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Lie Groups of Measurable Mappings

Helge Glöckner

Abstract. We describe new construction principles for infinite-dimensional Lie groups. In particular, given any measure space (X, Σ, μ) and (possibly infinite-dimensional) Lie group G, we construct a Lie group $L^\infty(X, G)$, which is a Fréchet-Lie group if G is so. We also show that the weak direct product $\prod^*_{i\in I} G_i$ of an arbitrary family $(G_i)_{i\in I}$ of Lie groups can be made a Lie group, modelled on the locally convex direct sum $\bigoplus_{i\in I} L(G_i)$.

Introduction

Many popular examples of infinite-dimensional Lie groups arise from finite-dimensional Lie groups G by general construction principles. For instance, for any $r \in \mathbb{N}_0 \cup \{\infty\}$ and compact smooth manifold K, the group $C^r(K, G)$ of G-valued C^r-maps on K is a Lie group [6], [23], [24], [31] and so is $C^r_c(M, G)$ when M is a non-compact, finite-dimensional smooth manifold M [1], [12], [23], [26]. It is remarkable that, although the G-valued mappings are of class C^r only, the group operations on the mapping groups are analytic, as a consequence of analyticity of the group operations of G (*cf.* [24, p. 1013]). Indeed, $C(K, G)$ is a smooth (resp., analytic) Lie group when K is an arbitrary compact topological space and G an arbitrary (possibly infinite-dimensional) smooth (resp., analytic) Lie group [12], [30]. Having passed from C^r-maps on manifolds to continuous mappings on topological spaces, it is a natural next step to consider groups of measurable mappings on measure spaces. Well-known examples are the Banach-Lie groups $L^\infty(X, A)^\times$ of invertible elements in the Banach algebra $L^\infty(X, A)$ of equivalence classes of essentially bounded measurable mappings on a measure space (X, Σ, μ), with values in a finite-dimensional Banach algebra A. For example, $L^\infty\big(\mathbb{S}^1, M_n(\mathbb{C})\big)^\times$ is encountered in [31], where it arises as the commutant of the multiplication operator by $\mathrm{id}\colon \mathbb{S}^1 \to \mathbb{S}^1$ in $\mathrm{GL}\big(L^2(\mathbb{S}^1, \mathbb{C}^n)\big)$. We also mention the Lie groups associated with Sobolev completions of loop algebras (see [1], [31]). In this article, we construct Lie groups $L^\infty(X, G)$ for arbitrary (not necessarily finite-dimensional) Lie groups G. The Lie groups $L^\infty(X, G)$ are natural generalizations of the unit groups just described, as $L^\infty(X, A)^\times = L^\infty(X, A^\times)$.

Another way to obtain new groups from given ones is the formation of direct limits. Construction principles for direct limits (mainly) of finite-dimensional Lie groups are described in [25]–[27] (see also [28, appendix]) and [13]. Examples of direct limit Lie groups of infinite-dimensional Lie groups can be found, *e.g.*, in [12], [20], [21], [26], [27], but no general construction principles or criteria ensuring the existence of direct limit Lie groups of directed systems of infinite-dimensional Lie

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groups (in the strong sense used in this article) seem to be available. As we show, at least for weak direct products

$$G = \prod_{i\in I}^{*} G_i = \varinjlim_{\substack{F\subseteq I,\\ |F|<\infty}} \prod_{i\in F} G_i$$

no pathologies occur and a Lie group structure can always be constructed, even for uncountable families $(G_i)_{i\in I}$ of Lie groups. We also show that G has the desired universal property of direct limit in suitable categories of Lie groups, at least when each G_i has a globally defined exponential function, which is diffeomorphic on some zero-neighbourhood (for generalizations, see [16]).

The material is organized as follows.

We begin with a brief description of the precise setting of differential calculus used in the article (Section 1). Section 2 provides specific results from topology and measure theory which are essential for our constructions. In Section 3, we first define $L^\infty(X, G)$ when G is a Hausdorff topological group. It is the group of equivalence classes (modulo functions $\equiv 1$ a.e.) of Borel measurable mappings $\gamma\colon X \to G$ the closure of whose image is compact and metrizable. $L^\infty(X, G)$ is a Hausdorff topological group in a natural way. If E is a Hausdorff locally convex space, then so is $L^\infty(X, E)$. If E is a Fréchet space, then also $L^\infty(X, E)$ is a Fréchet space; in this case, a mapping $\gamma\colon X \to E$ belongs to $L^\infty(X, E)$ if and only if it is a uniform limit of a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of finitely-valued, measurable mappings $\gamma_n\colon X \to E$. In Section 4, we show that the mapping $L^\infty(X, f)\colon L^\infty(X, E) \to L^\infty(X, F)$, $\gamma \mapsto f\circ\gamma$ is smooth (resp., $\mathbb{K}$-analytic), for every smooth (resp., $\mathbb{K}$-analytic) mapping $f\colon E \to F$ between locally convex $\mathbb{K}$-vector spaces (where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$), and prove various related results. These considerations allow us to turn $L^\infty(X, G)$ into a smooth (resp., $\mathbb{K}$-analytic) Lie group, for every smooth (resp., $\mathbb{K}$-analytic) Lie group G (Section 5). In Section 6, we have a closer look at the special case of ℓ^∞-spaces. We show that $\ell^\infty(X, E)$ need not be complete (nor quasi-complete) when E is a complete locally convex space (Example 6.5); its completion is the space $\widetilde{\ell^\infty}(X, E)$ of all functions $X \to E$ with relatively compact image (equipped with the topology of uniform convergence). Given a Lie group G, it is also possible to turn the group $\widetilde{\ell^\infty}(X, G)$ of all G-valued mappings with relatively compact image into a Lie group. However, measure-theoretic pathologies prevent us from defining Lie groups "$\widetilde{L^\infty}(X, G)$" based on measurable mappings with relatively compact image, for general measure spaces (X, Σ, μ): the metrizability condition in the definition of $L^\infty(X, G)$ is essential for our arguments. In Section 7, we construct a smooth (resp., $\mathbb{K}$-analytic) Lie group structure on the weak direct product $\prod_{i\in I}^{*} G_i$ of an arbitrary family $(G_i)_{i\in I}$ of smooth (resp., $\mathbb{K}$-analytic) Lie groups, modelled on the locally convex direct sum $\bigoplus_{i\in I} L(G_i)$. This allows us to turn the subgroup $L_c^\infty(X, G) \subseteq L^\infty(X, G)$ of compactly supported mappings into a smooth (resp., $\mathbb{K}$-analytic) Lie group modelled on $L_c^\infty\big(X, L(G)\big)$, for every Borel measure μ on a σ-compact locally compact space X (Section 8).

To tackle weak direct products of Lie groups, we provide technical results concerning mappings between locally convex direct sums, which are of independent interest. In [15], they serve as the basis of a theory of "patched" locally convex spaces, which ensures differentiability properties for suitable mappings between spaces of compactly supported sections in vector bundles. Variants are used in [16] to construct Lie group structures on diffeomorphism groups of finite-dimensional smooth manifolds over totally disconnected local fields.

For further connections between Lie theory and measure theory, *cf.* also [4].

1 The Setting of Differential Calculus

We use the framework of differential calculus of smooth and analytic mappings between open subsets of locally convex spaces outlined by J. Milnor [24], slightly generalized however as we do not presume sequential completeness of the locally convex spaces. See [10] for a detailed exposition of this generalized framework. Background material can also be found in [2], [5], [17], [21], [22], and [29]. We briefly recall various basic definitions and facts.

1.1 Suppose that E and F are Hausdorff real locally convex spaces, U is an open subset of E, and $f\colon U \to F$ a map. We say that f is *of class* C^0 if it is continuous, and set $d^0 f := f$. If f is continuous, we say that f is *of class* C^1 if the (two-sided) directional derivative $df(x,h) := \lim_{t\to 0} t^{-1}\big(f(x+th) - f(x)\big)$ exists for all $(x,h) \in U \times E$ (where $t \in \mathbb{R}\setminus\{0\}$ with $|t|$ sufficiently small), and the mapping $df\colon U\times E \to F$ is continuous. Recursively, we define f to be *of class* C^k for $2 \le k \in \mathbb{N}$ if it is of class C^{k-1} and $d^{k-1}f\colon U \times E^{2^{k-1}-1} \to F$ (having been defined recursively) is a mapping of class C^1 on the open subset $U \times E^{2^{k-1}-1}$ of the locally convex space $E^{2^{k-1}}$. We then set $d^k f := d(d^{k-1}f)\colon U \times E^{2^k-1} \to F$. The mapping f is called *smooth* or *of class* C^∞ if it is of class C^k for all $k \in \mathbb{N}$.

1.2 Equivalently, set $d^{(0)}f := f$ and, having defined C^j-maps and $d^{(j)}f\colon U\times E^j \to F$ for $0 \le j < k \in \mathbb{N}$, call f a mapping of class C^k if it is of class C^{k-1}, the limit

$$\begin{aligned} &d^{(k)}f(x,h_1,\dots,h_k)\\ &\qquad := \lim_{t\to 0} t^{-1}\big(d^{(k-1)}f(x+th_k,h_1,\dots,h_{k-1}) - d^{(k-1)}f(x,h_1,\dots,h_{k-1})\big)\end{aligned}$$

exists for all $x \in U$ and $h_1,\dots,h_k \in E$, and the mapping $d^{(k)}f\colon U\times E^k \to F$ so obtained is continuous. This is the usual definition of C^k-maps in the sense of Michal-Bastiani [6, p. 24], [10, Definition 1.8]. It is equivalent to the definition given in Subsection 1.1 (which is particularly well-suited for inductive arguments) by [10, Lemma 1.14].[1] For later use, we abbreviate $\delta_x^{(k)}f(h) := d^{(k)}f(x,h,\dots,h)$ for $x \in U$, $h \in E$.

1.3 Since compositions of C^r-maps are of class C^r for $0 \le r \le \infty$ [10, Proposition 1.15], C^r-manifolds modelled on Hausdorff locally convex spaces can be defined in the

[1] The "iterated differentials" $d^k f$ defined above are denoted $D^k f$ in [10]; $d^{(k)}f$ is denoted $d^k f$ there.

usual way, using an atlas of charts with C^r-transition functions. A *smooth Lie group* is a group, equipped with a smooth manifold structure modelled on a Hausdorff locally convex space, with respect to which the group multiplication and inversion are smooth mappings.

1.4 Let X be a C^r-manifold (where $1 \leq r \leq \infty$), and $f\colon X \to E$ a mapping of class C^r into a Hausdorff real locally convex space. Then the tangent map $Tf\colon TX \to TE = E \times E$ has the form $(x, v) \mapsto \big(f(x), df(x; v)\big)$ for $x \in X$ and $v \in T_xX$, where $df := \mathrm{pr}_2 \circ Tf\colon TX \to E$. We set $d^0 f := f$, $T^0 X := X$, and define $d^k f\colon T^k X \to E$ recursively via $d^k f := d(d^{k-1} f)$ for all $k \in \mathbb{N}$, $k \leq r$.

1.5 Let X be a Hausdorff topological space, E and F be Hausdorff locally convex spaces, U an open subset of E, and $f\colon X \times U \to F$ be a mapping. Given $r \in \mathbb{N}_0 \cup \{\infty\}$, we say that f is *partially C^r in the second argument* if $f(x, \bullet)\colon U \to F$ is a mapping of class C^r for all $x \in X$, and the functions $d_2^k f\colon X \times U \times E^{2^k - 1} \to F$, defined via $d_2^k f(x, \bullet) := d^k\big(f(x, \bullet)\big)$ for $x \in X$, are continuous for all $k \in \mathbb{N}_0$, $k \leq r$.

1.6 Let E and F be Hausdorff complex locally convex spaces, and $U \subseteq E$ be an open subset. A function $f\colon U \to F$ is called *complex analytic* or $\mathbb{C}$-analytic if it is continuous and for every $x \in U$, there exists a 0-neighbourhood V in E such that $x + V \subseteq U$ and $f(x + h) = \sum_{n=0}^{\infty} \beta_n(h)$ for all $h \in V$ as a pointwise limit, where $\beta_n\colon E \to F$ is a continuous homogeneous polynomial over $\mathbb{C}$ of degree n, for each $n \in \mathbb{N}_0$ [5, Definition 5.6].

1.7 A mapping f as in Subsection 1.6 is complex analytic if and only if it is smooth and $df(x, \bullet)\colon E \to F$ is complex linear for all $x \in U$ [10, Lemma 2.5].

1.8 Let E and F be Hausdorff real locally convex spaces, U be an open subset of E, and $f\colon U \to F$ be a map. Following Milnor's lines, we call f *real analytic* or $\mathbb{R}$-analytic if it extends to a complex analytic map $V \to F_{\mathbb{C}}$ on some open neighbourhood V of U in $E_{\mathbb{C}}$.

Throughout this article, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

1.9 Compositions of composable $\mathbb{K}$-analytic mappings are $\mathbb{K}$-analytic [10, Proposition 2.7, Proposition 2.8]. Thus complex (analytic) manifolds and real analytic manifolds, as well as complex (analytic) Lie groups and real analytic Lie groups modelled on Hausdorff locally convex spaces can be defined in the usual way.

2 Background Material and Preparatory Results

In this section, we assemble background material concerning compact metrizable spaces, continuous semi-metrics, and measurable mappings.

Lemma 2.1 *If K is a metrizable compact topological space and $f: K \to X$ a continuous mapping into a Hausdorff topological space X, then* $\operatorname{im}(f) = f(K)$ *is a metrizable compact subset of X.*

Proof It is well-known that $Q := f(K)$ is compact; the co-restriction $q := f|^Q$ is a closed surjection and thus a quotient map. Inverse images of points being compact, [9, Theorem 4.2.13] shows that $f(K)$ is metrizable. ■

Lemma 2.2 *Suppose that K_1 and K_2 are metrizable compact subsets of a Hausdorff topological group G. Then $K_1 \cup K_2$ and $K_1 \cdot K_2$ are metrizable compact subsets of G, and so is zK_1 when $G = E$ is a topological $\mathbb{K}$-vector space and $z \in \mathbb{K}$.*

Proof Since K_1 and K_2 are compact and metrizable, so is their topological direct sum $K_1 \coprod K_2$. For $i \in \{1,2\}$, let $\varepsilon_i: K_i \to K_1 \coprod K_2$ be the canonical embedding, and $\lambda_i: K_i \to K_1 \cup K_2$ be the inclusion map. Let $f: K_1 \coprod K_2 \to K_1 \cup K_2$ be the unique continuous mapping such that $f \circ \varepsilon_i = \lambda_i$ for $i \in \{1,2\}$. Then $K_1 \cup K_2 = \operatorname{im}(f)$, and thus $K_1 \cup K_2$ is metrizable by Lemma 2.1.

Note that $K_1 \times K_2$ is compact and metrizable, and $K_1 \cdot K_2 = m(K_1 \times K_2)$, where multiplication $m: G \times G \to G$, $(x,y) \mapsto xy$ is a continuous mapping. By Lemma 2.1, the compact set $K_1 \cdot K_2$ is metrizable.

Let $G = E$ be a topological $\mathbb{K}$-vector space now and $z \in \mathbb{K}$. The map $m_z: E \to E$, $v \mapsto zv$ being continuous, $zK_1 = m_z(K_1)$ is compact and metrizable by Lemma 2.1. ■

Given a semi-metric (*i.e.*, finite quasi-metric) $d: X \times X \to [0,\infty[$ on a set X, $\varepsilon > 0$, and $x \in X$, we let $B_d(x,\varepsilon) := \{y \in X : d(x,y) < \varepsilon\}$ denote the open ball of radius ε about x with respect to d. A family $(d_i)_{i\in I}$ of semi-metrics on X is called *directed* if, for all $i, j \in I$, there exists $k \in I$ such that $d_k \geq d_i$ and $d_k \geq d_j$ pointwise on $X \times X$.

Lemma 2.3 *Let K be a metrizable compact topological space, and $(d_i)_{i\in I}$ be a directed family of semi-metrics on K determining its topology. Then the following holds:*

(a) *If $d: K \times K \to [0,\infty[$ is any continuous semi-metric on K, and $\varepsilon > 0$, then there exists $i_0 \in I$ and $\delta > 0$ such that*

$$(\forall x \in K) \quad B_{d_{i_0}}(x,\delta) \subseteq B_d(x,\varepsilon). \tag{1}$$

(b) *There is a countable subset $J \subseteq I$ such that $(d_j)_{j\in J}$ determines the topology on K.*

Proof (a) We define $A := \{(x,y) \in K \times K : d(x,y) \geq \varepsilon\}$, $B_{i,\rho} := \{(x,y) \in K \times K : d_i(x,y) \leq \rho\}$ for $i \in I$, $\rho > 0$. As K is Hausdorff and the family $(d_i)_{i\in I}$ determines the topology of K, we have $\bigcap_{i\in I,\rho>0} B_{i,\rho} = \Delta$, where $\Delta = \{(x,x) : x \in K\}$ is the diagonal in $K \times K$. Thus A and each $B_{i,\rho}$ are compact subsets of $X \times X$, and

$$\bigcap_{i\in I,\rho>0} (A \cap B_{i,\rho}) = A \cap \Delta = \varnothing.$$

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By the finite intersection property of compact sets, there are finite subsets $F \subseteq I$ and $R \subseteq\ ]0,\infty[$ such that $\bigcap_{i\in F,\rho\in R}(A \cap B_{i,\rho}) = \varnothing$. We choose $i_0 \in I$ such that $d_{i_0} \geq d_i$ (pointwise) for all $i \in F$, and set $\delta := \min R$. Then $A \cap B_{i_0,\delta} = \varnothing$ and thus $B_{i_0,\delta} \subseteq \{(x,y) \in K \times K : d(x,y) < \varepsilon\}$, entailing (1).

(b) Let $d\colon K \times K \to [0,\infty[$ be a metric defining the topology on K. By (a), for every $n \in \mathbb{N}$ we find some $i_n \in I$ and $\delta_n > 0$ such that, for every $x \in X$, we have $B_{d_{i_n}}(x,\delta_n) \subseteq B_d(x, 2^{-n})$. Set $J := \{i_n : n \in \mathbb{N}\}$. Then J is countable, and it readily follows from the definition of the elements i_n that the family $(d_j)_{j\in J}$ of continuous semi-metrics on K determines the topology on K. ■

Remark 2.4 Recall in this connection that a semi-metric $d\colon G \times G \to [0,\infty[$ on a group G is *left invariant* if $d(gx, gy) = d(x,y)$ for all $x,y,g \in G$. Then $q(x) := d(x,e)$ (where $e \in G$ is the identity element) defines a *semi-norm* on G, *i.e.*, $q\colon G \to [0,\infty[$ satisfies $q(e) = 0$, $q(x) = q(x^{-1})$, and $q(xy) \leq q(x) + q(y)$, for all $x, y \in G$. Conversely, any semi-norm $q\colon G \to [0,\infty[$ on G gives rise to a left invariant semi-metric

(2) $$d_q\colon G \times G \to [0,\infty[\,, \quad d_q(x,y) := q(y^{-1}x).$$

Let Γ be a directed set of semi-norms on a group G and assume that, for all $x \in G$, $q \in \Gamma$ and $\varepsilon > 0$, there exists $p \in \Gamma$ and $\delta > 0$ such that $q(xyx^{-1}) < \varepsilon$ for all $y \in G$ satisfying $p(y) < \delta$ (the latter condition is vacuous if G is abelian). Then there exists a coarsest topological group topology on G making all $q \in \Gamma$ continuous (the topology defined by the family of semi-metrics $(d_q)_{q\in\Gamma}$); the sets $q^{-1}([0,\varepsilon[)$ (where $q \in \Gamma$, $\varepsilon > 0$) form a basis for its filter of identity neighbourhoods. The topology of any topological group can be obtained in this way, for a suitable family Γ (*cf.* [18, Theorem 8.2]).

If H is a Hausdorff topological space, we let $\mathcal{B}(H)$ denote its Borel σ-algebra, generated by the collection of open subsets of H. Measurability of functions on H or into H always refers to the measurable space $\big(H, \mathcal{B}(H)\big)$.

Lemma 2.5 *Suppose $(\gamma_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions $\gamma_n\colon X \to H$ from a measurable space (X,Σ) to a Hausdorff topological space H, converging pointwise to a function $\gamma\colon X \to H$. Suppose that $M := \operatorname{im}\gamma$ is separable, and suppose there exists a continuous semi-metric d on H such that $\delta := d|_{M\times M}$ is a metric on M defining its topology. Then γ is measurable.*

Proof Since $\mathcal{B}(M) = \{\omega \cap M : \omega \in \mathcal{B}(H)\}$ (*cf.* [3, Section 7, Exercise 2]), we only need to show that the co-restriction $\gamma|^M\colon X \to M$ is measurable. Let D be a countable dense subset of M. Every open subset of M being a countable union of balls $B_\delta(x, \frac{1}{k})$ for suitable $x \in D$ and $k \in \mathbb{N}$, the Borel σ-algebra $\mathcal{B}(M)$ is generated by the sets $B_\delta(x,\frac{1}{k})$. It therefore is initial with respect to the family $\big(\delta(x,\bullet)\big)_{x\in D}$ of mappings $\delta(x,\bullet)\colon M \to [0,\infty[$. Thus γ is measurable if and only if $\delta(x,\bullet)\circ\gamma|^M$ is measurable for each $x \in D$. Due to the continuity of $d(x,\bullet)$, we have $d(x,\bullet)\circ\gamma = \lim_{n\to\infty} d(x,\bullet)\circ\gamma_n$ pointwise; by [33, Theorem 1.14], $d(x,\bullet)\circ\gamma = \delta(x,\bullet)\circ\gamma|^M$ is measurable, as required. ■

Proposition 2.6 *Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions $\gamma_n\colon X \to H$ from a measurable space (X,Σ) to a completely regular topological space H, converging pointwise to a function $\gamma\colon X \to H$. If $K := \overline{\operatorname{im}\gamma}$ is compact and metrizable, then γ is measurable.*

Proof Since H is completely regular, its topology is determined by a set Γ of continuous semi-metrics (see [35, Section II.2.7, Satz 1 and Satz 2]). Lemma 2.3 entails that there is a sequence $(d_i)_{i\in\mathbb{N}}$ in Γ such that $(d_i|_{K\times K})_{i\in\mathbb{N}}$ defines the topology of K. Then $d\colon H\times H \to [0,1]$, $d(x,y) := \sum_{i=1}^{\infty} 2^{-i}\frac{d_i(x,y)}{1+d_i(x,y)}$ is a continuous semi-metric on H whose restriction to $K\times K$ is a metric on K defining the topology of K. By Lemma 2.5, γ is measurable. ■

Lemma 2.7 *If X and Y are Hausdorff topological spaces and X is second countable, then the Borel σ-algebra $\mathcal{B}(X\times Y)$ of the direct product $X\times Y$ of topological spaces coincides with the product σ-algebra $\mathcal{B}(X)\otimes\mathcal{B}(Y)$.*

Proof The inclusion $\mathcal{B}(X)\otimes\mathcal{B}(Y)\subseteq\mathcal{B}(X\times Y)$ holds for any Hausdorff spaces X and Y, the coordinate projections being continuous and thus measurable with respect to the Borel σ-algebras. Now if X is second countable, we let C be a countable basis of open sets for the topology of X. Let $U\subseteq X\times Y$ be an open subset. For any $p=(x,y)\in U$, we find $V_p\in C$ and an open subset $W_p\subseteq Y$ such that $p\in V_p\times W_p\subseteq U$. Given $V\in C$, we set $P_V := \{p\in U : V = V_p\}$, and set $W_V := \bigcup_{p\in P_V} W_p$. Then V and W_V are open subsets of X, resp., Y and thus Borel measurable, and so $V\times W_V\in\mathcal{B}(X)\otimes\mathcal{B}(Y)$. Thus also $U=\bigcup_{V\in C} V\times W_V$ is a member of $\mathcal{B}(X)\otimes\mathcal{B}(Y)$, being a countable union of members of $\mathcal{B}(X)\otimes\mathcal{B}(Y)$. We deduce that $\mathcal{B}(X\times Y)\subseteq\mathcal{B}(X)\otimes\mathcal{B}(Y)$. ■

As a consequence:

Lemma 2.8 *Let (X,Σ) be a measurable space, $Y_1, Y_2,\dots,Y_n$ and Z be Hausdorff topological spaces (where $n\in\mathbb{N}$), $f\colon Y_1\times\cdots\times Y_n \to Z$ be a continuous mapping, and $\gamma_i\colon (X,\Sigma)\to\big(Y_i,\mathcal{B}(Y_i)\big)$ be a measurable mapping such that $K_i := \overline{\operatorname{im}(\gamma_i)}$ is a metrizable compact subset of Y_i, for $i=1,\dots,n$. Then*

$$f\circ(\gamma_1,\dots,\gamma_n)\colon X\to Z,\quad x\mapsto f\big(\gamma_1(x),\dots,\gamma_n(x)\big)$$

is measurable as a mapping from (X,Σ) to $\big(Z,\mathcal{B}(Z)\big)$.

Proof It is easy to see that $(\gamma_1,\dots,\gamma_n)$ is measurable as a mapping into $K_1\times\cdots\times K_n$, equipped with the product σ-algebra $\mathcal{B}(K_1)\otimes\cdots\otimes\mathcal{B}(K_n)$. Since metrizable compact spaces are second countable, Lemma 2.7 gives $\mathcal{B}(K_1)\otimes\cdots\otimes\mathcal{B}(K_n) = \mathcal{B}(K_1\times\cdots\times K_n)$. Since $f|_{K_1\times\cdots\times K_n}\colon K_1\times\cdots\times K_n\to Z$ is continuous and thus Borel measurable, we deduce that the composition $f\circ(\gamma_1,\dots,\gamma_n) = f|_{K_1\times\cdots\times K_n}\circ(\gamma_1,\dots,\gamma_n)|^{K_1\times\cdots\times K_n}$ is measurable. ■

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3 The Topological Groups $\mathfrak{L}^\infty(X,G)$ and $L^\infty(X,G)$; The Spaces $\mathfrak{L}^\infty(X,E)$ and $L^\infty(X,E)$

Throughout this section, G denotes a Hausdorff topological group, E a Hausdorff, locally convex topological $\mathbb{K}$-vector space, and (X,Σ,μ) an arbitrary (not necessarily σ-finite) measure space. We shall define a Hausdorff topological group $L^\infty(X,G)$ and Hausdorff locally convex space $L^\infty(X,E)$, and study some of their properties.

General Convention When considering topological groups as uniform spaces, we shall always refer to the *left* uniform structure (as in [18, Definition 4.11]).

Definition 3.1 We let $\mathfrak{L}^\infty(X,G)$ be the set of all mappings $\gamma\colon X\to G$ such that

(a) γ is measurable as a mapping $(X,\Sigma)\to\big(G,\mathcal{B}(G)\big)$, and
(b) the closure of $\operatorname{im}(\gamma)$ is a metrizable, compact subset of G.

Remark 3.2 It follows readily from Lemma 2.2 and Lemma 2.8 that $\mathfrak{L}^\infty(X,G)$ is a subgroup of G^X. For example, if $\gamma_1,\gamma_2\in\mathfrak{L}^\infty(X,G)$, then $\gamma_1\cdot\gamma_2=m\circ(\gamma_1,\gamma_2)$ in terms of the continuous multiplication map $m\colon G\times G\to G$, whence $\gamma_1\cdot\gamma_2$ is measurable (Lemma 2.8). Its image being contained in the metrizable compact set $\overline{\operatorname{im}(\gamma_1)}\cdot\overline{\operatorname{im}(\gamma_2)}$ (Lemma 2.2), it has metrizable compact closure.

Remark 3.3 A particularly important special case of the preceding definition is obtained by choosing $G=E$; then $\mathfrak{L}^\infty(X,E)$ is a vector subspace of E^X.

For many familiar locally convex spaces E, compact subsets are automatically metrizable, making it unnecessary to require metrizability of $\overline{\operatorname{im}(\gamma)}$ in the definition of $\mathfrak{L}^\infty(X,E)$ for such spaces:

Proposition 3.4 *Suppose that E satisfies at least one of the following conditions:*

(a) *E is metrizable (e.g., E is a Fréchet space or a Banach space);*
(b) *$E=\varinjlim E_n$ is the locally convex direct limit of an ascending sequence $E_1\subseteq E_2\subseteq\cdots$ of metrizable locally convex spaces, such that E_n is a closed vector subspace of E_{n+1} and equipped with the induced topology (e.g., E might be any LF-space);*
(c) *$E=(F',\mathcal{T})$ is the dual of some separable locally convex space F, equipped with any locally convex vector topology $\mathcal{T}$ which is finer than the weak-$*$-topology.*

Then every compact subset of E is metrizable.

Proof (a) is trivial.

(b) If K is a compact subset of E, then K is contained in E_n for some $n\in\mathbb{N}$ by [34, Assertion 6.5]. As E induces the given topology on E_n (*loc. cit.*, Assertion 6.4) and E_n is metrizable, its subspace K is metrizable.

(c) Let K be a compact subset of $E=(F',\mathcal{T})$, and D be a countable dense subset of F. Then the family $(d_x)_{x\in D}$ of continuous semi-metrics $d_x\colon K\times K\to[0,\infty[$, $d_x(\lambda_1,\lambda_2):=|\lambda_1(x)-\lambda_2(x)|$ determines a Hausdorff topology $\mathcal{O}$ on K which is coarser than the given compact topology, and thus coincides with the latter. The family $(d_x)_{x\in D}$ being countable, $\mathcal{O}$ is metrizable. ■

In particular, (c) applies if $E = F_b'$ is the strong dual of a separable locally convex space (where the index "b" indicates the topology of uniform convergence on bounded sets).

Example 3.5 Let Ω be an open subset of $\mathbb{R}^n$. Then the space of test functions $\mathcal{D}(\Omega)$ is a separable LF-space. By Proposition 3.4, every compact subset of $\mathcal{D}(\Omega)$ is metrizable, and so is every compact subset of the distribution space $\mathcal{D}'(\Omega) := \mathcal{D}(\Omega)_b'$.

Recall that the *essential supremum* of a non-negative measurable function $f\colon X \to [0,\infty[$ on the measure space (X,Σ,μ) is defined as

$$\operatorname{ess\,sup}_\mu(f) := \min\{\sup f(X \setminus A) : A \in \Sigma \text{ s.t. } \mu(A) = 0\} \in [0,\infty].$$

3.6 Let Γ be a set of continuous semi-norms on G defining its topology (see Remark 2.4); if $G = E$ is a locally convex $\mathbb{K}$-vector space, we assume that each $q \in \Gamma$ is a semi-norm on E considered as a vector space, *i.e.*, furthermore $q(zx) = |z| \cdot q(x)$ for all $z \in \mathbb{K}$, $x \in E$.

