)q(s)f\big(s, y(s), p(s)y'(s)\big)\,ds\,dv \left(\int_0^k \frac{ds}{p(s)}\right)^{-1} \int_0^x \frac{ds}{p(s)} \\ & - \int_0^x \frac{1}{p(v)} \int_v^k p(s)q(s)f\big(s, y(s), p(s)y'(s)\big)\,ds\,dv + y(k)\frac{x}{k} \end{aligned}$$

and so $\big(p(x)y'(x)\big)' = p(x)q(x)f\big(x, y(x), p(x)y'(x)\big)$. Consequently $(py')' \in C(0,\infty)$ with $\frac{1}{p}(py')' = qf(t,y,py')$, $0 < t < \infty$. It also follows immediately that $|y(t)| \leq M$ and $|p(t)y'(t)| \leq M_1$ for $0 \leq t < \infty$. ∎

THEOREM 3.2. *Suppose (1.3), (1.4), (1.5), (2.3) with* $u \in [-M, 0]$, *(2.8) or (2.9), (2.11) and (3.1) are satisfied.*

(i) Suppose (2.10) holds. Then (1.1) has at least one solution $y \in \mathrm{BC}^2[0,\infty)$ *with* $-M \leq y(t) \leq 0$, $s_1 \leq p(t)y'(t) \leq r_1$ *for* $t \in [0,\infty)$. *If in addition (2.12) is satisfied then* $(py')' \geq 0$ *on* $(0,\infty)$.

(ii) Suppose (2.4), (2.13) and (2.14) are satisfied. Then (1.1) has at least one solution $y \in \mathrm{BC}^2[0,\infty)$ *with* $-M \leq y(t) \leq 0$, $-J^{-1}(N) \leq p(t)y'(t) \leq r_1$ *for* $t \in [0,\infty)$. *If in addition (2.15) is satisfied then* $(py')' \geq 0$ *on* $(0,\infty)$.

(iii) Suppose (2.4), (2.14) and (2.16) are satisfied. Then (1.1) has at least one solution $y \in \mathrm{BC}^2[0,\infty)$ *with* $-M \leq y(t) \leq 0$, $s_1 \leq p(t)y'(t) \leq J^{-1}(N)$ *for* $t \in [0,\infty)$. *If in addition (2.17) is satisfied then* $(py')' \geq 0$ *on* $(0,\infty)$.

(iv) Suppose (2.4) and (2.14) are satisfied. Then (1.1) has at least one solution $y \in \mathrm{BC}^2[0,\infty)$ *with* $-M \leq y(t) \leq 0$, $-J^{-1}(N) \leq p(t)y'(t) \leq J^{-1}(N)$ *for* $t \in [0,\infty)$. *If in addition (2.18) is satisfied then* $(py')' \geq 0$ *on* $(0,\infty)$.

PROOF. Essentially the same reasoning as in Theorem 3.1 (except we now use Theorem 2.2) yields the result, *i.e.*, we obtain a solution $y \in \mathrm{BC}^2[0,\infty)$ with the appropriate bounds on $|y|$ and $|py'|$. ∎

In addition we have

THEOREM 3.3. *Suppose (1.3), (1.4), (1.5), (2.1) or (2.2), (2.3), (2.25) and (3.1) are satisfied. Then (1.1) has at least one solution* $y \in \mathrm{BC}^2[0,\infty)$ *with* $-M \leq y(t) \leq M$, $s_1 \leq p(t)y'(t) \leq r_1$ *for* $t \in [0,\infty)$.

EXAMPLES. (i) Consider the boundary value problem

$$(3.2)\qquad \begin{cases} \frac{1}{t^\alpha}(t^\alpha y')' = t^\beta(1+y)(y+c)^2(2-t^\alpha y')^m(3+t^\alpha y')^n, & 0 < t < \infty \\ y(0) = 0,\ y(t) \text{ bounded on } [0,\infty) \end{cases}$$

with $0 \leq \alpha < 1$, $\alpha+\beta > -1$, $m \geq 1$, $n \geq 1$ and $c \neq 0$.

To show that (3.2) has a solution $y \in \mathrm{BC}^2[0,\infty)$ we will apply Theorem 3.2(i). Let $p(t) = t^\alpha$, $q(t) = t^\beta$

$$f(t,u,z) = (u+1)(u+c)^2(2-z)^m(3+z)^n.$$

It follows easily that (1.3), (1.4), (1.5) and (3.1) are satisfied since $0 < \alpha < 1$ and $\alpha+\beta > -1$. Now if $-1 \leq c \leq 1$, with $c \neq 0$, then (2.8) holds with $M = 1$. However if $c > 1$ or $c < -1$, (2.9) holds with $M = 1$, $\sigma = 1$ say, and $c_1 = c$.

REMARK. It is not a good idea to choose $M = |c|$ in (2.8) if $c > 1$ or $c < -1$ because of (2.12). This example illustrates why (2.8) may be too restrictive in some situations when examining the semi-infinite problem (see Section 4).

Let $s_1 = -3$ and $r_1 = 2$. Certainly (2.3), (2.10) and (2.11) are satisfied. Also (2.12) is true since $M = 1$.

Thus Theorem 3.2(i) implies that (3.2) has at least one solution $y \in \mathrm{BC}^2[0,\infty)$ with $-1 \leq y(t) \leq 0$, $-3 \leq t^\alpha y'(t) \leq 2$ for $t \in [0,\infty)$ and $(t^\alpha y')' \geq 0$ on $(0,\infty)$.

(ii) Next consider

$$(3.3)\qquad \begin{cases} \frac{1}{t^\alpha}(t^\alpha y')' = t^\beta(1+y)(2-t^\alpha y')^m(3+t^\alpha y')^n, & 0<t<\infty \\ y(0)=0,\ y(t) \text{ bounded on } [0,\infty) \end{cases}$$

with $0 \le \alpha < 1, \alpha+\beta > -1, 2\alpha+\beta = 0, m \ge 0, n \ge 0$ and $0 \le m+n \le 2$.

To show that (3.3) has a solution $y \in BC^2[0,\infty)$ we will apply Theorem 3.1. It is easy to check that (1.3), (1.4), (1.5), (3.1) and (2.1), with $M = 1$, hold.

Let $p(t) = t^\alpha$, $q(t) = t^\beta$, $\phi(t) = 2$ and $\psi(|z|) = (2+|z|)^m(3+|z|)^n$. Certainly (2.3) is true and (2.4) follows since $2\alpha+\beta = 0$. Finally (2.5) is satisfied since $\int_0^\infty \frac{u}{\psi(u)}\,du = \infty$. Consequently Theorem 3.1 implies (3.3) has at least one solution $y \in BC^2[0,\infty)$.

(iii) Consider

$$(3.4)\qquad \begin{cases} y'' = (1+y)(A+y')^m, & 0<t<\infty \\ y(0)=0,\ y(t) \text{ bounded on } [0,\infty) \end{cases}$$

with $A > 0, m \ge 1$ and $A > 0, 1 < \frac{A^{2-m}}{(m-2)(m-1)}$ if $m > 2$.

To see that (3.4) has a solution $y \in BC^2[0,\infty)$ we will apply Theorem 3.2(iii). Let $p = q = \phi = 1, f(t,u) = (u+1)(A+z)^m$ and $s_1 = -A$. It is also easy to check that (1.3), (1.4), (1.5), (3.1), (2.8) with $M = 1$, (2.11), (2.16) and (2.17) hold. Also with $\psi(z) = (A+z)^m, z > 0$ we have

$$\int_0^\infty \frac{u}{(A+u)^m}\,du = \infty \quad \text{if } 1 \le m \le 2$$

whereas

$$\int_0^\infty \frac{u}{(A+u)^m}\,du = \frac{A^{2-m}}{(m-2)(m-1)} \quad \text{if } m > 2.$$

Thus (2.14) is satisfied. Consequently Theorem 3.2(iii) implies that (3.4) has a solution $y \in BC^2[0,\infty)$.

4. **Semi infinite problem.** We will now use the results of Section 3 to discuss the boundary value problem

$$\begin{cases} \frac{1}{p}(py')' = qf(t,y,py'), & 0<t<\infty \\ y(0)=0,\ \lim_{t\to\infty} y(t) \text{ exists.} \end{cases}$$

THEOREM 4.1. *Suppose (1.3), (1.4), (1.5), (2.3) with $u \in [-M,0]$, (2.8) or (2.9), (2.11) and (3.1) are satisfied.*

(i) Suppose (2.4), (2.14) and (2.18) hold. In addition assume the following:

(4.1) *Let $\beta > 0$ and $c > 0$ be fixed. Then for all u with $M \ge u+M \ge \beta$ and $t \ge c$ there exists a constant $K > 0$ (which may depend on β and c) with $f(t,u,z) \ge K$ for $z \in [-J^{-1}(N), J^{-1}(N)]$. Here J and N are as described in (2.14)*

and

(4.2) $\lim_{t\to\infty}\left(A\int_b^t \frac{1}{p(s)}\int_b^s p(z)q(z)\,dz\,ds - B\int_b^t \frac{ds}{p(s)} =\right) +\infty$ *for any constants $A > 0$, $B > 0$ and $b > 0$.*

Then the boundary value problem

$$(4.3)\qquad \begin{cases} \frac{1}{p}(py')' = qf(t,y,py'), & 0 < t < \infty \\ y(0) = 0,\ \lim_{t\to\infty} y(t) = -M \end{cases}$$

has at least one solution $y \in \mathrm{BC}^2[0,\infty)$.

(ii) Suppose (2.4), (2.13), (2.14) and (2.15) are satisfied. In addition assume the following:

$$(4.4)\qquad \int_b^\infty \frac{ds}{p(s)} = \infty \quad \textit{for any } b > 0$$

$$(4.5)\qquad \begin{cases} \textit{Let } \beta > 0 \textit{ and } c > 0 \textit{ be fixed. Then for all } u \textit{ with } M \ge u+M \ge \beta \textit{ and} \\ t \ge c \textit{ there exists a constant } K > 0 \textit{ (which may depend on } \beta \textit{ and } c\textit{) with} \\ f(t,u,z) \ge Kg(z) \textit{ for } z \in \big[-J^{-1}(N), r_1\big). \textit{ Here } J \textit{ and } N \textit{ are as described in} \\ \textit{(2.14); also } g\colon \mathbf{R} \to \mathbf{R} \textit{ is such that } g(0) > 0 \textit{ and } g \textit{ has no negative zero's} \\ \textit{and its first positive zero is } r_1 \end{cases}$$

and

$$(4.6)\qquad \begin{cases} \textit{Let } G(z) = \int_{-J^{-1}(N)}^z \frac{du}{g(u)},\ -J^{-1}(N) \le z < r_1. \textit{ Assume for any constants} \\ A > 0 \textit{ and } b > 0 \textit{ that } \lim_{t\to\infty} \int_b^t \frac{1}{p(s)} G^{-1}\big(A \int_b^s p(z)q(z)\,dz\big)\,ds = +\infty. \end{cases}$$

Then (4.3) has at least one solution $y \in \mathrm{BC}^2[0,\infty)$.

(iii) Suppose (2.4), (2.14), (2.16), (2.17), (4.2) and (4.4) are satisfied. In addition assume the following:

$$(4.7)\qquad \begin{cases} \textit{Let } \beta > 0,\ c > 0 \textit{ and } a, \textit{ with } s_1 < a \le J^{-1}(N), \textit{ be fixed. Then for all } u \\ \textit{with } M \ge u+M \ge \beta \textit{ and } t \ge c \textit{ there exists a constant } K > 0 \textit{ (which may} \\ \textit{depend on } \beta,\ c \textit{ and } a\textit{) with } f(t,u,z) \ge K \textit{ for } z \in [a, J^{-1}(N)]. \textit{ Here } J \textit{ and} \\ N \textit{ are as described in (2.14).} \end{cases}$$

Then (4.3) has at least one solution $y \in \mathrm{BC}^2[0,\infty)$.

(iv) Suppose (2.10), (4.2) and (4.4) are satisfied. In addition assume the following:

$$(4.8)\qquad \begin{cases} \textit{Let } \beta > 0,\ c > 0 \textit{ and } a, \textit{ with } s_1 < a < r_1, \textit{ be fixed. Then for all } u \textit{ with} \\ M \ge u+M \ge \beta \textit{ and } t \ge c \textit{ there exists a constant } K > 0 \textit{ (which may} \\ \textit{depend on } \beta,\ c \textit{ and } a \textit{ and a function } g_1\colon \mathbf{R} \to \mathbf{R} \textit{ with } f(t,u,z) \ge Kg_1(z) \\ \textit{for } z \in [a, r_1). \textit{ Here } g_1, g_1(0) > 0, \textit{ has no negative zero's and its first} \\ \textit{positive zero is } r_1 \end{cases}$$

and

$$(4.9)\qquad \begin{cases} \textit{Let } G_1(z) = \int_{s_1}^z \frac{du}{g_1(u)},\ s_1 \le z < r_1. \textit{ Assume } \int_0^{r_1} \frac{du}{g_1(u)} = \infty \textit{ and} \\ \lim_{t\to\infty} \int_b^t \frac{1}{p(s)} G_1^{-1}\big(A \int_b^s p(z)q(z)\,dz\big)\,ds = +\infty \textit{ for any constants } A > 0 \\ \textit{and } b > 0. \end{cases}$$

Then (4.3) has at least one solution $y \in \mathrm{BC}^2[0,\infty)$.

PROOF. Theorem 3.2 implies there exists $y \in \mathrm{BC}^2[0,\infty)$ with $\frac{1}{p}(py')' = qf(t,y,py')$, $0 < t < \infty$, $y(0) = 0$ and in addition $-M \le y(t) \le 0$ on $[0,\infty)$ and $(py')' \ge 0$ on $(0,\infty)$.

Thus py' is nondecreasing on $(0,\infty)$. If there exists a $\xi \in (0,\infty)$ with $p(\xi)y'(\xi) = 0$ then $p(t)y'(t) \geq 0$ for $t \geq \xi$ so y is monotonic for $t \geq \xi$. On the other hand if no such ξ exists then y is monotonic. The above together with $-M \leq y(t) \leq 0$ on $[0,\infty)$ implies $\lim_{t\to\infty} y(t)$ exists. Consequently $\lim_{t\to\infty} y(t) = \alpha$ with $\alpha \in [-M, 0]$. It remains to show $\alpha = -M$. To do this we assume $-M < \alpha \leq 0$.

(i) From Theorem 3.2 we know that $-J^{-1}(N) \leq p(t)y'(t) \leq J^{-1}(N)$ for $t \in [0,\infty)$. Also since $\lim_{t\to\infty} y(t) = \alpha$ there exists a $c > 0$ with $y(t) + M \geq \frac{1}{2}(\alpha + M) > 0$ for $t \geq c$. Now assumption (4.1) guarantees the existence of a constant $K > 0$ with $(py')' \geq pqK$ for $t \geq c$. Integration from c to t ($t > c$) yields

$$p(t)y'(t) \geq p(c)y'(c) + K\int_c^t p(z)q(z)\,dz \geq -J^{-1}(N) + K\int_c^t p(z)q(z)\,dz$$

and another integration from c to t yields

$$y(t) \geq \frac{1}{2}(\alpha - M) - J^{-1}(N)\int_c^t \frac{ds}{p(s)} + K\int_c^t \frac{1}{p(s)}\int_c^s p(z)q(z)\,dz\,ds.$$

Now (4.2) implies $y(t)$ is unbounded on $[0,\infty)$, a contradiction. Thus $\lim_{t\to\infty} y(t) = -M$.

(ii) From Theorem 3.2 we know that $-J^{-1}(N) \leq p(t)y'(t) \leq r_1$ for $t \in [0,\infty)$. We now claim that in fact $-J^{-1}(N) \leq p(t)y'(t) < r_1$ for $t \in [0,\infty)$. To see this suppose $p(\eta)y'(\eta) = r_1$ for some $\eta \in [0,\infty)$. Then since $(py')' \geq 0$ we have $p(t)y'(t) = r_1$ for $t \geq \eta$ and so $y(t) = r_1 \int_\eta^t \frac{ds}{p(s)} + y(\eta)$ for $t \geq \eta$. Now (4.4) implies $y(t)$ is unbounded on $[0,\infty)$, a contradiction. Consequently $-J^{-1}(N) \leq p(t)y'(t) < r_1$ for $t \in [0,\infty)$. Also as in (i), there exists a $c > 0$ with $y(t) + M \geq \frac{1}{2}(\alpha + M) > 0$ for $t \geq c$. Now assumption (4.5) guarantees the existence of a constant $K > 0$ with $(py')' \geq pqKg(py')$ for $t \geq c$. Integration from c to t ($t > c$) yields

$$\int_{-J^{-1}(N)}^{p(t)y'(t)} \frac{du}{g(u)} \geq \int_{p(c)y'(c)}^{p(t)y'(t)} \frac{du}{g(u)} \geq K\int_c^t p(z)q(z)\,dz.$$

Now this implies for $t \geq c$ that

$$p(t)y'(t) \geq G^{-1}\Big(K\int_c^t p(z)q(z)\,dz\Big)$$

since $G\colon \big[-J^{-1}(N), r_1\big) \to [0,\infty)$ is strictly increasing. Another integration from c to t yields

$$y(t) \geq \frac{1}{2}(\alpha - M) + \int_c^t \frac{1}{p(s)} G^{-1}\Big(K\int_c^s p(z)q(z)\,dz\Big)\,ds.$$

Assumption (4.6) implies $y(t)$ is unbounded on $[0,\infty)$, a contradiction. Thus $\lim_{t\to\infty} y(t) = -M$.

(iii) From Theorem 3.2 we know that $s_1 \leq p(t)y'(t) \leq J^{-1}(N)$ for $t \in [0,\infty)$. We now claim that there exists $t_0 \in [0,\infty)$ with $s_1 < p(t)y'(t) \leq J^{-1}(N)$ for $t > t_0$. To see this suppose $p(\eta)y'(\eta) = s_1$ for some $\eta \in [0,\infty)$. Now $(py')' \geq 0$, $t \in [0,\infty)$ implies either $p(t)y'(t) = s_1$ for $t \geq \eta$ or there exists $\xi \geq \eta$ with $p(t)y'(t) = s_1$ on $[\eta,\xi]$ and

$p(t)y'(t) > s_1$ on (ξ,∞). If $p(t)y'(t) = s_1$ for $t \geq \eta$ then $y(t) = s_1 \int_\eta^t \frac{ds}{p(s)} + y(\eta)$, which contradicts the boundedness of y on $[0,\infty)$. Thus our claim is established with $t_0 = \xi$.

Also as in (i), there exists a $c > 0$ with $y(t) + M \geq \frac{1}{2}(\alpha + M) > 0$ for $t \geq c$. Let $d = \max\{c, t_0\} + 1$. Note $p(t)y'(t) \geq p(d)y'(d) > s_1$ for $t \geq d$. Assumption (4.7) guarantees the existence of a constant $K > 0$ (which may depend on d) with $(py')' \geq pqK$ for $t \geq d$. Integration from d to t ($t > d$) yields

$$p(t)y'(t) \geq s_1 + K \int_d^t p(z)q(z)\,dz.$$

Another integration from d to t yields

$$y(t) \geq \frac{1}{2}(\alpha - M) + s_1 \int_d^t \frac{ds}{p(s)} + K \int_d^t \frac{1}{p(s)} \int_d^s p(z)q(z)\,dz\,ds,$$

which contradicts the boundedness of y on $[0,\infty)$. Thus $\lim_{t\to\infty} y(t) = -M$.

(iv) From Theorem 3.2 we know that $s_1 \leq p(t)y'(t) \leq r_1$ for $t \in [0,\infty)$. As in (ii) and (iii) there exists $t_0 \in [0,\infty)$ with $s_1 < p(t)y'(t) < r_1$ for $t > t_0$. Also there exists a $c > 0$ with $y(t) + M \geq \frac{1}{2}(\alpha + M) > 0$ for $t \geq c$. Let $d = \max\{c, t_0\} + 1$. Note $p(t)y'(t) \geq p(d)y'(d) > s_1$ for $t \geq d$. Now assumption (4.8) guarantees the existence of a constant $K > 0$ (which may depend on d) with $(py')' \geq pqKg_1(py')$ for $t \geq d$. Integration from d to t ($t > d$) yields

$$\int_{s_1}^{p(t)y'(t)} \frac{du}{g_1(u)} \geq \int_{p(d)y'(d)}^{p(t)y'(t)} \frac{du}{g_1(u)} \geq K \int_d^t p(z)q(z)\,dz.$$

Now this implies for $t \geq d$ that

$$p(t)y'(t) \geq G_1^{-1}\Big(K \int_d^t p(z)q(z)\,dz\Big)$$

since $G_1\colon [s_1, r_1) \longrightarrow [0,\infty)$ is strictly increasing. Another integration from d to t yields

$$y(t) \geq \frac{1}{2}(\alpha - M) + \int_d^t \frac{1}{p(s)} G_1^{-1}\Big(K \int_d^s p(z)q(z)\,dz\Big)\,ds,$$

which contradicts the boundedness of y on $[0,\infty)$. Thus $\lim_{t\to\infty} y(t) = -M$. ∎

REMARK. From the above analysis we see that if (4.6) is relaxed to

$\lim_{t\to\infty} \int_b^t \frac{1}{p(s)} G^{-1}\big(A \int_b^s p(z)q(z)\,dz\big)\,ds > M$, for any constants $A > 0$ and $b > 0$

then existence of a solution in Theorem 4.1(ii) is again guaranteed. A similar remark applies to (4.2) and (4.9).

EXAMPLES. (i) Consider the semi infinite problem

$$\text{(4.10)} \qquad \begin{cases} y'' = (1+y)(A+y')^m, & 0 < t < \infty \\ y(0) = 0,\ \lim_{t\to\infty} y(t) = -1 \end{cases}$$

with $A > 0$, $m \geq 1$ and $A > 0$, $a < \frac{A^{2-m}}{(m-2)(m-1)}$ if $m > 2$.

Now example (iii) in Section 3 implies

$$\begin{cases} y'' = (1+y)(A+y')^m, & 0 < t < \infty \\ y(0) = 0, \ y(t) \text{ bounded on } [0,\infty) \end{cases}$$

has a solution $y \in BC^2[0,\infty)$. To see that (4.10) has a solution we apply Theorem 4.1(iii). In this case $p = q = 1$, $s_1 = -A$ and $M = 1$. Clearly (4.2) and (4.4) are true and (4.7) follows with $K = \beta(A+a)^m$. Consequently (4.10) has a solution $y \in BC^2[0,\infty)$.

(ii) Consider

$$\begin{cases} y'' = (1+y)(2-y')^m(3+y')^n, & 0 < t < \infty \\ y(0) = 0, \ \lim_{t\to\infty} y(t) = -1 \end{cases} \tag{4.11}$$

with $m \geq 1$ and $n \geq 1$.

Now example (i) in Section 3 implies

$$\begin{cases} y'' = (1+y)(2-y')^m(3+y')^n, & 0 < t < \infty \\ y(0) = 0, \ y(t) \text{ bounded on } [0,\infty) \end{cases}$$

has a solution $y \in BC^2[0,\infty)$. To see that (4.11) has a solution we apply Theorem 4.1(iv). In this case $p = q = 1$, $s_1 = -3$, $r_1 = 2$ and $M = 1$. Clearly (4.2) and (4.4) are true. To see that (4.8) is satisfied let $g_1(z) = (2-z)^m$ and $K = \beta(3+a)^n$. Now $G_1(z) = \int_{-3}^{z} \frac{du}{(2-u)^m}$, $-3 \leq z < 2$ so

$$G_1^{-1}(z) = 2 - 5e^{-z} \quad \text{if } m = 1 \text{ whereas } G_1^{-1}(z) = 2 - \left((m-1)z + 5^{1-m}\right)^{\frac{-1}{m-1}} \quad \text{if } m > 1.$$

In addition if $m = 1$ then

$$\int_b^1 \frac{1}{p(s)} G_1^{-1}\left(A \int_b^s p(z)q(z)\,dz\right) ds = \int_b^t G_1^{-1}\left(A(s-b)\right) ds = 2(t-b) + \frac{5}{A}(e^{-A(t-b)} - 1)$$

whereas if $m > 1$ and $m \neq 2$ then

$$\int_b^t G_1^{-1}\left(A(s-b)\right) ds = 2(t-b) + \frac{1}{2-m}\left((m-1)t + 5^{1-m}\right)^{\frac{m-2}{m-1}} - \frac{1}{2-m}\left((m-1)b + 5^{1-m}\right)^{\frac{m-2}{m-1}}$$

while if $m = 2$ then

$$\int_b^t G_1^{-1}\left(A(s-b)\right) ds = 2(t-b) - \ln(t + 5^{-1}) + \ln(b + 5^{-1}).$$

It is now easy to see that (4.9) is satisfied. Existence of a solution to (4.11) now follows from Theorem 4.1(iv).

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