ON THE GAUSS MAP OF RULED SURFACES

by CHRISTOS BAIKOUSSIS† and DAVID E. BLAIR

(Received 25 June, 1991)

Let M^2 be a (connected) surface in Euclidean 3-space E^3, and let $G: M^2 \to S^2(1) \subset E^3$ be its Gauss map. Then, according to a theorem of E. A. Ruh and J. Vilms [**3**], M^2 is a surface of constant mean curvature if and only if, as a map from M^2 to $S^2(1)$, G is harmonic, or equivalently, if and only if

$$\Delta G = \|dG\|^2 G \tag{1.1}$$

where Δ is the Laplace operator on M^2 corresponding to the induced metric on M^2 from E^3 and where G is seen as a map from M^2 to E^3. A special case of (1.1) is given by

$$\Delta G = \lambda G, \qquad (\lambda \in \mathbb{R}) \tag{1.2}$$

i.e., the case where the Gauss map $G: M^2 \to E^3$ is an eigenfunction of the Laplacian Δ on M^2.

On the other hand, F. Dillen, J. Pas and L. Verstraelen [**2**] recently proved that among the surfaces of revolution in E^3, the only ones whose Gauss map satisfies the condition

$$\Delta G = \Lambda G, \qquad (\Lambda \in \mathbb{R}^{3\times 3}) \tag{1.3}$$

are the planes, the spheres and the circular cylinders.

We observe that from the surfaces of revolution in E^3 which satisfy (1.3) the planes and the circular cylinders are ruled surfaces. On the other hand, for the helicoid $X(s,t) = (t\cos s, t\sin s, \alpha s)$, $\alpha \neq 0$ the Guass map is given by

$$G = \frac{1}{\sqrt{t^2+\alpha^2}}(-\alpha\sin s, \alpha\cos s, -t).$$

Then it is easy to show that the Laplacian ΔG of the Gauss map G can be expressed as follows

$$\Delta G = \frac{2\alpha^2}{(t^2-\alpha^2)^{5/2}}(-\alpha\sin s, \alpha\cos s, -t)$$

which clearly doesn't satisfy condition (1.3).

A question which arises now is: What are the ruled surfaces satisfying condition (1.3)?

In particular, we will prove the following:

THEOREM. *Among the ruled surfaces in E^3, the only ones whose Gauss map satisfies* (1.3) *are the planes and the circular cylinders.*

We first study cylindrical surfaces M^2. Let $X(s,t) = \alpha(s) + t\beta$ be the position vector of M^2 in E^3 where $\alpha(s)$ is the plane curve $\alpha = (\alpha_1, \alpha_2, 0)$ parameterized by arc-length and β is the constant vector $\beta = (0,0,1)$. We have the following lemma.

LEMMA. *The only cylindrical surfaces whose Gauss map satisfies* (1.3) *are the planes and the circular cylinders.*

† This work was done while the first author was a visiting scholar at Michigan State University.

Glasgow Math. J. **34** (1992) 355–359.

Proof. The Gauss map of M^2 is $G = \alpha' \times \beta = (\alpha'_2, -\alpha'_1, 0)$ and the Laplacian of G is $\Delta G = (-\alpha'''_2, \alpha'''_1, 0)$. Thus from the condition (1.3) we have

$$\begin{array}{rl} \text{(i)} & -\alpha'''_2 = \lambda_{11}\alpha'_2 - \lambda_{12}\alpha'_1 \\ \text{(ii)} & \alpha'''_1 = \lambda_{21}\alpha'_2 - \lambda_{22}\alpha'_1 \\ \text{(iii)} & 0 = \lambda_{31}\alpha'_2 - \lambda_{32}\alpha'_1 \end{array}$$

where $\Lambda = [\lambda_{ij}]$ is a constant matrix. Since $|\alpha'| = 1$ we can put

$$\alpha'_1 = \cos\theta, \qquad \alpha'_2 = \sin\theta \tag{2.2}$$

where $\theta = \theta(s)$. Then from (2.1)(i), (ii) we obtain

$$\theta'' \cos\theta - \theta'^2 \sin\theta = -\lambda_{11} \sin\theta + \lambda_{12}\cos\theta$$

$$\theta'' \sin\theta + \theta'^2 \cos\theta = -\lambda_{21} \sin\theta + \lambda_{22}\cos\theta$$

which give

$$\theta'^2 = -(\lambda_{12} + \lambda_{21})\sin\theta\cos\theta + \lambda_{11}\sin^2\theta + \lambda_{22}\cos^2\theta \tag{2.3}$$

$$\theta'' = (\lambda_{22} - \lambda_{11})\sin\theta\cos\theta + \lambda_{12}\cos^2\theta - \lambda_{21}\sin^2\theta. \tag{2.4}$$

Taking the derivative of (2.3) and using (2.4) we obtain

$$\theta'[4(\lambda_{22} - \lambda_{11})\sin\theta\cos\theta + (3\lambda_{12} + \lambda_{21})\cos^2\theta - (\lambda_{12} + 3\lambda_{21})\sin^2\theta] = 0$$

If $\theta' = 0$, the Gauss map G is constant and hence M^2 is a plane. So, suppose $\theta' \neq 0$. Since $\sin^2\theta, \cos^2\theta$ and $\sin\theta\cos\theta$ are linearly independent functions of $\theta = \theta(s)$, we obtain from (2.5).

$$\lambda_{11} = \lambda_{22}, 3\lambda_{12} + \lambda_{21} = 0, \lambda_{12} + 3\lambda_{21} = 0.$$

Thus $\lambda_{12} = \lambda_{21} = 0$. Substitution into (2.3) then gives $\theta'^2 = \frac{1}{r^2}$, where $\frac{1}{r^2} = \lambda_{11} = \lambda_{22} = \text{const.}$ Now from (2.1)(i) and (ii) we conclude that the curve α is the circle

$$\alpha = (r\sin(rs + c) + d_1, -r\cos(rs + c) + d_2, 0) \tag{2.6}$$

where c, d_1, and d_2 are constants. Also from (2.1)(iii) we obtain $\lambda_{31} = \lambda_{32} = 0$.

REMARK. The matrix $\Lambda = [\lambda_{ij}]$ in the condition (1.3) when M^2 is the circular cylinder on the circle (2.6) is given by

$$\Lambda = \begin{bmatrix} \frac{1}{r^2} & 0 & \lambda_{13} \\ 0 & \frac{1}{r^2} & \lambda_{23} \\ 0 & 0 & \lambda_{33} \end{bmatrix}$$

where λ_{i3}, $i = 1, 2, 3$ are arbitrary constants.

Proof of the theorem. We suppose that M^2 is a non-cylindrical ruled surface in E^3. The surface M^2 can be expressed in terms of a directrix curve $\alpha(s)$ and a unit vector field $\beta(s)$ pointing along the rulings as

$$X(s, t) = \alpha(s) + t\beta(s).$$

Moreover, we can choose the parameter s to be arc length along the spherical curve $\beta(s)$. Thus for the curves α, β we have

$$\langle \beta, \beta\rangle = 1, \langle \beta', \beta'\rangle = 1, \langle \alpha', \beta\rangle = 0. \tag{2.7}$$

If we define a function q by

$$q = \|\alpha' + t\beta'\|^2 = t^2 + 2ut + v \tag{2.8}$$

where $u = \langle \alpha', \beta'\rangle$ and $v = \langle \alpha', \alpha'\rangle$, then the Gauss map of the surface is given by

$$G = q^{-1/2}((\alpha' + t\beta') \times \beta).$$

It is easy to shwo that the Laplacian Δ of M can be expressed as (see [**1**])

$$\Delta = -\frac{\partial^2}{\partial t^2} - \frac{1}{q}\frac{\partial^2}{\partial s^2} + \frac{1}{2}\frac{\partial q}{\partial s}\frac{1}{q^2}\frac{\partial}{\partial s} - \frac{1}{2}\frac{\partial q}{\partial t}\frac{1}{q}\frac{\partial}{\partial t}. \tag{2.10}$$

For convenience we put

$$G = (G_1, G_2, G_3) = q^{-1/2}(A_1 + tB_1, A_2 + tB_2, A_2 + tB_3) \tag{2.11}$$

where

$$\begin{aligned}(A_1, A_2, A_3) &= \alpha' \times \beta \\ (B_1, B_2, B_3) &= \beta' \times \beta.\end{aligned} \tag{2.12}$$

We now compute the Laplacian of the functions G_i. We have

$$\frac{\partial G_i}{\partial t} = q^{-3/2}[B_i q - (A_i + tB_i)(t+u)] = q^{-3/2}C_i$$

$$\frac{\partial^2 G_i}{\partial t^2} = q^{-5/2}[(B_i u - A_i)q - 3(B_i q - (A_i + tB_i)(t+u))(t+u)] = q^{-5/2}D_i$$

$$\frac{\partial G_i}{\partial s} = \tfrac{1}{2}q^{-3/2}[2(A_i' + tB_i')q - (A_i + tB_i)(2u't + v')] = \tfrac{1}{2}q^{-3/2}E_i$$

$$\begin{aligned}\frac{\partial^2 G_i}{\partial s^2} &= \tfrac{1}{2}q^{-5/2}\{[2(A_i'' + tB_i'')q + (A_i' + tB_i')(2u't + v') - (A_i + tB_i)(2u''t + v'')]q \\ &\quad - \tfrac{3}{2}[2(A_i' + tB_i')q - (A_i + tB_i)(2u't + v')](2u't + v')\} \\ &= \tfrac{1}{2}q^{-5/2}F_i.\end{aligned}$$

Thus, from the above relations and (2.10),

$$\Delta G_i = -q^{-5/2}D_i - \tfrac{1}{2}q^{-7/2}F_i + \tfrac{1}{4}q^{-7/2}(2u't + v')E_i - q^{-5/2}(t+u)C_i.$$

Now if we put $\Lambda = [\lambda_{ij}]$ from (1.3) and (2.11) we have

$$-4qD_i - 2F_i + (2u't + v')E_i - 4q(t+u)C_i = 4\sum_{j=1}^{3}\lambda_{ij}(A_j + tB_j)q^3, \qquad i = 1, 2, 3. \tag{2.13}$$

We consider the powers of t in equation (2.13). From the coefficient of t^7 we have

$$\sum_{j=1}^{3}\lambda_{ij}B_j = 0, \qquad i = 1, 2, 3. \tag{2.14}$$

Considering the coefficients of the other powers of t and using (2.14) we obtain for any $i = 1, 2, 3$

$$\sum_{j=1}^{3} \lambda_{ij} A_j = 0 \tag{2.15}$$

$$B_i'' = 0 \tag{2.16}$$

$$A_i - B_i u - 3B_i' u' + A_i'' - B_i u'' = 0 \tag{2.17}$$

$$-8A_i u + 4B_i u^2 + 4B_i v - 8A_i'' u + 6A_i' u' + 12B_i' uu' + 3B_i' v' + 2A_i u'' + B_i v'' + 4B_i uu'' - 8B_i u'^2 = 0 \tag{2.18}$$

$$12B_i uv - 12A_i u^2 - 4A_i'' v + 3A_i' v' + A_i v'' - 8A_i'' u^2 + 12A_i' uu' + 6B_i' u'v + 6B_i' uv' + 4A_i uu'' + 2B_i uv'' + 2B_i u'' v - 8A_i u'^2 - 8B_i u'v' = 0 \tag{2.19}$$

$$4B_i u^2 v + 4B_i v^2 - 8A_i u^3 - 8A_i'' uv + 6A_i' uv' + 2A_i uv'' + 6A_i' u'v + 3B_i' vv' + 2A_i u'' v + B_i vv'' - 8A_i u'v' - 2B_i v'^2 = 0 \tag{2.20}$$

$$2B_i uv^2 + 2A_i v^2 - 4A_i u^2 v - 2A_i'' v^2 + 3A_i' vv' + A_i vv'' - 2A_i v'^2 = 0. \tag{2.21}$$

We remark that $\det \Lambda = 0$, for if we assume $\det \Lambda \neq 0$, then from (2.14), $B_i = 0$, $i = 1, 2, 3$. Thus, from (2.12) we have $\beta' \times \beta = 0$, contradicting (2.7).

From (2.16) and (2.12) we have $\beta' \times \beta = cs + d$, where c and d are constant vectors. So $1 = \|\beta' \times \beta\|^2 = \langle c, c\rangle s^2 + 2\langle c, d\rangle s + \langle d, d\rangle$, from which we conclude that $\langle c, c\rangle = 0$, $\langle d, d\rangle = 1$, or equivalently $\beta' \times \beta = d$, where d is a constant unit vector. Since β is a spherical curve, this implies that β is a great circle. Let $\beta = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, where $\theta = \theta(s)$ and $\varphi = \text{const}$. From $\beta' \times \beta = d$ we conclude that $\theta'^2 = 1$ and so

$$\beta' \times \beta = (B_1, B_2, B_3) = (\sin\varphi, -\cos\varphi, 0).$$

Now, from (2.17) we have $A_i - B_i u + (A_i - B_i u)'' = 0$. If we put $A_i - B_i u = w_i$, $i = 1, 2, 3$, then $w_i + w_i'' = 0$ and $(\alpha' - u\beta') \times \beta = w$ where $w = (w_1, w_2, w_3)$. So $\|\alpha' - u\beta'\|^2 = \langle w, w\rangle$, or

$$v = u^2 + w^2 \tag{2.22}$$

where $w^2 = \langle w, w\rangle$.

Since $A_3 = w_3$ and $w_3'' = -w_3$, we have from (2.18)

$$3w_3' u' + w_3 u'' = 0. \tag{2.23}$$

By using (2.23), from (2.19) we find that

$$-4w_3 u^2 + 4w_3 v + 3w_3' v' + w_3 v'' - 8w_3 u'^2 = 0. \tag{2.24}$$

Using (2.22), (2.23) and (2.24), equations (2.20) and (2.21) can be written as

$$w_3 (w^2)' u' = 0 \tag{2.25}$$

$$4w_3 u'^2 v - w_3 v'^2 = 0. \tag{2.26}$$

Now, using the equations (2.22)–(2.26) we will prove that $w = 0$. Suppose, for the moment, that $w_3 \neq 0$. From (2.25) we have $(w^2)'u' = 0$. If $u' = 0$, from (2.26) we have $v' = 0$ and hence from (2.24) $u^2 = v$. Thus (2.22) implies $w = 0$, a contradiction. Thus $u' \neq 0$ and so $(w^2)' = 0$. From (2.22) $v' = 2uu'$ and from (2.26) $v = u^2$. Again (2.22) implies $w = 0$, a contradiction. So, we have $w_3 = 0$. This means that the vector w lies in the xy plane. But $w = \alpha' \times \beta - u\beta' \times \beta$ and the vector $\beta' \times \beta$ lies in the xy plane. So $\alpha' \times \beta$ lies in the xy plane. This means that the vectors $\alpha' \times \beta$ and $\beta' \times \beta$ are parallel. If we put $\alpha' \times \beta = \mu\beta' \times \beta$, then $(\alpha' - \mu\beta') \times \beta = 0$ or $\alpha' = \mu\beta'$. So $\mu = \langle \alpha', \beta' \rangle = u$ and $\alpha' = u\beta'$, namely $w = 0$.

Now we conclude that $q = (t + u)^2$ and the Gauss map is constant, which means that M^2 is a plane.

REFERENCES

1. B. Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken, Ruled surfaces of finite type, *Bull. Austral. Math. Soc.* **42** (1990), 447–453.

2. F. Dillen, J. Pas and L. Verstraelen, On the Gauss map of surfaces of revolution, *Bull. Inst. Math. Acad. Sinica*, **18** (1990), 239–246.

3. E. A. Ruh and J. Vilms, The tension field of the Gauss map, *Trans. Amer. Math. Soc.* **149** (1970), 569–573.

Department of Mathematics
University of Ioannina
Ioannina 45110
Greece

Department of Mathematics
Michigan State University
East Lansing, MI 48824
U.S.A.