Given $q \in \Gamma$, we define $\widetilde{q}\colon \mathfrak{L}^\infty(X,G) \to [0,\infty[$ via

$$\widetilde{q}(\gamma) := \operatorname{ess\,sup}_\mu(q \circ \gamma) \quad \text{for } \gamma \in \mathfrak{L}^\infty(X,G).$$

We set $N := \bigl\{\gamma \in \mathfrak{L}^\infty(X,G) : \mu\bigl(\gamma^{-1}(G \setminus \{e\})\bigr) = 0\bigr\}$.

3.7 It is clear that $\widetilde{q}$ is a semi-norm on the group $\mathfrak{L}^\infty(X,G)$, for all $q \in \Gamma$, and clearly N is a normal subgroup of $\mathfrak{L}^\infty(X,G)$. Given $q \in \Gamma$, $\gamma \in \mathfrak{L}^\infty(X,G)$ and $\varepsilon > 0$, in view of the compactness of $K := \overline{\operatorname{im}\gamma}$ there exists $p \in \Gamma$ and $\delta > 0$ such that $q(xyx^{-1}) \leq \varepsilon$ for all $x \in K$ and $y \in G$ such that $p(y) \leq \delta$. As a consequence, $\widetilde{q}(\gamma\eta\gamma^{-1}) \leq \varepsilon$ for all $\eta \in \mathfrak{L}^\infty(X,G)$ such that $\widetilde{p}(\eta) \leq \delta$. We give $\mathfrak{L}^\infty(X,G)$ the (usually non-Hausdorff) group topology determined by the family of semi-norms $(\widetilde{q})_{q\in\Gamma}$ (see Remark 2.4). It is easily verified that the topology on $\mathfrak{L}^\infty(X,G)$ is independent of the choice of Γ in Subsection 3.6.

3.8 If $G = E$, then each $\widetilde{q}$ is a semi-norm on $\mathfrak{L}^\infty(X,E)$ as a vector space, whence the semi-norms $\widetilde{q}$ give rise to a locally convex vector topology on $\mathfrak{L}^\infty(X,E)$. In this case, N is the set of those $\gamma \in \mathfrak{L}^\infty(X,E)$ vanishing μ-almost everywhere, which is a vector subspace of $\mathfrak{L}^\infty(X,E)$

Then we have:

Lemma 3.9 *Let $\gamma \in \mathfrak{L}^\infty(X,G)$. Then $\gamma \in N$ if and only if $\widetilde{q}(\gamma) = 0$ for all $q \in \Gamma$. Thus $N = \overline{\{e\}}$ in $\mathfrak{L}^\infty(X,G)$.*

Proof If $\gamma \in N$, then apparently $\widetilde{q}(\gamma) = 0$ for all $q \in \Gamma$.

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Conversely, assume that $\gamma \in \mathfrak{L}^\infty(X,G)$ and $\widetilde{q}(\gamma) = 0$ for all $q \in \Gamma$. The topology on the compact metrizable set $K := \overline{\operatorname{im}(\gamma)} \cup \{e\} \subseteq G$ is defined by the family of semi-metrics $(d_q)_{q\in\Gamma}$, where

$$d_q \colon K \times K \to [0,\infty[\,, \quad d_q(x,y) := q(y^{-1}x).$$

By Lemma 2.3, the topology on K is determined by $(d_q)_{q\in J}$, for some countable subset $J \subseteq \Gamma$. As $\widetilde{q}(\gamma) = 0$ for $q \in J$, we have $\mu(A_q) = 0$, where $A_q := \{x \in X : q(\gamma(x)) \neq 0\}$. As $A := \bigcup_{q\in J} A_q$ is a countable union of μ-null sets, we have $\mu(A) = 0$. However, as $(d_q)_{q\in J}$ determines the topology on $\operatorname{im}(\gamma) \cup \{e\}$, we have $A = \{x \in X : \gamma(x) \neq e\}$. Thus $\gamma \in N$.

The remainder is obvious. ■

The hypothesis that the compact sets $\overline{\operatorname{im}(\gamma)}$ be *metrizable* is essential for the validity of Lemma 3.9, as the following example shows.

Example 3.10 Given an uncountable set X, let $\Sigma := \mathcal{P}(X)$ be its power set and define a measure $\mu \colon \Sigma \to [0,\infty]$ via $\mu(A) := 0$ for countable subsets $A \subseteq X$ and $\mu(A) := \infty$ for uncountable ones. Let $E := \mathbb{R}^X$, equipped with the product topology, and consider the function

$$\gamma \colon X \to E, \quad x \mapsto \delta_{x,\bullet} = \mathbf{1}_{\{x\}}.$$

Then $\operatorname{im}(\gamma) \subseteq \{0,1\}^X$ and thus $\overline{\operatorname{im}(\gamma)}$ is compact. As $\Sigma = \mathcal{P}(X)$, γ is measurable. The semi-norms $q_x \colon \mathbb{R}^X \to [0,\infty[$, $q_x(f) := |f(x)|$ (for $x \in X$) determine the locally convex topology on $E = \mathbb{R}^X$, and as $q_x \circ \gamma = \delta_{x,\bullet}$ vanishes outside a finite set and thus μ-almost everywhere, we have $\widetilde{q_x}(\gamma) = 0$, for all $x \in X$. However, $\gamma^{-1}(E \setminus \{0\}) = X$ is a set of infinite (and thus non-zero) measure.

Definition 3.11 We define $L^\infty(X,G) := \mathfrak{L}^\infty(X,G)/N$ and give $L^\infty(X,G)$ the quotient topology, which makes it a Hausdorff topological group. If $G = E$, apparently $L^\infty(X,E)$ is a Hausdorff locally convex space.

Remark 3.12 In the present section, we strictly distinguish functions $\gamma \in \mathfrak{L}^\infty(X,G)$ and the associated equivalence classes $[\gamma] := \gamma N \in L^\infty(X,G)$. Following the general custom, for convenience of formulations we shall occasionally abandon this strict distinction in later sections, when no confusion can arise.

For $q \in \Gamma$, the continuous semi-norm $\widetilde{q}$ on $\mathfrak{L}^\infty(X,G)$ gives rise to a continuous semi-norm on $L^\infty(X,G)$, also denoted $\widetilde{q}$, via $\widetilde{q}([\gamma]) := \widetilde{q}(\gamma)$ for $\gamma \in \mathfrak{L}^\infty(X,G)$.

Definition 3.13 Given an open subset $U \subseteq G$, we define

$$\mathfrak{L}^\infty(X,U) := \{\gamma \in \mathfrak{L}^\infty(X,G) : \overline{\operatorname{im}\gamma} \subseteq U\} \subseteq \mathfrak{L}^\infty(X,G) \quad \text{and}$$
$$L^\infty(X,U) := \{[\gamma] : \gamma \in \mathfrak{L}^\infty(X,U)\} \subseteq L^\infty(X,G).$$

Lemma 3.14

(a) $L^\infty(X,U)$ *is open in* $L^\infty(X,G)$*, for every open subset* $U \subseteq G$*.*
(b) *When* U *ranges through the open identity neighbourhoods of* G*, the sets* $L^\infty(X,U)$ *form a basis for the filter of identity neighbourhoods of* $L^\infty(X,G)$*.*

Proof (a) If $f \in L^\infty(X,U)$, there exists $\gamma \in \mathfrak{L}^\infty(X,G)$ such that $[\gamma] = f$ and $K := \overline{\gamma(X)} \subseteq U$. The set K being compact and U being open, there exists an open identity neighbourhood V in G such that $KV \subseteq U$. There is $q \in \Gamma$ and $\varepsilon > 0$ such that $q^{-1}([0,\varepsilon[) \subseteq V$. If $h \in L^\infty(X,G)$ such that $\widetilde{q}(h) < \varepsilon$, there exists $\eta \in \mathfrak{L}^\infty(X,G)$ such that $[\eta] = h$ and $\sup q\big(\eta(X)\big) < \varepsilon$. Noting that $\sup q\big(\eta(X)\big) = \sup q\big(\overline{\eta(X)}\big)$, we deduce that the compact set $M := \overline{\eta(X)}$ is contained in V. Thus $\overline{\operatorname{im}(\gamma\eta)} \subseteq KM \subseteq KV \subseteq U$, and thus $fh \in L^\infty(X,U)$. We have shown that $f\widetilde{q}^{-1}([0,\varepsilon[) = B_{d_{\widetilde{q}}}(f,\varepsilon) \subseteq L^\infty(X,U)$, which is a neighbourhood of f.

(b) The assertion easily follows from (a) and the observation that

$$L^\infty\big(X, q^{-1}([0,\varepsilon[)\big) = \widetilde{q}^{-1}([0,\varepsilon[),$$

for every $q \in \Gamma$ and $\varepsilon > 0$. ■

If U is an open subset of E, we shall consider the open subset $L^\infty(X,U) \subseteq L^\infty(X,E)$ as an open $\mathbb{K}$-analytic submanifold of $L^\infty(X,E)$.

Lemma 3.15 *If* $\phi\colon G \to H$ *is a continuous homomorphism between Hausdorff topological groups, then also* $L^\infty(X,\phi)\colon L^\infty(X,G) \to L^\infty(X,H)$*,* $[\gamma] \mapsto [\phi \circ \gamma]$ *is a continuous homomorphism.*

Proof In view of Lemma 2.1, we have $\phi \circ \gamma \in \mathfrak{L}^\infty(X,H)$ for all $\gamma \in \mathfrak{L}^\infty(X,G)$. Clearly $[\phi \circ \gamma]$ only depends on $[\gamma]$, and $L^\infty(X,\phi)$ is a homomorphism. Given an open identity neighbourhood U in H, the homomorphism $L^\infty(X,\phi)$ takes the open identity neighbourhood $L^\infty\big(X,\phi^{-1}(U)\big)$ into $L^\infty(X,U)$. In view of Lemma 3.14, this means that $L^\infty(X,\phi)$ is continuous at the identity and thus continuous. ■

As a consequence:

Lemma 3.16 $L^\infty(X, G\times H) \cong L^\infty(X,G) \times L^\infty(X,H)$ *canonically, for all Hausdorff topological groups* G *and* H*. If* E *is a real locally convex space, then* $L^\infty(X,E)_{\mathbb{C}} = L^\infty(X,E_{\mathbb{C}})$*.* ■

For the remainder of this section, we investigate completeness properties of $L^\infty(X,G)$, and alternative characterizations of the functions $\gamma \in \mathfrak{L}^\infty(X,G)$.

Definition 3.17 $\mathcal{F}(X,G)$ denotes the group of all measurable mappings $\gamma\colon X \to G$ with finite image.

Proposition 3.18 *For every $\gamma \in \mathfrak{L}^\infty(X, G)$, there exists a sequence $(\gamma_n)_{n\in\mathbb{N}}$ in $\mathcal{F}(X, G)$ converging uniformly to γ (and thus also with respect to the topology on $\mathfrak{L}^\infty(X, G)$). In particular, the group $\mathcal{F}(X, G)$ of all finitely-valued measurable mappings is dense in $\mathfrak{L}^\infty(X, G)$. If, conversely, $\gamma\colon X \to G$ is a uniform limit of a sequence $(\gamma_n)_{n\in\mathbb{N}}$ in $\mathcal{F}(X, G)$ (or, more generally, in $\mathfrak{L}^\infty(X, G)$), then $\operatorname{im}\gamma$ is pre-compact; hence if $\overline{\operatorname{im}\gamma}$ is compact and metrizable, then $\gamma \in \mathfrak{L}^\infty(X, G)$.*

Proof Given $\gamma \in \mathfrak{L}^\infty(X, G)$, set $K := \overline{\operatorname{im}(\gamma)}$. Then K is a metrizable compact subset of G. The family $(d_q)_{q\in\Gamma}$ of semi-metrics $d_q\colon K \times K \to [0, \infty[$, $d_q(v, w) := q(v^{-1}w)$ determines the topology on K. By Lemma 2.3 and since Γ is directed, we find an ascending sequence $q_1 \le q_2 \le \cdots$ of semi-norms $q_n \in \Gamma$ such that $(d_n)_{n\in\mathbb{N}}$ determines the topology on K, where $d_n := d_{q_n}$. Given $v \in K$ and $\varepsilon > 0$, let $B_{d_n}(v, \varepsilon)$ be the open d_n-ball of radius ε around v in K. Due to compactness, for every $n \in \mathbb{N}$ we have $K = \bigcup_{i=1}^{m(n)} B_{d_n}(v_i^n, \frac{1}{n})$ for finitely many elements $v_1^n, \ldots, v_{m(n)}^n \in K$. Then

$$A_{n,i} := B_{d_n}(v_i^n, \tfrac{1}{n}) \setminus \bigcup_{j=1}^{i-1} B_{d_n}(v_j^n, \tfrac{1}{n})$$

is a measurable subset of K, for every $n \in \mathbb{N}$, $i = 1, \ldots, m(n)$. Since $K = A_{n,1} \cup \cdots \cup A_{n,m(n)}$ is a disjoint union, and $\operatorname{im}\gamma \subseteq K$, setting $\gamma_n(x) := v_i^n$ for $x \in \gamma^{-1}(A_{n,i})$ we obtain a function $\gamma_n \in \mathcal{F}(X, G)$. By definition, $\operatorname{im}\gamma_n \subseteq K$. We claim that $\gamma_n \to \gamma$ uniformly as $n \to \infty$. To see this, let $q \in \Gamma$ and $\varepsilon > 0$. By Lemma 2.3, there exists $k \in \mathbb{N}$ and $\delta > 0$ such that

$$(\forall v, w \in K) \quad d_k(v, w) < \delta \Rightarrow d_q(v, w) < \varepsilon.$$

We find $n_0 \ge k$ such that $\frac{1}{n_0} < \delta$. For every $n \in \mathbb{N}$ such that $n \ge n_0$ and $x \in X$, there exists a unique $i \in \{1, \ldots, m(n)\}$ such that $\gamma(x) \in A_{n,i}$, and then $\gamma_n(x) = v_i^n$. As $A_{n,i} \subseteq B_{d_n}(v_i^n, \frac{1}{n})$, we deduce that $d_k\big(\gamma_n(x), \gamma(x)\big) \le d_n\big(\gamma_n(x), \gamma(x)\big) < \frac{1}{n} < \delta$ and thus $d_q\big(\gamma_n(x), \gamma(x)\big) < \varepsilon$. Consequently, $\sup\big\{ d_q\big(\gamma_n(x), \gamma(x)\big) : x \in X\big\} \le \varepsilon$. Thus $\gamma_n \to \gamma$ uniformly indeed. Furthermore, $q\big(\gamma(x)^{-1}\gamma_n(x)\big) = d_q\big(\gamma_n(x), \gamma(x)\big) < \varepsilon$ entails that $\widetilde{q}(\gamma^{-1}\gamma_n) \le \varepsilon$ for all $n \ge n_0$, whence $\gamma_n \to \gamma$ in $\mathfrak{L}^\infty(X, G)$.

To prove the partial converse, suppose that $\gamma\colon X \to G$ is the uniform limit of a sequence $(\gamma_n)_{n\in\mathbb{N}}$ in $\mathfrak{L}^\infty(X, G)$. Given an identity neighbourhood U in G, choose an identity neighbourhood V in G such that $VV \subseteq U$. As $\gamma_n \to \gamma$ uniformly, there exists $n \in \mathbb{N}$ such that, for all $x \in X$, $\gamma(x) \in \gamma_n(x)V$. Now $\overline{\operatorname{im}\gamma_n}$ being compact, we have $\operatorname{im}\gamma_n \subseteq FV$ for some finite subset F of G. Then $\operatorname{im}\gamma \subseteq (\operatorname{im}\gamma_n)V \subseteq FVV \subseteq FU$; we have shown that $\operatorname{im}\gamma$ is pre-compact.

In the preceding situation, assume in addition that $\overline{\operatorname{im}\gamma}$ is compact and metrizable. The Hausdorff group G being completely regular [18, Theorem 8.4], Proposition 2.6 shows that γ is measurable. Thus $\gamma \in \mathfrak{L}^\infty(X, G)$. ■

Corollary 3.19 *If every pre-compact subset of G has metrizable, compact closure, then a function $\gamma\colon X \to G$ belongs to $\mathfrak{L}^\infty(X, G)$ if and only if it is the uniform limit of a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of finitely-valued, measurable functions $\gamma_n \in \mathcal{F}(X, G)$.* ■

Remark 3.20 Note that the hypotheses of Corollary 3.19 are satisfied for any Fréchet space, and for any LF-space (*cf.* [34, Section II.6.5]). For example, they are satisfied for $G = \mathcal{D}(\Omega)$, for any open subset Ω of $\mathbb{R}^n$. They are also satisfied for $G = \mathcal{D}'(\Omega)$, since by completeness every pre-compact subset of $\mathcal{D}'(\Omega)$ has compact closure, and the latter is metrizable (see Example 3.5).

Proposition 3.21 *If G is metrizable, then so is $L^\infty(X, G)$. If G is metrizable and complete, then so is $L^\infty(X, G)$. If E is a Fréchet space (resp., a Banach space), then so is $L^\infty(X, E)$.*

Proof If G is metrizable (resp., if E is normable), we can choose a continuous group-norm (resp., vector space norm) q on G (resp., E) determining the group topology (resp., locally convex vector topology), and then $\widetilde{q}$ is a group-norm (resp., vector space norm) on $L^\infty(X, G)$ (resp., $L^\infty(X, E)$) determining its group topology (resp., locally convex vector topology), whence the latter group is metrizable (resp., a normed space).

Thus, it only remains to assume that G is complete and metrizable, and show that every Cauchy sequence $(g_n)_{n\in\mathbb{N}}$ in $L^\infty(X, G)$ converges. For every $n \in \mathbb{N}$, choose $\gamma_n \in \mathfrak{L}^\infty(X, G)$ such that $[\gamma_n] = g_n$. Let q be a continuous norm on G determining the group topology. Then, for all $n, m \in \mathbb{N}$, there exists $A_{n,m} \in \Sigma$ such that $\mu(A_{n,m}) = 0$ and

$$\widetilde{q}(g_m^{-1} g_n) = \widetilde{q}(\gamma_m^{-1}\gamma_n) = \sup\big(q \circ (\gamma_m^{-1}\gamma_n)\big)(X \setminus A_{n,m}).$$

The set $\mathbb{N}^2$ being countable, we have $A := \bigcup_{n,m\in\mathbb{N}} A_{n,m} \in \Sigma$, and $\mu(A) = 0$. Then $[\eta_n] = [\gamma_n] = g_n$ for $\eta_n \in \mathfrak{L}^\infty(X, G)$ defined via $\eta_n|_{X\setminus A} := \gamma_n|_{X\setminus A}$ and $\eta_n|_A := e$. We have

$$(3) \qquad (\forall n, m \in \mathbb{N}) \quad \widetilde{q}(g_m^{-1} g_n) = \sup\big(q \circ (\eta_m^{-1}\eta_n)\big)(X),$$

entailing that $\big(\eta_n(x)\big)_{n\in\mathbb{N}}$ is a Cauchy sequence in G for each $x \in X$, and thus convergent to some $\eta(x) \in G$. We claim that $\eta_n \to \eta$ uniformly. In fact, let $\varepsilon > 0$. There is $n_0 \in \mathbb{N}$ such that $\widetilde{q}(g_m^{-1} g_n) < \varepsilon$ for all $n, m \ge n_0$. In view of equation (3), we then have $q\big(\eta_m(x)^{-1}\eta_n(x)\big) < \varepsilon$ for all $x \in X$ and $n, m \ge n_0$. Letting $m \to \infty$, we deduce that

$$(\forall n \ge n_0)\ (\forall x \in X) \quad q\big(\eta(x)^{-1}\eta_n(x)\big) \le \varepsilon.$$

Thus $\eta_n \to \eta$ uniformly indeed. By Proposition 3.18, $\operatorname{im}\eta$ is pre-compact. Thus, being a closed, pre-compact subset of the complete uniform space G, the set $\overline{\operatorname{im}\eta}$ is compact; since G is metrizable, so is $\overline{\operatorname{im}\eta}$. By Proposition 3.18, $\eta \in \mathfrak{L}^\infty(X, G)$. As $\eta_n \to \eta$ uniformly, we have $g_n = [\eta_n] \to [\eta]$ *a fortiori.* ■

Thus, if E is a Fréchet space, then $L^\infty(X, E)$ is the completion of $\mathcal{F}(X, E)$ (modulo functions vanishing μ-almost everywhere), as a consequence of Proposition 3.18 and Proposition 3.21. As a special case of Corollary 3.19, we have:

Remark 3.22 If E is a Fréchet space, then a function $\gamma\colon X \to E$ belongs to $\mathfrak{L}^\infty(X, E)$ if and only if there exists a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of finitely-valued measurable functions

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converging uniformly to γ. Thus, we have a close conceptual similarity between the $\mathfrak{L}^\infty(X,E)$-functions defined here and the so-called strongly measurable functions with values in a Banach space E, which can be characterized as uniform limits of *countably-valued* measurable functions (almost everywhere), see [19, Corollary 1 to Theorem 3.5.3].[2]

Remark 3.23 Further motivation for our specific definition of $\mathfrak{L}^\infty(X,E)$ came from [37]. In this paper, differentiability properties of mappings of the type $\mathcal{E}^m_\bullet(X,E_1) \to \mathcal{E}^m_\bullet(X,E_2)$, $\gamma \mapsto f\circ\gamma$ are analyzed for smooth maps $f\colon E_1 \to E_2$ between quasi-complete locally convex spaces (or open subsets thereof), where $\mathcal{E}^m_\bullet(X,E_i)$ denotes the space of m times k-continuously differentiable E_i-valued mappings with relatively compact image on an open subset X of a locally convex space F. Our most important technical tools (to be developed in the next section) are analogues for L^∞-spaces of Thomas' results.

4 Differentiable Mappings Between L^∞-Spaces

In order to turn the topological group $L^\infty(X,G)$ into a Lie group when G is a Lie group, we need to understand differentiability properties of certain types of mappings on $L^\infty\big(X,L(G)\big)$ (and open subsets thereof).

As before, (X,Σ,μ) denotes an arbitrary measure space.

Proposition 4.1 *Suppose that E and F are locally convex Hausdorff spaces, $U \subseteq E$ an open subset, P a Hausdorff topological space, and $\sigma\colon X \to P$ a measurable mapping such that $K := \overline{\operatorname{im}(\sigma)}$ is compact and metrizable. Let furthermore $\widetilde{f}\colon P\times U \to F$ be a mapping which is partially C^k in the second argument for some $k \in \mathbb{N}_0 \cup \{\infty\}$, and define*

$$f\colon = \widetilde{f}\circ(\sigma\times \mathrm{id}_U)\colon X\times U \to F.$$

Then

$$f_*\colon L^\infty(X,U)\to L^\infty(X,F), \quad f_*([\gamma]) := [f\circ(\mathrm{id}_X,\gamma)]$$

is a mapping of class C^k.

Proof We may assume that $k \in \mathbb{N}_0$. The definition of $f_*[\gamma]$ makes sense, since $f_*\gamma := f\circ(\mathrm{id}_X,\gamma) = \widetilde{f}\circ(\sigma,\gamma) \in \mathfrak{L}^\infty(X,F)$ as a consequence of Lemma 2.1 and Lemma 2.8, for any $\gamma \in \mathfrak{L}^\infty(X,U)$; furthermore, apparently $[f\circ(\mathrm{id}_X,\gamma)]$ only depends on $[\gamma]$.

Step 1: f_* is continuous. To see this, suppose that $\gamma \in \mathfrak{L}^\infty(X,E)$ such that $M := \overline{\operatorname{im}\gamma} \subseteq U$, and suppose that V is an open zero-neighbourhood in F. Let $V_1 \subseteq V$ be a

[2] Apparently, if we replace μ by its Lebesgue completion $\overline{\mu}$, we obtain an isomorphic space $L^\infty(X,E)$, and then $\gamma\colon X\to E$ is an $\mathfrak{L}^\infty(X,E)$-function on $(X,\overline{\Sigma},\overline{\mu})$ if and only if it is a uniform limit of finitely-valued measurable functions on (X,Σ) μ-almost everywhere. So, for our purposes nothing is lost by considering measurable functions on (X,Σ) only instead of $\overline{\mu}$-measurable ones. Rather, we profit from shorter proofs.

closed zero-neighbourhood in F. Exploiting the compactness of K and M, we easily find an open zero-neighbourhood $W \subseteq E$ such that $M + W \subseteq U$, and such that

$$(\forall p \in K,\ y \in M) \quad \widetilde{f}\big(\{p\} \times (y + W)\big) \subseteq \widetilde{f}(p, y) + V_1.$$

Now if $\eta \in \mathfrak{L}^\infty(X, W)$, then $\overline{\operatorname{im}(\gamma + \eta)} \subseteq M + \overline{\operatorname{im}\eta} \subseteq M + W \subseteq U$, and

$$\big(f_*(\gamma + \eta)\big)(x) = \widetilde{f}\big(\sigma(x), \gamma(x) + \eta(x)\big) \in \widetilde{f}\big(\sigma(x), \gamma(x)\big) + V_1 = (f_*\gamma)(x) + V_1$$

for all $x \in X$, entailing that $\overline{\operatorname{im}\big(f_*(\gamma + \eta) - f_*\gamma\big)} \subseteq V_1 \subseteq V$ and therefore $f_*(\gamma + \eta) - f_*\gamma \in \mathfrak{L}^\infty(X, V)$. We have shown that f_* takes $[\gamma] + L^\infty(X, W)$ into $f_*[\gamma] + L^\infty(X, V)$. In view of Lemma 3.14 (b), the continuity of f_* follows.

Now suppose that $k \geq 1$.

Step 2: f_* is of class C^1. Let $\gamma \in \mathfrak{L}^\infty(X, U)$ and $\eta \in \mathfrak{L}^\infty(X, E)$ be given. The sets $M := \overline{\operatorname{im}\gamma} \subseteq U$ and $N := \overline{\operatorname{im}\eta}$ being compact, there is $\varepsilon > 0$ such that

$$\overline{\operatorname{im}\gamma} + [-\varepsilon, \varepsilon] \cdot \overline{\operatorname{im}\eta} \subseteq U.$$

Then

$$(4) \qquad \tfrac{1}{h}\big(f_*(\gamma + h\eta) - f_*\gamma\big)(x) = \tfrac{1}{h}\Big(\widetilde{f}\big(\sigma(x), \gamma(x) + h\eta(x)\big) - \widetilde{f}\big(\sigma(x), \gamma(x)\big)\Big)$$
$$= \int_0^1 d_2\widetilde{f}\big(\sigma(x), \gamma(x) + th\eta(x), \eta(x)\big)\, dt$$
$$= \int_0^1 H\big(\sigma(x), \gamma(x), \eta(x), th\big)\, dt$$

for all $h \in [-\varepsilon, \varepsilon]$ and $x \in X$, where $H\colon K \times M \times N \times [-\varepsilon, \varepsilon] \to F$ is defined via

$$H(u, v, w, s) := d_2\widetilde{f}(u, v + sw, w)$$

for $u \in K$, $v \in M$, $w \in N$, and $s \in [-\varepsilon, \varepsilon]$. Given an open zero-neighbourhood V in F, we choose a closed, convex, symmetric zero-neighbourhood $W \subseteq V$ of F. In view of the compactness of $K \times M \times N$, we find $\delta \in \,]0, \varepsilon]$ such that $H(u, v, w, s) - H(u, v, w, 0) \in W$ for all $u \in K$, $v \in M$, $w \in N$, and $s \in \,]-\delta, \delta[$. Using equation (4), we deduce that

$$h^{-1}\big(f_*(\gamma + h\eta) - f_*\gamma\big)(x) - H\big(\sigma(x), \gamma(x), \eta(x), 0\big) \in W$$

for all $h \in \,]-\delta, \delta[$ and $x \in X$. Note that

$$H\big(\sigma(x), \gamma(x), \eta(x), 0\big) = (d_2\widetilde{f})\big(\sigma(x), \gamma(x), \eta(x)\big)$$

here. Thus

$$\overline{\operatorname{im}\Big(h^{-1}\big(f_*(\gamma + h\eta) - f_*\gamma\big) - (d_2\widetilde{f}) \circ (\sigma, \gamma, \eta)\Big)} \subseteq W \subseteq V,$$

and thus $h^{-1}\big(f_*(\gamma+h\eta)-f_*\gamma\big)-(d_2\widetilde{f})\circ(\sigma,\gamma,\eta)\in\mathfrak{L}^\infty(X,V)$ for all $h\in\,]-\delta,\delta[$. We deduce that $h^{-1}\big(f_*(\gamma+h\eta)-f_*\gamma\big)\to d_2\widetilde{f}\circ(\sigma,\gamma,\eta)=g_*(\gamma,\eta)$ in $L^\infty(X,F)$ as $h\to 0$, where $g:=d_2\widetilde{f}\circ(\sigma\times\mathrm{id}_{U\times E})$ (*cf.* Remark 3.12). Thus $d(f_*)$ exists, and is given by

$$d(f_*)=g_*\colon L^\infty(X,U\times E)\to L^\infty(X,F), \tag{5}$$

identifying $L^\infty(X,U)\times L^\infty(X,E)$ with $L^\infty(X,U\times E)\subseteq L^\infty(X,E^2)$ (see Lemma 3.16). Note that g is obtained from the function $\widetilde{g}:=d_2\widetilde{f}\colon P\times(U\times E)\to F$ (which is partially C^{k-1} in the second argument) in the same way in which f is obtained from $\widetilde{f}$. Thus Step 1, applied to g, shows that $d(f_*)=g_*$ is continuous.

Step 3: Induction. Suppose the proposition holds for $k-1$ in place of $k\geq 1$. Step 2 shows that f is of class C^1, with $d(f_*)=g_*$ of class C^{k-1} by the induction hypothesis. Thus f is of class C^k. ■

Corollary 4.2 *Suppose that E and F are complex locally convex Hausdorff spaces, $U\subseteq E$ an open subset, P a Hausdorff topological space, and $\sigma\colon X\to P$ a measurable mapping such that $K:=\overline{\mathrm{im}(\sigma)}$ is compact and metrizable. Let $\widetilde{f}\colon P\times U\to F$ be a mapping which is partially C^∞ in the second argument and such that $\widetilde{f}(p,\bullet)\colon U\to F$ is complex analytic for each $p\in P$. Define $f:=\widetilde{f}\circ(\sigma\times\mathrm{id}_U)\colon X\times U\to F$. Then $f_*\colon L^\infty(X,U)\to L^\infty(X,F)$ is complex analytic.*

Proof By Proposition 4.1 and its proof, f_* is smooth and $df_*(\gamma,\eta)=d_2\widetilde{f}\circ(\sigma,\gamma,\eta)$, which is complex linear in η as $\widetilde{f}(p,\bullet)$ is complex analytic for each $p\in P$. By Subsection 1.7, f_* is complex analytic. ■

Proposition 4.3 *Suppose that E and F are Hausdorff real locally convex spaces, $U\subseteq E$ an open zero-neighbourhood, P a real analytic manifold, modelled on a Hausdorff locally convex space Z, and $\sigma\colon X\to P$ a measurable mapping such that $K:=\overline{\mathrm{im}(\sigma)}$ is compact and metrizable. Given a real analytic mapping $\widetilde{f}\colon P\times U\to F$, define $f:=\widetilde{f}\circ(\sigma\times\mathrm{id}_U)\colon X\times U\to F$. Then $f_*\colon L^\infty(X,U)\to L^\infty(X,F)$ is real analytic on $L^\infty(X,Q)$ for some open zero-neighbourhood $Q\subseteq U$ in E.*

Proof As P is a real analytic manifold, for every $a\in P$ we find a diffeomorphism $\phi_a\colon W_a\to P_a$ of real analytic manifolds from an open zero-neighbourhood W_a in Z onto an open neighbourhood P_a of a in P such that $\phi_a(0)=a$. Then the mapping $\theta_a\colon W_a\times U\to F$, $\theta_a(w,u):=\widetilde{f}\big(\phi_a(w),u\big)$ is real analytic and hence extends to a complex analytic mapping $\widetilde{\theta}_a\colon E_a\to F_{\mathbb{C}}$, defined on an open neighbourhood E_a of $W_a\times U$ in $Z_{\mathbb{C}}\times E_{\mathbb{C}}$. Shrinking W_a if necessary, we may assume that E_a contains a 0-neighbourhood of the form $W_a'\times(Q_a+iQ_a')$ for an open neighbourhood W_a' of W_a in $Z_{\mathbb{C}}$, an open zero-neighbourhood $Q_a\subseteq U$ in E, and an open, symmetric, convex zero-neighbourhood Q_a' in E. As $\overline{\mathrm{im}\,\sigma}$ is compact, we have $\mathrm{im}\,\sigma\subseteq\bigcup_{a\in A}P_a=:P'$ for some finite subset A of P. Then $Q':=\bigcap_{a\in A}Q_a'$ is an open, symmetric, convex

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zero-neighbourhood in E, and $Q := \bigcap_{a\in A} Q_a$ is an open zero-neighbourhood. If $a, a' \in A$, then for every $p \in P_a \cap P_{a'}$, the prescriptions $x \mapsto \widetilde{\theta}_a\big(\phi_a^{-1}(p), x\big)$ and $x \mapsto \widetilde{\theta}_{a'}\big(\phi_{a'}^{-1}(p), x\big)$ define complex analytic mappings $Q + iQ' \to F_{\mathbb{C}}$ which coincide on Q (where they coincide with $\widetilde{f}(p, \bullet)|_Q$), and which therefore coincide (*cf.* [5, Proposition 6.6]). We abbreviate $Q_1 := Q + iQ'$. By the preceding,

$$\widetilde{h}\colon P' \times Q_1 \to F_{\mathbb{C}}, \quad \widetilde{h}(p, x) := \widetilde{\theta}_a\big(\phi_a^{-1}(p), x\big) \quad \text{if } p \in P_a, \text{ where } a \in A$$

is a well-defined smooth mapping such that $\widetilde{h}(p, \bullet)\colon Q_1 \to F_{\mathbb{C}}$ is complex analytic for each $p \in P'$. Define $h := \widetilde{h} \circ (\sigma \times \mathrm{id}_{Q_1})$, $X \times Q_1 \to F_{\mathbb{C}}$. By Corollary 4.2, the mapping $h_*\colon L^\infty(X, Q_1) \to L^\infty(X, F_{\mathbb{C}}) = L^\infty(X, F)_{\mathbb{C}}$ is complex analytic. Thus $f_*|_{L^\infty(X,Q)} = h_*\big|_{L^\infty(X,Q)}^{L^\infty(X,F)}$ is real analytic. ■

Corollary 4.4 *Suppose that E and F are Hausdorff locally convex $\mathbb{K}$-vector spaces, U an open zero-neighbourhood in E, and $f\colon U \to F$ a smooth (resp., $\mathbb{K}$-analytic) mapping. Then*

$$L^\infty(X, f)\colon L^\infty(X, U) \to L^\infty(X, F),\ [\gamma] \mapsto [f \circ \gamma]$$

is a smooth (resp., $\mathbb{K}$-analytic) mapping, and $dL^\infty(X, f) = L^\infty(X, df)$.

Proof Let $\{0\}$ be a manifold consisting of single point. Define $\widetilde{h}\colon \{0\} \times U \to F$, $\widetilde{h}(0, u) := f(u)$ and $h := \widetilde{h} \circ (0 \times \mathrm{id}_U)\colon X \times U \to F$, $h(x, u) = f(u)$. Then $L^\infty(X, f) = h_*$.

If f is smooth, then $\widetilde{h}$ is smooth and thus partially C^∞ in the second argument, and thus the hypotheses of Proposition 4.1 are satisfied for $\widetilde{h}$ (with $k = \infty$). Thus $L^\infty(X, f) = h_*$ is smooth, and has the asserted derivative in view of (5).

If f is complex analytic, then $\widetilde{h}$ is complex analytic, whence the hypotheses of Corollary 4.2 are satisfied for $\widetilde{h}$, and thus h_* is complex analytic.

If f is real analytic, there exists a complex analytic function $g\colon \widetilde{U} \to F_{\mathbb{C}}$, defined on an open neighbourhood $\widetilde{U}$ of U in $E_{\mathbb{C}}$. By the preceding, the mapping $L^\infty(X, g)\colon L^\infty(X, \widetilde{U}) \to L^\infty(X, F_{\mathbb{C}}) = L^\infty(X, F)_{\mathbb{C}}$ is complex analytic. Possessing a complex analytic extension, $L^\infty(X, f) = L^\infty(X, g)\big|_{L^\infty(X,U)}^{L^\infty(X,F)}$ is real analytic. ■

5 The Lie Group $L^\infty(X, G)$

Let G be any smooth or $\mathbb{K}$-analytic Lie group. In this section, we show that $L^\infty(X, G)$ can be made a smooth, resp., $\mathbb{K}$-analytic Lie group, with Lie algebra $L^\infty\big(X, L(G)\big)$.

We shall use the following folklore fact:

Proposition 5.1 **(Local Characterization of Lie Groups)** *Suppose that a subset U of a group G is equipped with a smooth (resp., $\mathbb{K}$-analytic) manifold structure modelled on a Hausdorff locally convex space E, and suppose that V is an open subset of U such that $e \in V$, $V = V^{-1}$, $VV \subseteq U$, and such that the multiplication map $V \times V \to U$, $(g, h) \mapsto gh$ is smooth (resp., $\mathbb{K}$-analytic) as well as inversion $V \to V$, $g \mapsto g^{-1}$; here*

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V is considered as an open submanifold of U. Suppose that for every element x in a symmetric generating set of G, there is an open identity neighbourhood $W \subseteq U$ such that $xWx^{-1} \subseteq U$, and such that the mapping $W \to U$, $w \mapsto xwx^{-1}$ is smooth (resp., $\mathbb{K}$-analytic).[3] *Then there is a unique smooth (resp., $\mathbb{K}$-analytic) Lie group structure on G which makes V, equipped with the above manifold structure, an open submanifold of G.*

Proof The proof of [8, Chapter 3, Section 1.9, Proposition 18] can easily be adapted. ■

The Lie Group Structure on $L^\infty(X, G)$

Theorem 5.2 *Let G be a smooth (resp., $\mathbb{K}$-analytic) Lie group, and (X, Σ, μ) be a measure space. Then there is a uniquely determined smooth (resp., $\mathbb{K}$-analytic) Lie group structure on the group $L^\infty(X, G)$, modelled on $L^\infty\big(X, L(G)\big)$, such that $L^\infty(X, V_1)$ is an open identity neighbourhood and*

$$\Phi := L^\infty(X, \phi)\colon L^\infty(X, V_1) \to L^\infty(X, V) \subseteq L^\infty\big(X, L(G)\big)$$

is a diffeomorphism of smooth (resp., $\mathbb{K}$-analytic) manifolds, for some chart $\phi\colon V_1 \to V$ from an open identity neighbourhood $V_1 \subseteq G$ onto an open zero-neighbourhood in $L(G)$ such that $\phi(e) = 0$. Then in fact $L^\infty(X, \psi)$ is a diffeomorphism, for every chart ψ of G whose domain is contained in V_1 and which satisfies $\psi(e) = 0$. The topological group underlying the Lie group $L^\infty(X, G)$ coincides with the one described in Definition 3.11. Furthermore, if ϕ is chosen such that $d\phi(e, \bullet) = \mathrm{id}_{L(G)}$, then

$$d\Phi(e, \bullet)\colon L\big(L^\infty(X, G)\big) \to L^\infty\big(X, L(G)\big)$$

is an isomorphism of topological Lie algebras with respect to the "pointwise" Lie bracket on $L^\infty\big(X, L(G)\big)$.

Proof The construction will be given in steps.

5.3 Let $\kappa\colon U_1 \to U$ be a chart of G, defined on an open identity neighbourhood U_1 in G, with values in an open zero-neighbourhood U in $L(G)$, such that $\kappa(e) = 0$. Let V_1 be an open, symmetric identity neighbourhood in G such that $V_1V_1 \subseteq U_1$, and set $V := \kappa(V_1)$. Then the mappings

$$m\colon V \times V \to U, \quad m(x, y) := \kappa\big(\kappa^{-1}(x) \cdot \kappa^{-1}(y)\big) \quad \text{and}$$
$$\iota\colon V \to V, \quad \iota(x) := \kappa\big(\kappa^{-1}(x)^{-1}\big)$$

are smooth, resp., $\mathbb{K}$-analytic. We equip $L^\infty(X, U_1) \subseteq L^\infty(X, G)$ with the smooth (resp., $\mathbb{K}$-analytic) manifold structure making the bijection

$$\beta\colon L^\infty(X, U_1) \to L^\infty(X, U), \quad [\gamma] \mapsto [\kappa \circ \gamma] \tag{6}$$

a diffeomorphism of smooth (resp., $\mathbb{K}$-analytic) manifolds.

[3]This condition is automatically satisfied if V generates G.

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5.4 Since $L^\infty(X,V) \times L^\infty(X,V) \cong L^\infty(X,V\times V)$ and the mapping $L^\infty(X,m)\colon L^\infty(X,V\times V) \to L^\infty(X,U)$ is smooth (resp., $\mathbb{K}$-analytic) by Corollary 4.4, we deduce that the group multiplication of $L^\infty(X,G)$ restricts to a smooth (resp., $\mathbb{K}$-analytic) mapping $L^\infty(X,V_1)\times L^\infty(X,V_1) \to L^\infty(X,U_1)$. Similarly, inversion is smooth (resp., $\mathbb{K}$-analytic) on $L^\infty(X,V_1)$.

5.5 Let $[\gamma] \in L^\infty(X,G)$ now. As $\overline{\operatorname{im}\gamma}$ is compact, there is an open identity neighbourhood $W_1 \subseteq V_1$ in G and an open neighbourhood P of $\overline{\operatorname{im}\gamma}$ in G such that $pW_1p^{-1} \subseteq U_1$ for all $p \in P$. Set $W := \kappa(W_1)$. The mapping $h\colon P\times W_1 \to U_1$, $h(p,w) := pwp^{-1}$ being smooth (resp., $\mathbb{K}$-analytic), we deduce that so is $\tilde f := \kappa\circ h\circ(\mathrm{id}_P\times\kappa^{-1})\colon P\times W \to U$. Define

$$f := \tilde f\circ(\gamma\times\mathrm{id}_W)\colon X\times W \to U,\quad f(x,y) = \kappa\bigl(\gamma(x)\kappa^{-1}(y)\gamma(x)^{-1}\bigr).$$

In the case where G is a smooth or complex analytic Lie group, we deduce from Proposition 4.1 (resp., Corollary 4.2) that the mapping $f_*\colon L^\infty(X,W)\to L^\infty(X,U)$ is smooth (if G is a smooth Lie group), resp., complex analytic (if G is a complex Lie group). Thus conjugation I_γ by γ is smooth (resp., complex analytic) on the open identity neighbourhood $L^\infty(X,W_1) \subseteq L^\infty(X,V_1)$, noting that $I_\gamma\big|^{L^\infty(X,U_1)}_{L^\infty(X,W_1)} = \beta^{-1}\circ f_*\circ\beta\big|^{L^\infty(X,W)}_{L^\infty(X,W_1)}$. If G is a real analytic Lie group, Proposition 4.3 shows that $f_*|_{L^\infty(X,Q)}$ is real analytic for some open zero-neighbourhood $Q \subseteq W$, and thus conjugation by γ is real analytic on $L^\infty\bigl(X,\kappa^{-1}(Q)\bigr) \subseteq L^\infty(X,V_1)$. In either case, Proposition 5.1 provides a unique smooth, resp., complex analytic, resp., real analytic Lie group structure on $L^\infty(X,G)$ making $L^\infty(X,V_1)$ an open submanifold. Thus $L^\infty(X,\phi)$ is a chart for $L^\infty(X,G)$, where $\phi := \kappa\big|^{V}_{V_1}\colon V_1\to V$.

5.6 The assertion concerning ψ follows from Corollary 4.4, applied to the diffeomorphism $f := \psi\circ\phi\big|^{\operatorname{dom}\psi}_{\phi^{-1}(\operatorname{dom}\psi)}$ and its inverse.

5.7 To see that the topological group underlying the Lie group $L^\infty(X,G)$ coincides with the one described in Definition 3.11, note first that the bijection β defined in (6) is a homeomorphism with respect to the topology induced on $L^\infty(X,U_1)$ by the topological group $L^\infty(X,G)$ (*cf.* proof of Proposition 4.1, Step 1). As a consequence, the group topology on $L^\infty(X,G)$ defined in Definition 3.11 and the group topology underlying the Lie group $L^\infty(X,G)$ just defined have the same filter of identity neighbourhoods, and thus coincide.

5.8 As a topological vector space, we identify the Lie algebra of $L^\infty(X,G)$ with $L^\infty\bigl(X,L(G)\bigr)$ by means of the isomorphism of topological vector spaces

$$dL^\infty(X,\phi)(e,\bullet)\colon T_e\bigl(L^\infty(X,G)\bigr) \to L^\infty\bigl(X,L(G)\bigr),$$

where we assume that ϕ is chosen such that $d\phi(e,\bullet)\colon T_e(G) = L(G)\to L(G)$ is the identity map. We have to show that, with respect to this identification, the Lie bracket on $L\big(L^\infty(X,G)\big)$ is the mapping $L^\infty(X,[\,.\,,.\,])\colon L^\infty\big(X,L(G)^2\big)\cong L^\infty\big(X,L(G)\big)^2\to L^\infty\big(X,L(G)\big)$. To see this, let $\sigma,\eta\in L^\infty\big(X,L(G)\big)$. As a consequence of Proposition 4.1 and its proof (equation (5)), $\mathrm{Ad}_\gamma(\eta):=(T_eI_\gamma).\eta\in L^\infty\big(X,L(G)\big)$ is given by $\mathrm{Ad}_\gamma(\eta)=f_*(\gamma)$ for $\gamma\in L^\infty(X,G)$, where $\widetilde{f}\colon L(G)\times G\to L(G)$, $(y,g)\mapsto \mathrm{Ad}_g(y)$ is smooth and $f:=\widetilde{f}\circ(\eta\times\mathrm{id}_G)\colon X\times G\to L(G)$. Using that $d_2\widetilde{f}(y,e,x)=d\big(\mathrm{Ad}_\bullet(y)\big)(e,x)=[x,y]$ for all $x,y\in L(G)$ by definition of the Lie bracket (see [24, pp. 1035–1037]), we find that

$$[\sigma,\eta]=d\big(\mathrm{Ad}_\bullet(\eta)\big)(e,\sigma)=d_2\widetilde{f}\circ(\eta,e,\sigma)=L^\infty(X,[\,.\,,.\,])(\sigma,\eta),$$

where the second equality is a consequence of Proposition 4.1 and its proof (equation (5)). This completes the proof of Theorem 5.2. ■

Functoriality of $L^\infty(X,\bullet)$

The following variant of Corollary 4.4 will be essential for our discussions of universal complexifications.

Proposition 5.9 *Suppose that G_1 and G_2 are smooth (resp., $\mathbb{K}$-analytic) Lie groups, and suppose that $f\colon G_1\to G_2$ is a smooth (resp., $\mathbb{K}$-analytic) mapping. Then*

$$L^\infty(X,f)\colon L^\infty(X,G_1)\to L^\infty(X,G_2),\quad [\gamma]\mapsto[f\circ\gamma]$$

is a smooth (resp., $\mathbb{K}$-analytic) mapping.

Proof For $j\in\{1,2\}$, let $\phi_j\colon U_j\to V_j$ be a diffeomorphism of smooth (resp., $\mathbb{K}$-analytic) manifolds from an open zero-neighbourhood U_j in $L(G_j)$ onto an open identity neighbourhood V_j in G_j, such that $\phi_j(0)=e$, $L^\infty(X,V_j)$ is an open identity neighbourhood in $L^\infty(X,G_j)$, and $\Phi_j:=L^\infty(X,\phi_j)\colon L^\infty(X,U_j)\to L^\infty(X,V_j)$ is a diffeomorphism of smooth (resp., $\mathbb{K}$-analytic) manifolds. The mapping

$$\tilde{g}\colon G_1\times U_1\to G_2,\quad \tilde{g}(a,u):=f(a)^{-1}f\big(a\cdot\phi_1(u)\big)$$

is smooth (resp., $\mathbb{K}$-analytic), and $\tilde{g}(a,0)=e$ for every $a\in G_1$. Suppose $\gamma\in\mathfrak{L}^\infty(X,G_1)$ is given. Then $\overline{\operatorname{im}\gamma}\times\{0\}$ being a compact subset of $G_1\times U_1$ on which $\tilde{g}=e$, we find an open neighbourhood P of $\overline{\operatorname{im}\gamma}$ in G_1 and an open zero-neighbourhood $Q\subseteq U_1$ such that $\tilde{g}(P\times Q)\subseteq V_2$. Then $\tilde{h}:=\phi_2^{-1}\circ\tilde{g}\big|_{P\times Q}^{V_2}$, $P\times Q\to U_2\subseteq L(G_2)$ is a smooth (resp., $\mathbb{K}$-analytic) mapping. We define

$$h:=\tilde{h}\circ(\gamma|^P\times\mathrm{id}_Q),\quad X\times Q\to U_2. \tag{7}$$

Thus $h(x,q)=\phi_2^{-1}\Big(f\big(\gamma(x)\big)^{-1}f\big(\gamma(x)\phi_1(q)\big)\Big)$ for all $x\in X$ and $q\in Q$.

In the *smooth or complex analytic case*, the mapping $h_*\colon L^\infty(X,Q)\to L^\infty(X,U_2)$ is smooth (resp., complex analytic), by Proposition 4.1 (resp., Corollary 4.2); we set $Q':=Q$.

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In the *real analytic case*, $h_*|_{L^\infty(X,Q')}$ is real analytic for some open zero-neighbourhood $Q' \subseteq Q$, by Proposition 4.3.

In either case, we let $\lambda_\gamma \colon L^\infty(X, G_1) \to L^\infty(X, G_1)$, $\sigma \mapsto \gamma \cdot \sigma$ denote left translation by $[\gamma]$ on $L^\infty(X, G_1)$ and $\lambda_{f\circ\gamma}$ denote left translation by $[f \circ \gamma]$ on $L^\infty(X, G_2)$. We abbreviate $V_1' := \phi_1(Q')$. Since

$$\Phi_2^{-1} \circ \left(\lambda_{f\circ\gamma}^{-1} \circ L^\infty(X, f) \circ \lambda_\gamma\right)\Big|_{L^\infty(X,V_1')}^{L^\infty(X,V_2)} \circ \Phi_1\Big|_{L^\infty(X,Q')}^{L^\infty(X,V_1')} = h_*|_{L^\infty(X,Q')}$$

is a smooth (resp., $\mathbb{K}$-analytic) mapping, it follows that $\lambda_{f\circ\gamma}^{-1} \circ L^\infty(X, f) \circ \lambda_\gamma$ is smooth (resp., $\mathbb{K}$-analytic) on the identity neighbourhood $L^\infty(X, V_1')$ of $L^\infty(X, G_1)$. Translations being diffeomorphisms, this entails that $L^\infty(X, f)$ is smooth (resp., $\mathbb{K}$-analytic) on some neighbourhood of $[\gamma]$. ∎

Remark 5.10 Applying Proposition 5.9 to smooth (resp., $\mathbb{K}$-analytic) homomorphisms in particular, we deduce that functors $L^\infty(X, \bullet)$ from the category of smooth Lie groups and smooth homomorphisms (resp., $\mathbb{K}$-analytic Lie groups and $\mathbb{K}$-analytic homomorphisms) into itself can be defined.

$L^\infty(X, G)$ When G is a BCH-Lie Group

Definition 5.11 A $\mathbb{K}$-analytic Lie group G modelled on a Hausdorff locally convex topological $\mathbb{K}$-vector space is called a *Baker-Campbell-Hausdorff Lie group* (or "BCH-Lie group" for short) if it has the following properties:

(a) The exponential function $\exp_G \colon L(G) \to G$ is defined on all of $L(G)$, and there is an open zero-neighbourhood U in $L(G)$ such that $V := \exp_G(U)$ is open in G and $\phi := \exp_G|_U^V \colon U \to V$ is a diffeomorphism of $\mathbb{K}$-analytic manifolds.

(b) There is a zero-neighbourhood $W \subseteq U$ in $L(G)$ with $\exp_G(W)\exp_G(W) \subseteq V$, such that $\phi^{-1}\big(\phi(x)\phi(y)\big) = \sum_{n=1}^\infty \beta_n(x, y) =: x * y$ is given by the BCH-series for $x, y \in W$ (with pointwise convergence).

Thus $\beta_1(x, y) = x+y$, $\beta_2(x, y) = \frac{1}{2}[x, y]$, $\beta_3(x, y) = \frac{1}{12}\big(\big[x, [x, y]\big] + \big[y, [y, x]\big]\big)$, *etc.*

Remark 5.12 The class of BCH-Lie groups includes all finite-dimensional Lie groups, Banach-Lie groups, mapping groups $C^r(K, G)$ and $C_c^\infty(M, G)$ (where K is a compact smooth manifold, M a σ-compact finite-dimensional smooth manifold, and G any BCH-Lie group) [12], as well as the direct limit Lie groups $\mathrm{GL}_\infty(\mathbb{K}) = \varinjlim_n \mathrm{GL}_n(\mathbb{K})$ and their analytic subgroups [25], [12]. Also the unit group of any sequentially complete (or, more generally, Mackey complete) continuous inverse algebra is a BCH-Lie group [11]. The general Lie theory of BCH-Lie groups (analytic subgroups, integration of Lie algebra homomorphisms, quotient groups, universal complexifications) is developed in [12]. See also the earlier paper [32] for information concerning the closely related class of CBH-Lie groups.

Theorem 5.13 *For every $\mathbb{K}$-analytic BCH-Lie group G, $L^\infty(X, G)$ is a $\mathbb{K}$-analytic BCH-Lie group, with Lie algebra $L^\infty\big(X, L(G)\big)$ and exponential function $L^\infty(X, \exp_G)$.*

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Proof Let us assume first that G is a complex BCH-Lie group.

5.14 We let U be a balanced open zero-neighbourhood in $L(G)$ such that the BCH-series converges on $U \times U$ to a complex analytic function $* := m\colon U^2 \to L(G)$, $m(x,y) := \sum_{n=1}^{\infty} \beta_n(x,y)$. Then $m(0,0) = 0$, and $M := L^\infty(X,m)\colon L^\infty(X,U^2) \to L^\infty\big(X, L(G)\big)$ is a complex analytic mapping by Corollary 4.4. Since $L^\infty(X,U)^2 \cong L^\infty(X,U^2)$ as complex analytic manifolds, we may consider M as a mapping on $L^\infty(X,U)^2$. As $P := L^\infty(X,U)^2$ is a balanced open zero-neighbourhood in $L^\infty\big(X,L(G)\big)^2$ and M is complex analytic on P, there is a sequence of continuous homogeneous polynomials $\alpha_n\colon L^\infty\big(X,L(G)\big)^2 \to L^\infty\big(X,L(G)\big)$ such that $M(\gamma,\eta) = \sum_{n=0}^{\infty} \alpha_n(\gamma,\eta)$ for all $\gamma,\eta \in P$ (as follows from [5, Proposition 5.5]); here $\alpha_0 = 0$ as $M(0,0) = 0$. Of course, using the notation introduced in Subsection 1.2, we have $\alpha_n = \frac{1}{n!}\delta^{(n)}_{(0,0)} M$ and $\beta_n = \frac{1}{n!}\delta^{(n)}_{(0,0)} m$ (see [5]). Inductively, we deduce from Corollary 4.4 that $d^n M = d^n L^\infty(X,m) = L(X,d^n m)$ for all $n \in \mathbb{N}_0$. As a consequence, $d^{(n)}M = L^\infty(X,d^{(n)}m)$ (*cf.* [10, Lemma 1.14]) and hence $\delta^{(n)}_{(0,0)}M = L^\infty(X,\delta^{(n)}_{(0,0)}m)$. Thus $\alpha_n(\gamma,\eta) = \frac{1}{n!}\delta^{(n)}_{(0,0)}M = L^\infty(X,\beta_n)(\gamma,\eta)$ is in fact the homogeneous term of order n in the BCH-series, evaluated at (γ,η).

5.15 In view of the connectedness of U, it follows from the Identity Theorems for analytic functions (*cf.* [5, Proposition 6.6]) that $\exp_G(x * y) = \exp_G(x)\exp_G(y)$ for all $x, y \in U$. Shrinking U if necessary, we may assume that $U_1 := \exp_G(U)$ is open in G, and that $\exp_G\big|_U^{U_1} =: \kappa^{-1}$ is a diffeomorphism of complex analytic manifolds. We let $V \subseteq U$ be any open symmetric zero-neighbourhood in $L(G)$ such that $V * V \subseteq U$ and set $V_1 := \exp_G(V)$. Then U, U_1, V, V_1, and κ can be used in Step 5.3 of the construction of the manifold structure on $L^\infty(X,G)$. We deduce that $L^\infty(X,\kappa|_{V_1}^{V})$ is a diffeomorphism of $\mathbb{K}$-analytic manifolds, and so is its inverse $L^\infty(X,\exp_G)\big|_{L^\infty(X,V)}^{L^\infty(X,V_1)}$. It now easily follows that $L^\infty(X,G)$ is a complex BCH-Lie group with the asserted properties.

Now assume that G is a real BCH-Lie group.

5.16 Since G is a real BCH-Lie group, there is an open balanced zero-neighbourhood W in $L(G)_{\mathbb{C}}$ such that the BCH-series converges to a complex analytic map $\widetilde{m}\colon W \times W \to L(G)_{\mathbb{C}}$. As in Subsection 5.14, we see that the mapping

$$L^\infty(X,W)^2 \cong L^\infty(X,W^2) \to L^\infty\big(X, L(G)_{\mathbb{C}}\big), \quad (\gamma,\eta) \mapsto \widetilde{m} \circ (\gamma,\eta)$$

is complex analytic, and is given by the BCH-series.

5.17 We let $U \subseteq W$ be an open zero-neighbourhood in $L(G)$ such that $(\exp_G|_U^{U_1}) =: \kappa^{-1}$ is a diffeomorphism of real analytic manifolds onto an open subset U_1 of G, and let $V \subseteq U$ be a balanced open zero-neighbourhood in $L(G)$ such that $\tilde{m}(V \times V) \subseteq U$. Then $m := \tilde{m}\big|_{V \times V}^{L(G)}$ is a real analytic function, which is the limit of the BCH-series of

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$L(G)$ on $V \times V$. We have $V_1V_1 \subseteq U_1$ for $V_1 := \exp_G(V)$, and $\phi\big(\exp_G(x)\exp_G(y)\big) = m(x,y)$ for $x, y \in V$. In view of the considerations in Subsection 5.16, the mapping $L^\infty(X,V)^2 \to L^\infty\big(X, L(G)\big)$, $(\gamma,\eta) \mapsto m \circ (\gamma,\eta)$ is real analytic and is the limit of the Campbell-Hausdorff series of $L^\infty\big(X, L(G)\big)$ on $L^\infty(X,V)^2$. As we may use U, U_1, V, V_1, and κ in Subsection 5.3, we deduce as in Subsection 5.15 that $L^\infty(X,G)$ is a real BCH-Lie group with the asserted properties. This completes the proof of Theorem 5.13.

The Universal Complexification of $L^\infty(X,G)$

Definition 5.18 Let G be a real BCH-Lie group, H be a complex BCH-Lie group, and $\phi\colon G \to H$ a smooth homomorphism. We say that *H has a polar decomposition with respect to ϕ* if the map $\Phi\colon G \times L(G) \to H$, $\Phi(g,x) := \phi(g)\exp_H\big(iL(\phi).x\big)$ is a diffeomorphism of smooth manifolds. We call Φ the *polar map* in this case. If G and ϕ are understood, we simply say that H has a polar decomposition.

If H has a polar decomposition with respect to $\phi\colon G \to H$, then (H,ϕ) is a universal complexification of G in the category of all complex Lie groups with complex analytic exponential functions, *i.e.*, for every smooth homomorphism $f\colon G \to S$ from G into a complex Lie group S whose exponential function is defined on all of $L(S)$ and is complex analytic, there exists a unique complex analytic homomorphism $\widetilde{f}\colon H \to S$ such that $\widetilde{f} \circ \phi = f$ [12, Theorem 8.8]. We deduce:

Proposition 5.19 *Let G be a real BCH-Lie group whose universal complexification $G_{\mathbb{C}}$ in the category of complex BCH-Lie groups exists and has a polar decomposition with respect to the universal smooth homomorphism $\gamma_G\colon G \to G_{\mathbb{C}}$. Let (X,Σ,μ) be any measure space. Then $L^\infty(X,G_{\mathbb{C}})$ has a polar decomposition with respect to $L^\infty(X,\gamma_G)\colon L^\infty(X,G) \to L^\infty(X,G_{\mathbb{C}})$, and*

$$L^\infty(X,G)_{\mathbb{C}} = L^\infty(X,G_{\mathbb{C}})$$

in the category of all complex Lie groups with complex analytic exponential functions.

Proof Noting that $L^\infty(X,G) \times L^\infty\big(X,L(G)\big) \cong L^\infty\big(X, G \times L(G)\big)$ (as a consequence of Proposition 5.9), the first assertion follows from the observation that $L^\infty(X,\Phi)\colon L^\infty\big(X, G\times L(G)\big) \to L^\infty(M, G_{\mathbb{C}})$ is a diffeomorphism of smooth manifolds by Proposition 5.9, where $\Phi\colon G \times L(G) \to G_{\mathbb{C}}$ is the polar map. The remainder follows from [12, Theorem 8.8]. ∎

6 The Lie Groups $\ell^\infty(X,G)$ and $\widetilde{\ell^\infty}(X,G)$

Except for metrizable, complete topological groups G, we could not say much about completeness properties of $L^\infty(X,G)$ in Section 3, for general measure spaces (X,Σ,μ). As we shall see in this section, much more information is available for $\ell^\infty(X,G)$. We shall also define a certain topological group $\widetilde{\ell^\infty}(X,G)$ containing $\ell^\infty(X,G)$ as a dense subgroup, and equip it with a Lie group structure when G is a Lie group.

6.1 Given a Hausdorff topological group G, we let $\ell^\infty(X,G) := \mathfrak{L}^\infty(X,G) \cong L^\infty(X,G)$, using the counting measure μ on $\bigl(X, \mathcal{P}(X)\bigr)$, where $\mathcal{P}(X)$ denotes the power set of X. We define a larger group $\widetilde{\ell^\infty}(X,G) := \{\gamma \in G^X : \overline{\operatorname{im}\gamma} \text{ is compact}\}$ by dropping the metrizability condition on $\overline{\operatorname{im}\gamma}$ (the group operations are pointwise). Then $\widetilde{\ell^\infty}(X,G)$ is a topological group with respect to the topology of uniform convergence, which contains $\ell^\infty(X,G)$ as a topological subgroup.

6.2 Every function on $\bigl(X, \mathcal{P}(X)\bigr)$ being measurable, apparently Proposition 4.1, Corollary 4.2, Proposition 4.3, and Corollary 4.4 remain valid when $L^\infty(X,E)$ and $L^\infty(X,F)$ are replaced with $\widetilde{\ell^\infty}(X,E)$ and $\widetilde{\ell^\infty}(X,F)$. Therefore Theorem 5.2 (and all of the results of Section 5) remain valid when $L^\infty(X,G)$ is replaced with $\widetilde{\ell^\infty}(X,G)$. In particular, we obtain a natural smooth (resp., $\mathbb{K}$-analytic) Lie group structure on $\widetilde{\ell^\infty}(X,G)$, for every smooth (resp., $\mathbb{K}$-analytic) Lie group G.

Proposition 6.3 *Let G be a Hausdorff topological group, E a Hausdorff real locally convex space.*

(a) *Let (X,Σ) be a measurable space such that $\{x\} \in \Sigma$ for all $x \in X$, and let μ be counting measure on (X,Σ). Then $\mathfrak{L}^\infty(X,G) \cong L^\infty(X,G)$. If every closed pre-compact subset of G is compact and metrizable, then $L^\infty(X,G)$ is sequentially complete.*

(b) *If G is complete, then so is $\widetilde{\ell^\infty}(X,G)$. If every closed, pre-compact subset of G is compact, then $\widetilde{\ell}(X,G)$ is sequentially complete. If E is quasi-complete, then so is $\widetilde{\ell^\infty}(X,E)$.*

(c) *If every compact subset of G is metrizable, then $\ell^\infty(X,G) = \widetilde{\ell^\infty}(X,G)$.*

Proof (a) The first assertion is obvious. To prove the second, let $(\gamma_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathfrak{L}^\infty(X,G)$. Then $\bigl(\gamma_n(x)\bigr)_{n\in\mathbb{N}}$ is a Cauchy sequence in G and thus converges to some $\gamma(x) \in G$, the set $\overline{\{\gamma_n(x) : n \in \mathbb{N}\}}$ being closed and pre-compact and thus compact, by hypothesis. Then $\gamma_n \to \gamma$ uniformly. By Proposition 3.18, $\operatorname{im}\gamma$ is pre-compact. Now $\overline{\operatorname{im}\gamma}$ being closed and pre-compact and thus compact and metrizable by hypothesis, Proposition 3.18 shows that $\gamma \in \mathfrak{L}^\infty(X,G)$.

(b) The assertion concerning sequential completeness can be shown along the lines of (a). Now suppose that G is a complete Hausdorff group (resp., a quasi-complete Hausdorff locally convex space). Let (γ_α) be a Cauchy net (resp., bounded Cauchy net) in $\widetilde{\ell^\infty}(X,G)$. Given $x \in X$, $\bigl(\gamma_\alpha(x)\bigr)$ is a Cauchy net (resp., bounded Cauchy net) in G and thus convergent to some $\gamma(x) \in G$. As in the proof of Proposition 3.18, we find that $\operatorname{im}\gamma$ is pre-compact. Hence $\overline{\operatorname{im}\gamma}$ is a closed pre-compact subset of G and therefore a complete, pre-compact uniform space and thus compact, as G is complete (resp., quasi-complete). Thus $\gamma \in \widetilde{\ell^\infty}(X,G)$, and apparently $\gamma_\alpha \to \gamma$ uniformly.

(c) is obvious from the definitions. ■

Lemma 6.4 *For every $\gamma \in \widetilde{\ell^\infty}(X,G)$, there exists a net (γ_α) in $\mathcal{F}(X,G)$ converging uniformly to γ, such that* $\operatorname{im}\gamma_\alpha \subseteq \operatorname{im}\gamma$ *for all α.*

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Proof Given an open identity neighbourhood U in G, we have $\operatorname{im}\gamma \subseteq \bigcup_{j=1}^{n} v_j U$ for some finite sequence $v_1, \ldots, v_n \in \operatorname{im}\gamma$. Set $X_j := \gamma^{-1}(v_j U) \setminus \bigcup_{i=1}^{j-1} \gamma^{-1}(v_i U)$ for $j = 1, \ldots, n$; then $X = \bigcup_{j=1}^{n} X_j$ as a disjoint union. We define $\gamma_U \in \mathcal{F}(X, G)$ via $\gamma_U(x) := v_j$ for $x \in X_j$. Then $(\gamma_U)_{U \in \mathcal{U}_e(G)}$ is a net with the required properties. ■

Thus $\ell^\infty(X, G)$ is non-complete whenever $\ell^\infty(X, G)$ is a proper subgroup of $\widetilde{\ell^\infty}(X, G)$. If $\ell^\infty(X, E)$ is a proper subspace of $\widetilde{\ell^\infty}(X, E)$, then $\ell^\infty(X, E)$ is not quasi-complete.

Example 6.5 Let J be a set of cardinality $\operatorname{card}(J) > 2^{\aleph_0}$. Then $X := [-1, 1]^J$ is a non-separable compact topological space in the product topology. *We claim that* $\ell^\infty(X, \mathbb{R}^J)$ *is not quasi-complete* (although $\mathbb{R}^J$ is complete). To see this, let $\gamma\colon X \hookrightarrow \mathbb{R}^J$ be the inclusion map. Then $\gamma \in \widetilde{\ell^\infty}(X, \mathbb{R}^J)$, but $\gamma \notin \ell^\infty(X, \mathbb{R}^J)$ as $\overline{\operatorname{im}\gamma} = \operatorname{im}\gamma = X$ is not separable. By Lemma 6.4, there is a net (γ_α) of functions $\gamma_\alpha \in \mathcal{F}(X, \mathbb{R}^J)$ converging uniformly to γ such that $\operatorname{im}\gamma_\alpha \subseteq \operatorname{im}\gamma = X$ for all α. Thus (γ_α) is a bounded Cauchy net in $\mathcal{F}(X, \mathbb{R}^J) \subseteq \ell^\infty(X, \mathbb{R}^J)$, which cannot converge in $\ell^\infty(X, \mathbb{R}^J)$ as $\gamma \notin \ell^\infty(X, \mathbb{R}^J)$.

7 Weak Direct Products of Lie Groups

In this section, we show that the "weak direct product"

$$\prod_{i\in I}^{*} G_i := \Big\{ (g_i)_{i\in I} \subseteq \prod_{i\in I} G_i : g_i = e \text{ for } i \text{ off some finite subset of } I \Big\}$$

of an arbitrary family $(G_i)_{i\in I}$ of smooth (resp., $\mathbb{K}$-analytic) Lie groups can be made a smooth (resp, $\mathbb{K}$-analytic) Lie group modelled on the locally convex direct sum $\bigoplus_{i\in I} L(G_i)$.

Mappings Between Locally Convex Direct Sums

Proposition 7.1 *Let $(E_i)_{i\in I}$ and $(F_i)_{i\in I}$ be families of Hausdorff locally convex spaces, with locally convex direct sums $E := \bigoplus_{i\in I} E_i$ and $F := \bigoplus_{i\in I} F_i$. Suppose that $k \in \mathbb{N}_0 \cup \{\infty\}$, and suppose that $f_i\colon U_i \to F_i$ is a mapping on an open zero-neighbourhood U_i of E_i for $i \in I$, such that $f_i(0) = 0$. If I is countable, we assume that each f_i is of class C^k; if I is uncountable, we assume that each f_i is of class C^{k+1}. Then $U := \bigoplus_{i\in I} U_i := E \cap \prod_{i\in I} U_i$ is an open subset of E, and*

$$f := \bigoplus_{i\in I} f_i := \Big(\prod_{i\in I} f_i\Big)\Big|_U^F : U \to F, \quad \sum_{i\in I} v_i \mapsto \sum_{i\in I} f_i(v_i)$$

is mapping of class C^k.

Proof It suffices to prove the assertion for $k \in \mathbb{N}_0$; first we assume that I is uncountable.

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Step 1: U is open, and f is continuous. To see that U is a neighbourhood of v and f is continuous at $v = \sum_{i\in I} v_i \in U$, it suffices to show that $U - v$ is a zero-neighbourhood and that $g := f(v+\cdot) - f(v)\colon U - v \to F$ is continuous at 0. Here $U - v = \bigoplus_{i\in I}(U_i - v_i)$ and $g = \bigoplus_{i\in I} g_i$ with $g_i = f_i(v_i+\cdot) - f_i(v_i)$, a function built up in the same way as f. Hence without loss of generality $v = 0$.

Each U_i contains some convex symmetric zero-neighbourhood C_i; then $\operatorname{conv}\bigcup_{i\in I} C_i \subseteq U$ is a zero-neighbourhood in the locally convex direct sum E.

Given a convex, symmetric, open zero-neighbourhood Q in F, we have $\operatorname{conv}\bigcup_{i\in I} Q_i \subseteq Q$, where $Q_i := Q \cap F_i$ for $i \in I$, which is a convex, symmetric, open zero-neighbourhood in F_i. Since $df_i(0,0) = 0$ and df_i is continuous, there is an open, convex, symmetric zero-neighbourhood $P_i \subseteq U_i$ such that $df_i(P_i \times P_i) \subseteq Q_i$. Thus, for all $u \in P_i$ and $t \in [0,1]$, noting that $f_i(0) = 0$:

$$f_i(tu) = f_i(0) + t\int_0^1 df_i(stu, u)\, ds \in tQ_i. \tag{8}$$

Equation (8) entails that $f(\operatorname{conv}\bigcup_{i\in I} P_i) \subseteq \operatorname{conv}\bigcup_{i\in I} Q_i \subseteq Q$; it only remains to note that $\operatorname{conv}\bigcup_{i\in I} P_i$ is a zero-neighbourhood in the locally convex direct sum E.

Step 2: f is of class C^1 (when $k \geq 2$). In fact, given $u \in U$ and $v \in E$, we have $u, v \in \bigoplus_{i\in J} E_i = \prod_{i\in J} E_i$ for some finite subset $J \subseteq I$. The mapping $\prod_{i\in J} f_i$ being of class C^1, we deduce that $df(u,v) = \lim_{t\to 0} t^{-1}\big(f(u+tv) - f(u)\big)$ exists in $\prod_{i\in J} F_i$ and thus in F; its i-coordinate is $df_i(u_i, v_i)$. Thus

$$df = \bigoplus_{i\in I} df_i, \tag{9}$$

identifying $E \times E = (\bigoplus_{i\in I} E_i)^2$ with $\bigoplus_{i\in I}(E_i \times E_i)$ in the natural way. As each df_i is a mapping of class C^k (where $k \geq 1$), in view of equation (9), df is continuous by Step 1.

Step 3: Induction. Suppose that the proposition holds for k replaced with $k-1$, and suppose that each f_i is of class C^{k+1}, where $k \geq 1$. By Step 2, f is of class C^1, with $df = \bigoplus_{i\in I} df_i$. In view of the latter formula, df is of class C^{k-1} by induction. Thus f is of class C^k.

The Case of Countable I. The assertion being trivial when I is finite, we may assume that $I = \mathbb{N}$. Let us show that f is continuous when f_n is so for each $n \in \mathbb{N}$. We only need to prove continuity at zero (see Step 1). Given an open, convex, symmetric zero-neighbourhood Q in F, set $Q_n := Q \cap F_n$ and $T_n := 2^{-n}Q_n$. By continuity, for each $n \in \mathbb{N}$ there is an open, convex, symmetric zero-neighbourhood $P_n \subseteq U_n$ such that $f_n(P_n) \subseteq T_n$. Then $P := \bigoplus_{n\in\mathbb{N}} P_n$ is an open zero-neighbourhood in E, and $f(P) \subseteq \bigoplus_{n\in\mathbb{N}} T_n \subseteq Q$. Thus f is continuous at 0. To complete the proof, we argue as in Steps 2 and 3 above. ■

Corollary 7.2 *In the situation of Proposition 7.1, suppose that E_i and F_i are Hausdorff locally convex $\mathbb{K}$-vector spaces for each $i \in I$, and suppose that f_i is $\mathbb{K}$-analytic. Then $f := \bigoplus_{i\in I} f_i$ is $\mathbb{K}$-analytic.*

Proof *The case* $\mathbb{K} = \mathbb{C}$. By Proposition 7.1 and its proof, f is smooth and $df = \bigoplus_{i \in I} df_i$, whence $df(x, \bullet)$ is complex linear for each $x \in \bigoplus_{i \in I} U_i$. Thus f is complex analytic.

The case $\mathbb{K} = \mathbb{R}$. For each $i \in I$, there is a complex analytic mapping $g_i \colon V_i \to (F_i)_{\mathbb{C}}$ extending f_i, defined on some open neighbourhood V_i of U_i in $(E_i)_{\mathbb{C}}$. Then $\bigoplus_{i \in I} g_i$ is complex analytic by the preceding, and extends f. ■

Lie Group Structure on Weak Direct Products

Proposition 7.3 *Let $(G_i)_{i \in I}$ be a family of smooth (resp., $\mathbb{K}$-analytic) Lie groups. Then there exists a uniquely determined smooth (resp., $\mathbb{K}$-analytic) Lie group structure on $\prod^*_{i \in I} G_i$, modelled on the locally convex direct sum $\bigoplus_{i \in I} L(G_i)$, such that, for some charts $\phi_i \colon R_i \to S_i \subseteq L(G_i)$ of G_i taking e to 0, the mapping*

$$\bigoplus_{i \in I} S_i \to \prod_{i \in I}^{*} G_i, \quad (x_i)_{i \in I} \mapsto \big(\phi_i^{-1}(x_i)\big)_{i \in I} \tag{10}$$

is a diffeomorphism of smooth (resp., $\mathbb{K}$-analytic) manifolds onto an open subset of $\prod^*_{i \in I} G_i$. If each G_i is a $\mathbb{K}$-analytic BCH-Lie group, then so is $\prod^*_{i \in I} G_i$.

Proof For each $i \in I$, let $\kappa_i \colon U_i \to V_i$ be a chart of G_i about the identity element, where V_i is an open subset of $L(G_i)$ and $\kappa_i(e) = 0$. Let $R_i \subseteq U_i$ be an open, symmetric identity neighbourhood such that $R_i R_i \subseteq U_i$; define $S_i := \kappa_i(R_i)$ and $\phi_i := \kappa_i\big|_{R_i}^{S_i}$. Proceeding similarly as in the proof of Theorem 5.2 (but using Proposition 7.1 and Corollary 7.2 instead of Proposition 4.1 and its consequences), we deduce with Proposition 5.1 that there is a unique smooth (resp., $\mathbb{K}$-analytic) Lie group structure on the group $\prod^*_{i \in I} G_i$ making the mapping described in (10) a diffeomorphism of smooth (resp., $\mathbb{K}$-analytic) manifolds onto an open subset of $\prod^*_{i \in I} G_i$. Also the remaining assertions follow along similar lines. ■

Proposition 7.4 *Let $(G_i)_{i \in I}$ be a family of $\mathbb{K}$-analytic BCH-Lie groups (resp., $\mathbb{K}$-analytic Lie groups G_i with $\mathbb{K}$-analytic globally defined exponential maps inducing a local diffeomorphism of $\mathbb{K}$-analytic manifolds on some zero-neighbourhood; resp., smooth Lie groups G_i with globally defined exponential maps inducing a local C^∞-diffeomorphism on some zero-neighbourhood). Then*

$$\prod_{i \in I}^{*} G_i = \varinjlim_{F} \prod_{i \in F} G_i \tag{11}$$

in the respective category of Lie groups (where F ranges through the set of finite subsets of I, directed via inclusion). Furthermore, (11) holds in the category of $\mathbb{K}$-analytic Lie groups with globally defined, $\mathbb{K}$-analytic exponential functions (in the first and second case) and in the category of smooth Lie groups with globally defined, smooth exponential functions (in all cases).

Proof Suppose that each G_i is a $\mathbb{K}$-analytic BCH-Lie group. Given a finite subset F of I, we identify $G_F := \prod_{i\in F} G_i$ with the subgroup $\{(g_i)_{i\in I} : g_i = e \text{ for } i \in I \setminus F\}$ of $G := \prod^*_{i\in I} G_i$. If H is a $\mathbb{K}$-analytic BCH-Lie group and $\phi_F \colon G_F \to H$ a $\mathbb{K}$-analytic homomorphism for each F such that $\phi_F|_{G_{F_1}} = \phi_{F_1}$ whenever $F_1 \subseteq F$, there is a unique homomorphism $\phi\colon G \to H$ such that $\phi|_{G_F} = \phi_F$ for each F, since $G = \varinjlim G_F$ as an abstract group. As $L(G) = \bigoplus_{i\in I} L(G_i) = \varinjlim_F \prod_{i\in F} L(G_i) = \varinjlim L(G_F)$ in the category of locally convex spaces, there is a unique continuous linear map $\psi\colon L(G) \to L(H)$ such that $\psi|_{L(G_F)} = L(\phi_F)$ for all F. The map $\exp_G$ inducing a local diffeomorphism at 0 and $\exp_H \circ\psi$ being $\mathbb{K}$-analytic, we deduce from $\phi \circ \exp_G = \exp_H \circ\psi$ that the homomorphism ϕ is $\mathbb{K}$-analytic on some identity-neighbourhood and thus $\mathbb{K}$-analytic. The other assertions can be proved similarly. ■

Remark 7.5 See [25]–[28], [12] and [13] for information on direct limits of Lie groups. Some intricacies inherent to the subject are explained in [13], [14], and [36]; *cf.* also [23, Example 10.8]. For direct limit properties of countable weak direct products of arbitrary Lie groups, see [16].

Exploiting Proposition 7.1, the following result can be obtained; we omit the details, which closely resemble the proof of Proposition 5.19:

Proposition 7.6 *Let $(G_i)_{i\in I}$ be a family of real BCH-Lie groups such that $(G_i)_{\mathbb{C}}$ exists in the category of complex BCH-Lie groups and has a polar decomposition, for each $i \in I$. Then the complex BCH-Lie group $\prod^*_{i\in I}(G_i)_{\mathbb{C}}$ is the universal complexification of $\prod^*_{i\in I} G_i$ in the category of all complex Lie groups with complex analytic exponential functions, and it has a polar decomposition.* ■

8 The Lie Group $L^\infty_c(X, G)$

Let X be a *hemi-compact* Hausdorff topological space now (*viz.*, there exists a sequence $K_1 \subseteq K_2 \subseteq \cdots$ of compact subsets of X such that every compact subset of X is contained in some K_n). For example, X might be any σ-compact, locally compact space. Let $\Sigma := \mathcal{B}(X)$ be the Borel σ-algebra of X, and μ a measure on Σ. Given a Hausdorff topological group G and compact (or, more generally, relatively compact, measurable) subset K of X, we define $\mathfrak{L}^\infty_K(X, G) := \{\gamma \in \mathfrak{L}^\infty(X, G) : \gamma|_{G\setminus K} = e\}$, which is a subgroup of $\mathfrak{L}^\infty(X, G)$. We set $L^\infty_K(X, G) := \{[\gamma] : \gamma \in \mathfrak{L}^\infty_K(X, G)\} \subseteq L^\infty(X, G)$, and equip this subgroup of $L^\infty(X, G)$ with the induced topology. Then, in an obvious way, $L^\infty_K(X, G) \cong L^\infty(K, G)$ as a topological group, using the measure $\mu|_{\mathcal{B}(K)}$ on K. When G is a smooth (resp., $\mathbb{K}$-analytic) Lie group, we use the preceding identification to make $L^\infty_K(X, G)$ a smooth (resp., $\mathbb{K}$-analytic) Lie group (isomorphic to $L^\infty(K, G)$). It is the goal of this section to equip the group

$$L^\infty_c(X, G) := \bigcup_K L^\infty_K(X, G) \subseteq L^\infty(X, G)$$

with a natural smooth (resp., $\mathbb{K}$-analytic) Lie group structure (where K ranges through the set $\mathcal{K}(X)$ of compact subsets of X).

The Spaces $L_c^\infty(X,E)$

Let $\mathcal{R}(X)$ denote the set of Borel measurable, relatively compact subsets of X. Then $\mathcal{R}(X)$ is directed under inclusion of sets, and contains $\mathcal{K}(X)$ as a co-final subset.

Given a Hausdorff locally convex $\mathbb{K}$-vector space E, the set $L_K^\infty(X,E)$ is a vector subspace of $L^\infty(X,E)$, for each $K \in \mathcal{R}(X)$. We equip

$$L_c^\infty(X,E) = \bigcup_{K\in\mathcal{R}(X)} L_K^\infty(X,E) = \bigcup_{K\in\mathcal{K}(X)} L_K^\infty(X,E)$$

with the locally convex direct limit topology. Choose an ascending sequence $K_1 \subseteq K_2 \subseteq \cdots$ of compact subsets of X which is co-final in $\mathcal{K}(X)$. Then $\{K_n : n \in \mathbb{N}\}$ is a co-final subset of $\mathcal{R}(X)$ and thus $L_c^\infty(X,E) = \varinjlim L_{K_n}^\infty(X,E)$ as a locally convex space. Set $R_1 := K_1$, $R_n := K_n \setminus K_{n-1}$ for $2 \le n \in \mathbb{N}$. Then $K_n = \bigcup_{j=1}^n R_j$ as a disjoint union, for each $n \in \mathbb{N}$, and thus $L_{K_n}^\infty(X,E) \cong \bigoplus_{j=1}^n L^\infty(R_j,E)$. Thus

$$L_c^\infty(X,E) = \varinjlim_{n\in\mathbb{N}} \bigoplus_{j=1}^n L^\infty(R_j,E) = \bigoplus_{n\in\mathbb{N}} L^\infty(R_n,E)$$

as a locally convex space.

The Lie Group Structure on $L_c^\infty(X,G)$

Along the preceding lines, we see that $L_c^\infty(X,G) = \prod^*_{n\in\mathbb{N}} L^\infty(R_n,G)$ as an abstract group, for every topological group G. Hence, if G is a smooth or $\mathbb{K}$-analytic Lie group, Proposition 7.3 provides a smooth (resp., $\mathbb{K}$-analytic) Lie group structure on $L_c^\infty(X,G) = \prod^*_{n\in\mathbb{N}} L^\infty(R_n,G)$. Using the cited proposition and Proposition 7.4, we obtain:

Proposition 8.1 *Let X be a hemi-compact Hausdorff space, μ be a measure on $\big(X,\mathcal{B}(X)\big)$, and G be a smooth (resp., $\mathbb{K}$-analytic) Lie group. Then there is a unique smooth (resp., $\mathbb{K}$-analytic) Lie group structure on $L_c^\infty(X,G)$ modelled on the locally convex direct limit $L_c^\infty\big(X,L(G)\big) = \varinjlim_{K\in\mathcal{K}(X)} L_K^\infty\big(X,L(G)\big)$ such that*

$$L_c^\infty(X,\phi^{-1})\colon L_c^\infty(X,V) \to L_c^\infty(X,G),\ [\gamma] \mapsto [\phi^{-1}\circ\gamma]$$

is an isomorphism of smooth (resp., $\mathbb{K}$-analytic) manifolds onto an open subset of $L_c^\infty(X,G)$, for some chart $\phi\colon U \to V \subseteq L(G)$ of G such that $e \in U$ and $\phi(e) = 0$. If G is a $\mathbb{K}$-analytic BCH-Lie group, then so is $L_c^\infty(X,G)$, and

$$L_c^\infty(X,G) = \varinjlim_{K\in\mathcal{K}(X)} L^\infty(K,G)$$

holds in the category of $\mathbb{K}$-analytic BCH-Lie groups, as well as in the category of $\mathbb{K}$-analytic Lie groups with $\mathbb{K}$-analytic exponential functions, and in the category of smooth Lie groups with smooth exponential functions. ■

Here $L_c^\infty(X,V) := L_c^\infty\big(X, L(G)\big) \cap L^\infty(X,V)$.

Remark 8.2 If μ is inner regular or X is second countable, then the "essential support" ess $\operatorname{supp}_\mu(\gamma)$ of a measurable function $\gamma\colon X \to G$ can be defined as the complement of the largest open subset U of X such that $\gamma(x) = e$ for μ-almost all $x \in U$. In this case, we may interpret $L_c^\infty(X,G)$ as the group of equivalence classes of $\mathfrak{L}^\infty(X,G)$-functions with compact essential support.

As an immediate consequence of Proposition 5.19 and Proposition 7.6, we obtain:

Proposition 8.3 *Suppose that X is a hemi-compact Hausdorff topological space, μ a measure on $\big(X, \mathcal{B}(X)\big)$, and G a real BCH-Lie group such that $G_{\mathbb{C}}$ exists in the category of complex BCH-Lie groups and has a polar decomposition. Then $L_c^\infty(X,G)_{\mathbb{C}} = L_c^\infty(X,G_{\mathbb{C}})$ in the category of complex Lie groups with complex analytic exponential functions, and the latter group has a polar decomposition.* ■

References

[1] S. Albeverio, R. J. Høegh-Krohn, J. A. Marion, D. H. Testard and B. S. Torrésani, *Noncommutative Distributions*. Marcel Dekker, 1993.

[2] A. Bastiani, *Applications différentiables et variétés différentiables de dimension infinie*. J. Analyse Math. **13**(1964), 1–114.

[3] H. Bauer, *Maß- und Integrationstheorie*. 2nd edition, de Gruyter, 1992.

[4] A. Bloch, M. O. El Hadrami, H. Flaschka and T. S. Ratiu, *Maximal tori of some symplectomorphism groups and applications to convexity*. In: Deformation Theory and Symplectic Geometry (Ascona, 1996), Math. Phys. Stud. **20**, Kluwer, Dordrecht, 1997, 201–222.

[5] J. Bochnak and J. Siciak, *Analytic functions in topological vector spaces*. Studia Math. **39**(1971), 77–112.

[6] H. Boseck, G. Czichowski and K.-P. Rudolph, *Analysis on Topological Groups – General Lie Theory*. Teubner, Leipzig, 1981.

[7] N. Bourbaki, *Topological Vector Spaces, Chapters 1–5*. Springer-Verlag, 1987.

[8] ———, *Lie Groups and Lie Algebras, Chapters 1–3*. Springer-Verlag, 1989.

[9] R. Engelking, *General Topology*. Heldermann-Verlag, Berlin, 1989.

[10] H. Glöckner, *Infinite-dimensional Lie groups without completeness restrictions*. In: Geometry and Analysis on Finite- and Infinite-Dimensional Lie Groups (eds. A. Strasburger *et al.*), Banach Center Publications, Vol. **55**, Warsaw, 2002, 43–59.

[11] ———, *Algebras whose groups of units are Lie groups*. Studia Math. **153**(2002), 147–177.

[12] ———, *Lie group structures on quotient groups and universal complexifications for infinite-dimensional Lie groups*. J. Funct. Anal. **194**(2002), 347–409.

[13] ———, *Direct limit Lie groups and manifolds*. Math. J. Kyoto Univ. **43**(2003), in print.

[14] ———, *Discontinuous non-linear mappings on locally convex direct limits*. Submitted.

[15] ———, *Patched locally convex spaces, almost local mappings, and diffeomorphism groups of non-compact manifolds*. In preparation.

[16] ———, *Lie groups over non-discrete topological fields*. In preparation.

[17] R. Hamilton, *The inverse function theorem of Nash and Moser*. Bull. Amer. Math. Soc. **7**(1982), 65–222.

[18] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis I*. Springer-Verlag, 1979.

[19] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*. Amer. Math. Soc., Providence, 1957.

[20] N. Kamran and T. Robart, *On the parametrization problem of Lie pseudogroups of infinite type*. C. R. Acad. Sci. Paris Sér. I Math. **331**(2000), 899–903.

[21] ———, *A manifold structure for analytic isotropy Lie pseudogroups of infinite type*. J. Lie Theory **11**(2001), 57–80.

[22] H. H. Keller, *Differential Calculus in Locally Convex Spaces*. Springer-Verlag, Berlin, 1974.

[23] A. Kriegl and P. W. Michor, *The Convenient Setting of Global Analysis*. Amer. Math. Soc., Providence R. I., 1997.

[24] J. Milnor, *Remarks on infinite dimensional Lie groups*. In: Relativity, Groups and Topology II (eds. B. DeWitt and R. Stora), North-Holland, 1983, 1008–1057.
[25] L. Natarajan, E. Rodríguez-Carrington and J. A. Wolf, *Differentiable structure for direct limit groups*. Lett. Math. Phys. **23**(1991), 99–109.
[26] ———, *Locally convex Lie groups*. Nova J. Algebra Geom. (1) **2**(1993), 59–87.
[27] ———, *New classes of infinite-dimensional Lie groups.* In: Algebraic Groups and their Generalizations, Proc. Sympos. Pure Math. **56**, Part II, 1994, 377–392.
[28] ———, *The Bott-Borel-Weil Theorem for direct limit groups.* Trans. Amer. Math. Soc. **353**(2001), 4583–4622.
[29] K.-H. Neeb, *Infinite-dimensional groups and their representations.* In: Infinite-dimensional Kähler manifolds (eds. A. T. Huckleberry and T. Wurzbacher), Birkhäuser Verlag, 2001, 131–178.
[30] ———, *Central extensions of infinite-dimensional Lie groups*. Ann. Inst. Fourier (Grenoble) **52**(2002), 1365–1442.
[31] A. Pressley and G. B. Segal, *Loop Groups*. Clarendon Press, Oxford, 1986.
[32] T. