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Characterization and enumeration of hypotraversable graphs

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Dedicated to Professor E.G. Straus

A graph G on $p \geq 2$ vertices is said to be hypotraversable if G is not traversable but $G - v$ is traversable, that is, $G - v$ has an open trail which contains all the vertices and edges of $G - v$, for each vertex v of G . In this paper we first characterize hypotraversable graphs and then, using this characterization, obtain a complete enumeration of these graphs on p vertices for each positive integer p .

Introduction

A graph G is said to have hypo-property P if G does not have property P but $G - v$ has property P for each vertex v in $V(G)$ Hypohamiltonian graphs have been studied among others in [2], [3], [4], [6], [8], and hypotraceable graphs have been constructed in [7]. Hypo-eulerian graphs and hypotraversable graphs have been studied in [5], where characterizations for hypo-eulerian graphs have been obtained. About hypo-traversable graphs, the author of [5] notes that further research work on these graphs would be necessary to obtain their complete classification.

In this paper we first characterize hypotraversable graphs and then, using this characterization, obtain a complete enumeration of these graphs on p vertices for each positive integer p

Preliminaries

Let G be a graph. A vertex v of G is called an even (odd)

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vertex if $\deg v$ is even (odd). Let $e(G)$ and $E(G)$ $(\varepsilon(G)$ and $O(G))$ denote the number and the set of even (odd) vertices in G, respectively; $e(O(G), E(G))$ and $E(O(G), E(G))$ will denote the number of edges and the set of edges joining $O(G)$ and $E(G)$ in G, respectively. If $v \in V$, let $\Gamma(v)$ denote the set of all vertices adjacent to v in G. If $|\Gamma(v) \cap E(G)| = a$ and $|\Gamma(v) \cap O(G)| = b$ we say that v has the degree pair (a, b). Also a (b) is called the even (odd) degree of v.

A graph G on $p \geq 2$ vertices is said to be hypotraversable if G is not traversable but $G - v$ is traversable, that is, $G - v$ has an open trail which contains all the vertices and edges of $G - v$, for each vectex v of G. The following theorem characterizes traversable graphs.

THEOREM 1 (Euler). *Let G be a connected graph. Then G is traversable if and only if* $\varepsilon(G) = 2$.

Let G be a hypotraversable graph on p vertices. Then $p \geq 3$, $\varepsilon(G) \neq 2$, and $\varepsilon(G-v) = 2$ for each vertex v of G. G being a block, we have $\delta(G) \geq 2$. We note further that $0 \leq \varepsilon(G) \leq p$ and $\varepsilon(G)$ is even.

We will make use of the following theorem.

THEOREM 2 (Chartrand, Kapoor and Kronk [1]). *Let G be a graph with $p \geq 2$ vertices and let $1 \leq n \leq p$. The following conditions are sufficient for G to be n-connected:*

(i) for every k such that $n-1 \leq k \leq (p+(n-3))/2$, the number of vertices of degree not exceeding k does not exceed $k + 1 - n$;

(ii) the number of vertices of degree not exceeding $(p+n-3)/2$ does not exceed $p - n$.

Characterization and enumeration

THEOREM 3. *A graph G is hypotraversable if and only if $\varepsilon(G) = k \neq 2$, G is a block, even vertices of G have degree pair $(0, k-2)$, $(1, k-1)$, or $(2, k)$, and odd vertices of G have degree pair $(0, k-3)$, $(1, k-2)$, or $(2, k-1)$.*

Proof. Let G be a hypotraversable graph. Then $\varepsilon(G) \neq 2$ and G is a block. Let v be any vertex of G and let the degree pair of v be

(a, b) . Since $\varepsilon(G-v) = 2$, a can at most be 2 , and further b shall have the unique value stated in the theorem for each of the values $0, 1, 2$ of a . Conversely, if the degree pair of each vertex v of G is as stated in the theorem, then $\varepsilon(G-v) = 2$ for each v in $V(G)$. Since G is a block, $G - v$ is connected. This together with $\varepsilon(G-v) = 2$ implies that $G - v$ has an open eulerian trail for each vertex v of G . As $\varepsilon(G) \neq 2$, G is hypotraversable.

From Theorem 3 it follows that if G is a graph with $\varepsilon(G) = 0$ then G is hypotraversable if and only if G is a cycle.

As $\varepsilon(G) \neq 2$, in what follows we assume that $\varepsilon(G) = k \geq 4$.

THEOREM 4. *If G is hypotraversable then*

$$e(G) \leq [2k/(k-2)] ,$$

where $[x]$ denotes the greatest integer function.

