Canad. Math. Bull. Vol. 17 (5), 1975

ON PRODUCT DECOMPOSITIONS

BY
K. VARADARAJAN

Introduction. Borsuk has asked whether there exists for each compact metric absolute neighbourhood retract X an integer l (depending only on X) with the property that if X is homotopy equivalent to a cartesian product of more than l spaces than at least one of these spaces is contractible. The answer to this question is still not known. The following theorem has been proved by Ganea and Hilton [Theorem 1.3 of [1]].

THEOREM. *Let X be a compact connected metric absolute neighbourhood retract. Let π be the fundamental group of X. If the set of primes which are orders of periodic elements in π is finite, then an integer l with the required property exists.*

The object of this paper is to prove that Borsuk's problem admits of an affirmative answer in case X is a compact connected manifold without boundary, with no restrictions on the fundamental group of X. Throughout this paper by a manifold we mean a topological manifold without boundary. The main theorem proved here can be stated as follows.

THEOREM. *Let M be a compact, connected manifold of dimension n. Suppose M is homotopy equivalent to*

$$X_1 \times \cdots \times X_k.$$

(a) *If M is orientable and $k>n$ at least one of the X_r's is contractible.*

(b) *In general (when M is not necessarily orientable) if $k>2n-1$ at least one of the X_r's is contractible.*

Section 1 of this paper deals with the proof of this theorem. This theorem enables us to define the Borsuk number of a compact manifold M to be the least integer l with the property that if M is of the homotopy type of $X_1 \times \cdots \times X_k$ with $k>l$ then at least one of the X_r's is contractible. In section 2 we determine the Borsuk numbers of some familiar closed manifolds. Actually using the classification theorem for compact connected two dimensional manifolds we determine their Borsuk numbers completely. We also determine the Borsuk numbers of spheres, projective spaces and Lens spaces.

§1. **Compact manifolds.** For any topological space X let $H_i(X)$ {respectively $H_i(X, Z_2)$} denote the i-th Singular homology group of X with coefficients in Z {respy Z_2}. The following Lemmas are consequences of the Kunneth formula.

Received by the editors April 11, 1973 and, in revised form, June 29, 1973.

https://doi.org/10.4153/CMB-1974-133-6 Published online by Cambridge University Press

LEMMA 1.1. *Let n be any integer* ≥ 0 *and X a* 0-*connected space satisfying* $H_j(X)=0$ *for* $j>n$ *and* $H_n(X)\simeq Z$. *Suppose X is homotopy equivalent to* $X_1\times\cdots\times X_k$. *Then there exist integers* $q_i\geq 0$ *for* $1\leq i\leq k$ *satisfying*

(a) $H_j(X_i)=0$ *for* $j>q_i$; $H_{q_i}(X_i)\simeq Z$ *and*
(b) $\sum_{i=1}^k q_i=n$

LEMMA 1.2. *Suppose X is a* 0-*connected space with* $H_j(X, Z_2)=0$ *for* $j>n$ *and* $H_n(X, Z_2)\simeq Z_2$. *If X is of the homotopy type of* $X_1\times\cdots\times X_k$ *then there exist integers* $q_i\geq 0$ *for* $1\leq i\leq k$ *satisfying*

(a) $H_j(X_i; Z_2)=0$ *for* $j>q_i$; $H_{q_i}(X_i; Z_2)\simeq Z_2$ *and*
(b) $\sum_{i=1}^k q_i=n$.

THEOREM 1.3. *Let M be a compact, connected manifold of dimension n. Suppose M is homotopy equivalent to* $X_1\times\cdots\times X_k$.

(*A*) *If M is orientable and* $k>n$ *then at least one of the* X_r*'s is contractible.*
(*B*) *In general, if* $k>2n-1$ *at least one of the* X_r*'s is contractible.*

LEMMA 1.4. *Let X be a* $K(\pi, 1)$ *space with* π *finite. If X is dominated by a manifold then* $\pi=\{1\}$.

Proof. Let X be dominated by a manifold M of dimension m. Let $f:X\to M$ and $g:M\to X$ be such that $gof\sim Id_X$. Given any π-module A it can be thought of as a local coefficient system over X. Let $g^*(A)$ denote the pull-back of the local system A to M by g and $f^*(g^*(A))$ the pull-back of $g^*(A)$ to X. Since $gof\sim Id_X$ it follows that the composite

$$H^i(X, A)\xrightarrow{g^*} H^i(M, g^*(A))\xrightarrow{f^*} H^i(X, f^*(g^*(A))$$

is an isomorphism for each i. Since M is of dimension m $H^i(M, g^*(A))=0$ for $i>m$. It follows that $H^i(X, A)=0$ for $i>m$. This in turn implies that $H^i(\pi, A)=0$ for $i>m$ and all π-modules A. Hence the cohomological dimension of $\pi\leq m$. But a finite group is known to be of infinite cohomological dimension unless $\pi=\{1\}$.

Proof of Theorem 1.3. Let M be orientable and $k>n$. Since $H_n(M)\simeq Z$ and $H_j(M)=0$ for $j>n$ Lemma 1.1 applies to M. Hence there exist integers $q_i\geq 0$ for $1\leq i\leq k$ satisfying $H_j(X_i)=0$ for $j>q_i$, $H_{q_i}(X_i)\simeq Z$ and $\sum_{i=1}^k q_i=n$. Since $k>n$ and each $q_i\geq 0$ it follows that $q_i=0$ for at least one i. Without loss of generality we can assume that $q_1=0$. It follows that X_1 is 0-connected and that $H_j(X_1)=0$ for $j\geq 1$. Since X_1 is dominated by a topological manifold it follows that X_1 is of the homotopy type of a CW-complex. Let $p:\tilde{X}_1\to X_1$ denote the universal covering of X_1.

Let $g:M\to X_1\times\cdots\times X_k$ be a homotopy equivalence. Let $\phi:P\to M$ denote the pull-back of the covering

$$p' = p\times Id_{X_2\times\cdots\times X_k}:\tilde{X}_1\times X_2\times\cdots\times X_k\to X_1\times X_2\times\cdots\times X_k$$

https://doi.org/10.4153/CMB-1974-133-6 Published online by Cambridge University Press

by means of g. Then P is a covering manifold of M. Hence P is a connected orientable n-dimensional topological manifold. Moreover, P is of the homotopy type of $\tilde{X}_1 \times X_2 \times \cdots \times X_k$. Hence $H_{q_2}(X_2) \otimes \cdots \otimes H_{q_k}(X_k) \simeq Z$ is a direct summand of $H_{q_2+\cdots+q_k}(P) = H_n(P)$. It follows from this that P is compact. Hence $P \xrightarrow{\phi} M$ is a finite covering from which we immediately see that $\tilde{X}_1 \xrightarrow{p} X_1$ is a finite covering.

Since P is a compact, connected, orientable manifold of dimension n we get $H_n(P) = Z$ and $H_j(P) = 0$ for $j > n$. Hence Lemma 1.1 applies to P as well. Since P is homotopy equivalent to $\tilde{X}_1 \times X \times \cdots \times X_k$ and since $H_{q_i}(X_i) \simeq Z$ for $i \geq 2$ with $q_2 + \cdots + q_k = n$, it follows from Lemma 1.1 applied to P that $H_j(\tilde{X}_1) = 0$ for $j \geq 1$.

Since $\tilde{X}_1$ is simply connected it follows immediately that $\tilde{X}_1$ is contractible. If π denotes the fundamental group of X_1 it follows from what we have proved so far that π is finite and that $X_1 = K(\pi, 1)$. Since X_1 is dominated by M, by Lemma 1.4 we see that $\pi = \{1\}$. Hence X_1 is contractible. This completes the proof of A.

Assume M to be non-orientable and let $N \xrightarrow{p} M$ be the orientable double covering of M. Suppose $f: X_1 \times \cdots \times X_k \to M$ is a homotopy equivalence. Choose once and for all base points x_i^0 in X_i $(1 \leq i \leq k)$. Choose $m^0 = f(x_1^0, \ldots, x_k^0)$ as the base point in M and a point $u^0 \in N$ satisfying $p(u^0) = m^0$ as the base point in N. For any set of indices $1 \leq \mu_1 < \mu_2 < \cdots < \mu_r \leq k$ we will identify $X_{\mu_1} \times \cdots \times X_{\mu_r}$ with the subspace of $X_1 \times \cdots \times X_k$ consisting of those elements with x_i^0 as the i-th coordinate for $i \neq \mu_j$ $(1 \leq j \leq r)$. The fundamental groups of the spaces involved will be at the chosen base points. We denote the subgroup $p^*(\pi_1(N))$ of $\pi_1(M)$ by G. Then G is a subgroup of index 2 in $\pi_1(M)$.

Since $X_1 \times \cdots \times X_k$ is of the homotopy type of M and since $H_n(M; Z_2) \simeq Z_2$, from Lemma 1.2 we see that there exist integers $q_i \geq 0$ $(1 \leq i \leq k)$ satisfying

(a) $H_{q_i}(X_i; Z_2) \simeq Z_2$, $H_j(X_i; Z_2) = 0$ for $j > q_i$ and

(b) $\sum_{i=1}^{k} q_i = n$.

Since each $q_i \geq 0$ and $\sum_{i=1}^{k} q_i = n$ we can at most have n of the q_i's strictly positive. Hence at least $k-n$ of the q_i's should be 0. Without loss of generality we can assume $q_i = 0$ for $1 \leq i \leq k-n$. Then for $1 \leq i \leq k-n$ we have $H_j(X_i; Z_2) = 0$ whenever $j \geq 1$. In particular $H^1(X_i; Z_2) = \mathrm{Hom}_{Z_2}(H_1(X_i; Z_2); Z_2) = 0$. It follows that any map $X_i \to K(Z_2, 1)$ is homotopically trivial $(1 \leq i \leq k-n)$.

Let $1 \leq i \leq k-n$. Any double covering over X_i can be considered as a principal Z_2-bundle. As $K(Z_2, 1)$ is a classifying space for the group Z_2 it follows that any double covering over X_i has to be trivial. If $f_i = f \mid X_i : X_i \to M$, then the pull-back of the double covering $N \xrightarrow{p} M$ by f_i is trivial. From this it follows that there exists a map $h_i : X_i \to N$ lifting f_i (i.e. $p \circ h_i = f_i$) and satisfying $h_i(x_i^0) = u^0$. As an immediate consequence, we get:

$$f_{i*}(\pi_1(X_i)) \subset p_*(\pi_1(N)) = G \tag{1}$$

Now, $f_*: \pi_1(X_1) \times \cdots \times \pi_1(X_k) \to \pi_1(M)$ is an isomorphism. Let

$$H = f_*^{-1}(G) \cap \{\pi_1(X_{k-n+1}) \times \pi_1(X_{k-n+2}) \times \cdots \times \pi_1(X_k)\}.$$

https://doi.org/10.4153/CMB-1974-133-6 Published online by Cambridge University Press

Using (1) we immediately see that:

$$\pi_1(X_1) \times \cdots \times \pi_1(X_{k-n}) \times H = f_*^{-1}(G) \tag{2}$$

Since G is of index 2 in $\pi_1(M)$ it follows that:

$$\pi_1(X_1) \times \cdots \times \pi_1(X_{k-n}) \times H$$

is of index 2 in

$$\pi_1(X_1) \times \cdots \times \pi_1(X_{k-n}) \times \cdots \times \pi_1(X_k).$$

In particular H is of index 2 in $\pi_1(X_{k-n+1}) \times \cdots \times \pi_1(X_k)$. Let $t: Y \to X_{k-n+1} \times \cdots \times X_k$ be the covering space corresponding to the subgroup H of $\pi_1(X_{k-n+1}) \times \cdots \times \pi_1(X_k)$. Let $\tau = Id_{X_1 \times \cdots \times X_{k-n}} \times t: X_1 \times \cdots \times X_{k-n} \times Y \to X_1 \times \cdots \times X_k$. Then τ is a covering projection and

$$f_* \circ \tau_*(\pi_1(X_1) \times \cdots \times \pi_1(X_{k-n}) \times \pi_1(Y)) = G = p_*(\pi_1(N)) \tag{3}$$

It follows that there exists a map $h: X_1 \times \cdots \times X_{k-n} \times Y \to N$ lifting $f \circ \tau$. (i.e. $p \circ h = f \circ \tau$). Thus diagram 1 below is commutative.

$$\begin{array}{ccc} X_1 \times \cdots \times X_{k-n} \times Y & \xrightarrow{h} & N \\ \downarrow \tau & & \downarrow p \\ X_1 \times \cdots \times X_{k-n+1} \times \cdots \times X_k & \xrightarrow{f} & M \end{array}$$

Using (3), commutativity of diagram 1 with p and τ covering projections and the fact that f is a homotopy equivalence it is easily seen that

$$h: X_1 \times \cdots \times X_{k-n} \times Y \to N$$

induces isomorphisms of homotopy groups. Since the spaces involved have the homotopy type of CW complexes it follows that h is a homotopy equivalence.

Now, N is a compact, connected, orientable topological manifold. In the product $X_1 \times \cdots \times X_{k-n} \times Y$ there are $(k-n+1)$ factors. If $k > 2n-1$ then $k-n+1 > n$. Hence by A of Theorem 1.3 at least one of the factors in $X_1 \times \cdots \times X_{k-n} \times Y$ is contractible.

We claim that Y is not contractible. Denoting $X_{k-n+1} \times \cdots \times X_k$ by E we see that Y is a double covering of E. If Y were contractible it would follow that $E = K(Z_2, 1)$. But E is dominated in homotopy by M contradicting Lemma 1.4. It follows that necessarily one of the X_i's with $1 \leq i \leq k-n$ is contractible. This completes the proof of B of Theorem 1.3.

§2. **Borsuk numbers of certain closed manifolds.** Theorem 1.3 enables us to introduce the following:

Definition 2.1. Let M be any compact manifold. The Borsuk number of M is the least integer l with the property that if M is of the homotopy type of $X_1 \times \cdots \times X_k$ with $k > l$ then at least one of the X_μ's is contractible.

https://doi.org/10.4153/CMB-1974-133-6 Published online by Cambridge University Press

We denote the Borsuk number of M by $l(M)$. Before stating the results proved in this section we recall the definition of Lens spaces briefly. For any integer $n\geq 1$ we consider S^{2n-1} as the unit sphere in the complex n-space C^n; that is to say:

$$S^{2n-1} = \left\{(z_1, z_2, \ldots, z_n) \,\middle|\, \sum_{\mu=1}^{n} |z_\mu|^2 = 1\right\}$$

Let n and m be integers ≥ 2. Let π denote the cyclic group of order m with t as a generator. Given any n-tupe $(a_1, \ldots, a_n)$ of primitive m-th roots of unity an action of π on S^{2n-1} can be defined by setting

$$t(z_1, \ldots, z_n) = (a_1 z_1, \ldots, a_n z_n) \tag{4}$$

For $m=2$ the only primitive mth root of unity is -1 and then the above action corresponds to the antipodal action of the cyclic group of order 2 on S^{2n-1}. In this case the quotient space is $P^{2n-1}(R)$. When $m\geq 3$ the quotient space got under the action of π on S^{2n-1} described by (4) is known as a Lens space. It is clear that there exists a unique n-tuple of integers $(q_1, \ldots, q_n)$ satisfying

$$1 \leq q_j < m, \qquad \alpha_j = \exp(2\pi i q_j/m)$$

and q_j relatively prime to m. Also there exists a unique n-tuple of integers $(r_1, \ldots, r_n)$ satisfying $1\leq r_j<m$ and $r_j q_j=1(\text{mod}\, m)$. The quotient space S^{2n-1}/π corresponding to the action described in (4) is usually denoted by $L_m(r_1, \ldots, r_n)$.

PROPOSITION 2.2. *For any integer $n\geq 1$ we have*

(i) $l(S^n)=1$ and
(ii) $l(P^n(R))=1$

Proof. (i) Suppose $S^n\sim X_1\times X_2$. By Lemma 1.1 there exist integers $q_i\geq 0$ $(i=1, 2)$ such that: $H_{q_1}(X_1)\simeq Z$; $H_{q_2}(X_2)\simeq Z$; $H_j(X_1)=0$ for $j>q_1$, $H_j(X_2)=0$ for $j>q_2$ and $q_1+q_2=n$. If each $q_i>0$ it follows that $q_i<n$ for each i. Then $H_{q_1}(X_1)\simeq Z$ is a subgroup of $H_{q_1}(S^n)=0$, a contradiction. Hence one of the q_i's is 0. Without loss of generality we can assume $q_1=0$. Then $q_2=n$. If $n=1$, $\pi_1(X_1)\times\pi_1(X_2)\simeq Z$. Hence one of the groups $\pi_1(X_i)\simeq Z$ and the other 0. Since $H_1(X_1)=0$ we can not have $\pi_1(X_1)\simeq Z$. Hence $\pi_1(X_1)=0$. Also $H_j(X_1)=0$ for $j>0$ (since $q_1=0$). This proves that X_1 is contractible. This proves that $l(S^n)=1$.

(ii) Suppose $P^n(R)\sim X_1\times X_2$. Since $\pi_1(P^n(R)\simeq Z_2$ we have one of the groups $\pi_1(X_i)\simeq Z_2$ and the other 0. Assume $\pi_1(X_1)=0$ and $\pi_1(X_2)\simeq Z_2$. Let $\tilde{X}_2$ be the universal covering of X_2. Then $X_1\times\tilde{X}_2$ is the universal covering of $X_1\times X_2$ and hence of the homotopy type of S^n. By (i) either X_1 is contractible or $\tilde{X}_2$ is contractible. If $\tilde{X}_2$ is contractible then $X_2=K(Z_2, 1)$ and is dominated in homotopy by $P^n(R)$ contradicting Lemma 1.4. This shows that X_1 is contractible and hence $l(P^n(R))=1$.

PROPOSITION 2.3. $l(L_m(r_1, \ldots, r_n))=1$.

Proof. Suppose $L_m(r_1, \ldots, r_n)\sim X_1\times X_2$. The fundamental group $\pi_1(L_m(r_1, \ldots, r_n))\simeq\pi$ the cyclic group of order m. It follows from this that each of the groups

https://doi.org/10.4153/CMB-1974-133-6 Published online by Cambridge University Press

$\pi_1(X_i)$ $(i=1, 2)$ is a finite cyclic group. Let d_i=order of $\pi_1(X_i)$. Denoting the universal covering of X_i by $\tilde{X}_i$ we see that $S^{2n-1}\sim\tilde{X}_1\times\tilde{X}_2$, because S^{2n-1} is the universal covering of $L_m(r_1, \ldots, r_n)$. From (i) of proposition 2.2 we see that one of the spaces $\tilde{X}_i$ is contractible. For definite mass sake assume $\tilde{X}_1$ contractible. Then $X_1=K(Z_{d_1}, 1)$ and is dominated by the manifold $L_m(r_1, \ldots, r_n)$. By Lemma 1.4 we see that $d_1=1$ and hence X_1 is contractible. This completes the proof of proposition 2.3.

Our next result is the determination of the Borsuk numbers of all compact connected 2-dimensional manifolds. For this we need the following well-known classification theorem. Many standard books contain a proof of this theorem. We refer our readers to Theorem 5.1 of Chapter one in [2].

THEOREM. *Any compact connected 2-dimensional manifold is either homeomorphic to S^2 or to the connected sum $S^1\times S^1\#\cdots\cdots\#S^1\times S^1$* of n copies of the torus or the connected sum $P^2(R)\#\cdots\cdots\#P^2(R)$ of n copies of the projective plane where n is an integer ≥ 1.

We denote the connected sum of h copies of $S^1\times S^1$ by M_h and the connected sum of h copies of $P^2(R)$ by N_h. The result that we prove can now be stated as follows:

THEOREM 2.4.

(i) $l(S^2)=1$

(ii) $l(S^1\times S^1)=2$

(iii) $l(M_h)=1$ *whenever* $h\geq 2$

(iv) $l(N_h)=1$ *whenever* $h\geq 1$.

Proof. (i) is already contained in proposition 2.2.

(ii) Since $S^1\times S^1$ is orientable from $\mathscr{A}$ of Theorem 1.3 it follows that $l(S^1\times S^1)\leq 2$. Also in the factorisation $S^1\times S^1=X_1\times X_2$ with $X_1=S^1=X_2$ both the factors X_1 and X_2 are non-contractible. Hence $l(S^1\times S^1)\geq 2$. This proves (ii).

(iii) Let $h\geq 2$ and suppose $M_h\sim X_1\times X_2$. From Lemma 1.1 we see that there exist integers $q_i\geq 0$ $(i=1, 2)$ satisfying $H_{q_i}(X_i)\simeq Z$; $H_j(X_i)=0$ for $j>q_i$ and $q_1+q_2=2$. If both q_1 and q_2 are different from 0, then $q_1=1$, $q_2=1$ in which case we get $H_1(X_1\times X_2)\simeq Z\oplus Z$. But it is known [Propostion 5.1, Chapter four [2]] that $H_1(M_h)$ is a free abelian group of rank $2h$. Since $h\geq 2$ the rank of $H_1(X_1\times X_2)$ should be ≥ 4, contrary to $H_1(X_1\times X_2)\simeq Z\oplus Z$. Thus one of the q_i's has to be zero. Without loss of generality we can assume $q_1=0$. Then $q_2=2$. Let $p:\tilde{X}_1\to X_1$ be the universal covering of X_1 and $\tau=p\times Id_{X_2}:\tilde{X}_1\times X_2\to X_1\times X_2$. Let $t: V\to M_h$ be the pull-back of the covering $\tau:\tilde{X}_1\times X_2\to X_1\times X_2$ by a homotopy equivalence $M_h\to X_1\times X_2$. Then V is an orientable 2-dimensional manifold homotopy equivalent to $\tilde{X}_1\times X_2$. Hence $H_2(X_2)\simeq Z$ is a subgroup of $H_2(V)$. It follows from this that V is compact. Now, Lemma 1.1 applied to the situation $V\sim\tilde{X}_1\times X_2$ combined with the fact $H_2(X_2)\simeq Z$ immediately yields $H_j(\tilde{X}_1)=0$ for $j\geq 1$. Hence $\tilde{X}_1$ is contractible. Since V is compact

https://doi.org/10.4153/CMB-1974-133-6 Published online by Cambridge University Press

$t: V \to M_h$ is a finite covering and hence $\tilde{X}_1 \xrightarrow{p} X_1$ is a finite covering. Thus $X_1 = K(\pi, 1)$ for some finite group π. Since X_1 is dominated by M_h by Lemma 1.4 we see that $\pi = \{1\}$ and hence X_1 is contractible. This proves that $l(M_h) = 1$ whenever $h \geq 2$.

(iv) For $h=1$, $N_h = P^2(R)$ and (ii) of proposition 2.2 gives $l(P^2(R)) = 1$. Let now $h \geq 2$ and suppose $N_h \sim X_1 \times X_2$. Since N_h is a non-orientable compact connected 2-dimensional manifold we have $H_2(N_h) = 0$ and $H_2(N_h; Z_2) \simeq Z_2$. It is known [Proposition 5.1, Chapter four [2]] that $H_1(N_h)$ is a direct sum of a free abelian group of rank $(h-1)$ with a copy of Z_2. From this it follows that

$$\text{(5)} \qquad \dim H_1(N_h) \otimes Z_2 \quad \text{over} \quad Z_2 = h$$

From Lemma 1.2 we see that there exist integers $q_i \geq 0$ $(i=1, 2)$ satisfying

$$H_{q_i}(X_i; Z_2) \simeq Z_2; \qquad H_j(X_i; Z_2) = 0$$

for $j > q_i$ and $q_1 + q_2 = 2$. Suppose each $q_i \neq 0$. Then $q_1 = 1 = q_2$. In this case $H_1(X_1; Z_2) \simeq Z_2$, $H_1(X_2; Z_2) \simeq Z_2$ and hence

$$\text{(6)} \qquad H_1(N_h; Z_2) \simeq H_1(X_1 \times X_2; Z_2) \simeq Z_2 \oplus Z_2$$

But $H_1(N_h; Z_2) = H_1(N_h) \otimes Z_2 \oplus \mathrm{Tor}(H_0(N_h), Z_2) = H_1(N_h) \otimes Z_2$ (since $H_0(N_h) = Z$ is free abelian). Whenever $h \geq 3$ the conditions (5) and (6) are incompatible. Therefore it follows that whenever $h \geq 3$ one of the q_i's should necessarily be zero.

Now consider the case $h=2$. In this case $H_1(N_h) \simeq Z \oplus Z_2$. Also from $N_h \sim X_1 \times X_2$ we get $H_1(N_h) \simeq H_1(X_1) \oplus H_1(X_2)$. Again suppose $q_1 = 1 = q_2$. Then $H_1(X_1; Z_2) \simeq Z_2$; $H_1(X_2; Z_2) \simeq Z_2$. It follows from this that $H_1(X_1) \neq 0$ and $H_1(X_2) \neq 0$. Since $H_1(X_1) \oplus H_1(X_2) \simeq H_1(N_h) \simeq Z \oplus Z_2$ we see that one of the groups $H_1(X_i) \simeq Z$ and the other isomorphic to Z_2. In this case $H_1(X_1) \otimes H_1(X_2) \simeq Z \otimes Z_2 \simeq Z_2$ will be a subgroup of $H_2(X_1 \times X_2) \simeq H_2(N_h)$. But $H_2(N_h) = 0$. This contradiction shows that even when $h=2$ one of the q_i's has to be necessarily equal to 0.

Without loss of generality we can assume $q_1 = 0$ and $q_2 = 2$. Then

$$\text{(7)} \qquad H_j(X_1; Z_2) = 0 \quad \text{for } j \geq 1$$

Hence

$$\text{(8)} \qquad H^j(X_1; Z_2) = \mathrm{Hom}_{Z_2}(H_j(X_1; Z_2), Z_2) = 0 \quad \text{for } j \geq 1$$

Let $p: W \to N_h$ be the two fold orientable covering of N_h. Choose a homotopy equivalence $f: X_1 \times X_2 \to N_h$. Using the fact that $H^1(X_1; Z_2) = 0$ and arguing in the same way as in the proof of Theorem 1.3, B of this paper we see that there exists a double covering $Y \to X_2$ with the property that W is homotopy equivalent to $X_1 \times Y$. Since W is a compact, connected, orientable 2-dimensional manifold it it homeomorphic to either S^2 or to an M_d for some integer $d \geq 1$. From (i) and (iii) (proved already) we see that $l(W) = 1$ unless $W = S^1 \times S^1$.

https://doi.org/10.4153/CMB-1974-133-6 Published online by Cambridge University Press

We first deal with the case $W \neq S^1 \times S^1$. Then, as $l(W)=1$ one of the spaces X_1 or Y should be contractible. If Y is contractible then $X_2=K(Z_2, 1)$ and X_2 is dominated by N_h contradicting Lemma 1.4. Hence, X_1 is necessarily contractible.

For completing the proof of (iv) we have to still deal with the case $W=S^1 \times S^1$. Then $S^1 \times S^1 \sim X_1 \times Y$. Hence $\pi_1(X_1)$ is a direct summand of $\pi_1(S^1 \times S^1)=Z \oplus Z$. Hence either $\pi_1(X_1) \simeq 0$ or Z or $Z \oplus Z$. If $\pi_1(X_1) \simeq Z$ or $Z \oplus Z$ we will have $H_1(X_1) \simeq Z$ or $Z \oplus Z$ in which case $H_1(X_1; Z_2)$ will be different from 0 contradicting (7). Therefore, $\pi_1(X_1)=0$. Applying Lemma 1.1 to the situation $S^1 \times S^1 \sim X_1 \times Y$ we see that there exist integers $r_1 \geq 0$, $r_2 \geq 0$ such that $H_{r_1}(X_1) \simeq Z$, $H_{r_2}(Y) \simeq Z$; $H_j(X_1)=0$ for $j > r_1$, $H_j(Y)=0$ for $j > r_2$ and $r_1+r_2=2$. From $\pi_1(X_1)=0$ we get $H_1(X_1)=0$. Hence $r_1 \neq 1$. If $r_1=2$ then $H_2(X_1)=Z$ and hence $H_2(X_1; Z_2) \neq 0$ contradicting (7). It follows that $r_1=0$ (and hence $r_2=2$). Then X_1 satisfies $\pi_1(X_1)=0$ and $H_j(X_1)=0$ for $j \geq 1$. Hence X_1 is contractible.

This completes the proof of (iv).

§3. **Conclusion.** *A* of Theorem 1.3 implies that the Borusk number $l(M)$ of a compact, connected, orientable manifold M satisfies the inequality $l(M) \leq \dim M$. In case M is not orientable, *B* of Theorem 1.3 gives the inequality $l(M) \leq 2 \dim M - 1$.

Problem. Are there non-orientable compact, connected manifolds M for which $l(M) > \dim M$?

References

1. T. Ganea and P. J. Hilton, *On decomposition of spaces into Cartesian products and unions*, P. J. Hilton Proc. Camb. Phil. Soc **55** (1959) pp. 248–256.

2. W. S. Massey, *Algebraic Topology: An introduction*, Harcourt, Brace and World, Inc., 1967.

The University of Calgary,
Calgary, Alberta, Canada

https://doi.org/10.4153/CMB-1974-133-6 Published online by Cambridge University Press