Proof. Let G be hypotraversable. By Theorem 3, each vertex is adjacent to at least $k - 2$ odd vertices and each odd vertex is adjacent to at most 2 even vertices. Therefore we have $e(G)\ (k-2) \leq 2k$, whence

$$e(G) \leq [2k/(k-2)] .$$

From $e(G) \leq [2k/(k-2)]$ we see that $e(G) = 0, 1, 2, 3$, or 4 .

Below we discuss each case separately.

CASE I. If $e(G) = 0$, then G is hypotraversable if and only if G is $(k-3)$-regular and $k \geq 6$.

Proof. Since $e(G) = 0$, G is hypotraversable if and only if each vertex of G has degree pair $(0, k-3)$ and G is a block, that is if and only if G is $(k-3)$-regular and $k \geq 6$, for then G is a block by Theorem 2.

CASE II. If $e(G) = 1$, then there are exactly two hypotraversable graphs for each $k \geq 6$.

Proof. Let $e(G) = 1$, $E(G) = \{v\}$, and $O(G) = \{w_1, w_2, \ldots, w_k\}$ As v is the only even vertex, it must have degree pair $(0, k-2)$. Let $O(G) - \Gamma(v) = \{w_1, w_2\}$ where $w_1 \neq w_2$. Each vertex in $\Gamma(v)$ must have degree pair $(1, k-2)$ while w_1 and w_2 must have degree pair $(0, k-3)$

so that $\deg w_1 = \deg w_2 = k - 3$. As $\delta(G) \geq 2$ this implies that $k \geq 6$. There are only two such graphs G_1, G_2 for each $k \geq 6$. In view of the high degree of the vertices, we describe their complements $\overline{G}_1$ and $\overline{G}_2$ below:

(i) the edge set of $\overline{G}_1$ consists of

$$vw_1,\ vw_2,\ w_1w_2,\ w_1w_3,\ w_2w_4;\ w_5w_6,\ w_7w_8,\ \ldots,\ w_{k-1}w_k\ ;$$

(ii) the edge set of $\overline{G}_2$ consists of

$$vw_1,\ vw_2,\ w_1w_3,\ w_1w_4,\ w_2w_5,\ w_2w_6;\ w_7w_8,\ w_9w_{10},\ \ldots,\ w_{k-1}w_k\ .$$

Both these graphs G_1 and G_2 are seen to be two-connected, using Theorem 2. So there are two hypotraversable graphs for each $k \geq 6$ in this case.

CASE III. If $e(G) = 2$ then there are four hypotraversable graphs for each $k \geq 6$, and for $k = 4$ there are three.

Proof. Let $e(G) = 2$, $e(G) = \{v_1, v_2\}$, and $O(G) = \{w_1, w_2, \ldots, w_k\}$.

If the vertices v_1 and v_2 are adjacent in G , then degree pair of v_1 and v_2 must be $(1, k-1)$. Let $\Gamma(v_i) \cap O(G) = H_i$ for $i = 1, 2$, so that $|H_1| = |H_2| = k - 1$. Let $H_1 = H_2$ and $O(G) - H_1 = O(G) - H_2 = \{w_1\}$. Then for $2 \leq i \leq k$, w_i must have degree pair $(2, k-1)$. But then this makes the degree pair of w_1 to be $(0, k-1)$, which is larger than its required pair $(0, k-3)$. This is not possible. Therefore $H_1 \neq H_2$. So let $H_1 - H_2 = \{w_1\}$ and $H_2 - H_1 = \{w_2\}$ where $w_1 \neq w_2$. Then for $3 \leq i \leq k$, w_i must have degree pair $(2, k-1)$, and this gives degree pair $(1, k-2)$ to w_1 and w_2 , which is the same as their required degree pairs. Hence there exist only one graph with the required degree pair constraints for this case. This graph is easily seen to be a block by using Theorem 2. Thus there is one hypotraversable graph for each $k \geq 4$ in this case.

If the vertices v_1 and v_2 are not adjacent in G, then the degree pair for v_1 and v_2 is $(0, k-2)$. Let $\Gamma(v_i) = H_i$ for $i = 1, 2$. Then $|H_1| = |H_2| = k - 2$. If $H_1 = H_2$ then

$$O(G) - H_1 = O(G) - H_2 = \{w_1, w_2\} ,$$

say. Each vertex of H_1 has degree pair $(2, k-1)$, which gives degree pair $(0, k-2)$ to w_i for $i = 1, 2$. But this is not possible, as their required degree pair is $(0, k-3)$. Hence $H_1 \neq H_2$. Now there are two possibilities.

(a) $H_1 - H_2 = \{w_1\}$ and $H_2 - H_1 = \{w_2\}$ where $w_1 \neq w_2$. Then $|H_1 \cap H_2| = k - 3$ and $O(G) - (H_1 \cup H_2) = \{w_3\}$. Each vertex of $H_1 \cap H_2$ must have degree pair $(2, k-1)$, so that the degree pair of w_3 becomes $(0, k-3)$, as was required for it. So w_1 and w_2 must be joined by an edge to get their required degree pair $(1, k-2)$. If $k = 4$, w_3 has degree pair $(0, 1)$, which contradicts the fact that the minimum degree in G is at least two. Hence $k \geq 6$. This graph is seen to be a block by using Theorem 2. Thus we get a hypotraversable graph for each $k \geq 6$ in this case.

(b) Let $H_1 - H_2 = \{w_1, w_2\}$ and $H_2 - H_1 = \{w_3, w_4\}$ so that $O(G) - (H_1 \cup H_2) = \emptyset$ and $|H_1 \cap H_2| = k - 4$. For $5 \leq i \leq k$, w_i has degree pair $(2, k-1)$. Hence for $1 \leq i \leq 4$, w_i has degree pair $(1, k-4)$. So w_1, w_2, w_3, w_4 should be joined by a 4-cycle to obtain their required degree pair $(1, k-2)$. Two distinct 4-cycles can be obtained according as w_1 is adjacent to w_2 or w_1 is not adjacent to w_2. Both these graphs are seen to be blocks by using Theorem 2 in the case $k \geq 6$. For $k = 4$, we can check directly that these are blocks. Hence there are two hypotraversable graphs for each $k \geq 4$ in this case.

CASE IV. If $e(G) = 3$ then there are three hypotraversable graphs for $k = 4$ and one hypotraversable graph for $k = 6$.

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Proof. Let $e(G) = 3$. By Theorem 4, $k \le 6$. We will consider the two cases $k = 4$ and $k = 6$ separately.

(a) Let $k = 4$, $E(G) = \{v_1, v_2, v_3\}$, and $O(G) = \{w_1, w_2, w_3, w_4\}$. Since each even vertex has odd degree at least 2 and each odd vertex has even degree at most 2 , $e(O(G), E(G)) = 6, 7$, or 8 . If $e(O(G), E(G)) = 6$, then v_1, v_2 , and v_3 must have odd degree two and hence each has degree pair $(0, 2)$. This can happen in the following three ways.

(i) $E(O(G), E(G))$ is $P_7 : w_1v_1w_2v_2w_3v_3w_4$. Then w_2 and w_3 have even degree 2 and hence degree pair $(2, 3)$. This also gives the required degree pair $(1, 2)$ to w_1 and w_4 . Thus $\langle E(G)\rangle = \overline{K}_3$ and $\langle O(G)\rangle = K_4 - x$. As this graph is a block, it is hypotraversable.

(ii) $E(O(G), E(G))$ forms a $C_4 : v_1w_1v_2w_2v_1$ and a $P_3 : w_3v_3w_4$. Here w_1 and w_2 must have degree pair $(2, 3)$ and it gives degree pair $(1, 2)$ to w_3 and w_4 as required. Again $\langle O(G)\rangle = K_4 - x$ and $\langle E(G)\rangle = \overline{K}_3$, and this graph, being a block, is hypotraversable.

(iii) $E(O(G), E(G))$ forms a $C_6 : w_1v_1w_2v_2w_3v_3w_1$. Here w_1, w_2 , and w_3 have degree pair $(2, 3)$, which gives degree pair $(0, 3)$ to w_1 , which is larger than its required degree pair $(0, 2)$. So there is no graph in this case.

If $e(O(G), E(G)) = 7$, then without loss of generality let v_1, v_2 have odd degree 2 and v_3 have odd degree 3 . Hence the degree pairs of v_1, v_2, v_3 are $(0, 2)$, $(0, 2)$, and $(1, 3)$ respectively. But this is not possible, as v_3 has even degree one, while none of v_1 and v_2 is adjacent to it.

If $e(O(G), E(G)) = 8$ then, as the degree pairs for v_1, v_2 , and v_3 can not be $(0, 2)$, $(0, 2)$, $(2, 4)$, their degree pairs must be $(0, 2)$, $(1, 3)$, $(1, 3)$, respectively. Hence $\langle E(G)\rangle = \{v_2, v_3\}$. If

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$\Gamma(v_2) = \{w_1, w_2, w_3\}$, then $\Gamma(v_3) \neq \Gamma(v_2)$; for otherwise w_1, w_2, w_3 have even degree two each, and hence they can not be adjacent to v_1 . But then the degree pair of v_1 is $(0, 1)$, which is a contradiction. Therefore $\Gamma(v_3) = \{w_1, w_2, w_3\}$ say. Join v_1 to w_1 and w_4 to get the required degree pair $(0, 2)$ for v_1 . Now w_i for $i = 1, 2, 3, 4$ have degree pairs $(2, 3)$, so that $\langle O(G)\rangle = K_4$. As this graph is a block, it is hypotraversable.

(b) Let $k = 6$, $E(G) = \{v_1, v_2, v_3\}$ and $O(G) = \{w_1, w_2, \ldots, w_6\}$. Since each even vertex must have odd degree at least four and each odd vertex can have even degree at most two, $e(O(G), E(G)) = 12$. Therefore w_i has degree pair $(2, 5)$ for $i = 1, 2, \ldots, 6$, and v_i has degree pair $(0, 4)$ for $i = 1, 2, 3$. Let $\Gamma(v_1) = \{w_1, w_2, w_3, w_4\}$. If $\Gamma(v_2) = \Gamma(v_1)$, then the odd degree of v_3 is at most 2 , which is a contradiction. If v_2 is adjacent to three vertices out of $\{w_1, w_2, w_3, w_4\}$, then again the odd degree of v_3 will be at most three, which is not possible. So $\Gamma(v_2) = \{w_1, w_2, w_5, w_6\}$, $\Gamma(v_3) = \{w_3, w_4, w_5, w_6\}$, $\langle O(G)\rangle = K_6$, and $\langle E(G)\rangle = \overline{K}_3$. This graph is easily seen to be a block, and hence it is hypotraversable.

CASE V. If $e(G) = 4$ there are two hypotraversable graphs.

Proof. If $e(G) = 4$, then $k = 4$. Let $E(G) = \{v_1, v_2, v_3, v_4\}$ and $O(G) = \{w_1, w_2, w_3, w_4\}$. Since an odd vertex can have even degree at most two and each even vertex must have odd degree at least two, $e(O(G), E(G)) = 8$. Therefore each even vertex must have degree pair $(0, 2)$, and each odd vertex must have degree pair $(2, 3)$. There are two such graphs.

(i) $E(O(G), E(G))$ consists of two 4-cycles: $v_1w_1v_2w_2v_1$ and $v_3w_3v_4w_4v_3$.

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(ii) $E(O(G), E(G))$ forms an 8-cycle: $v_1w_1v_2w_2v_3w_3v_4w_4v_1$. As $\langle O(G)\rangle = K_4$ and $\langle E(G)\rangle = \overline{K}_3$, both these graphs are seen to be blocks and hence both are hypotraversable.

This completes the classification of the hypotraversable graphs. To sum up, we state all these results in the following theorem.

THEOREM 5. *If* $f(p)$ *denotes the number of hypotraversable graphs on* p *vertices then*

$$f(p) = \begin{cases} 0 & \text{if } p = 1, 2 \\ 1 & \text{if } p = 3, 4, 5 \\ 6 & \text{if } p = 6, 7 \\ 9 & \text{if } p = 8 \\ 4 & \text{if } p = 9 \\ 5+t_p & \text{if } p = 2k \ , \ k \geq 5 \\ 3 & \text{if } p = 2k+1 \ , \ k \geq 5 \ , \end{cases}$$

where t_p *is the number of* 2*-regular graphs on* p *vertices.*

We note that the number t_p mentioned above is the number of partitions of p where each summand is at least 3 . We conclude this paper with the following theorem, from which we have

$$t_p = P(p) - P(p-1) - P(p-2) + P(p-3) \ ,$$

where $P(n)$ denotes the number of unrestricted partitions of n .

Let $A_i(n)$ denote the number of partitions of n with each part greater than or equal to i , and let $M_i(n)$ denote the number of partitions of n with minimum part exactly equal to i . Clearly $M_1(n) = P(n-1)$ and $P(n) = \sum_{i=1}^{k-1} M_i(n) + A_k(n)$. We have the following theorem.

THEOREM 6. $M_i(n) = P(n-i) - \sum_{j=1}^{i-1} M_j(n-i)$.

Proof. To find the number of partitions of n with minimum part exactly i , we take out i from n and write all partitions of $n - i$.

As we do not want any partition of $n - i$ which has a summand less than i, we have to subtract $\sum_{j=1}^{i-1} M_j(n-i)$ from $P(n-i)$ in order to get $M_i(n)$. Hence the theorem is proved.

This completes the enumeration of hypotraversable graphs for each positive integer p.

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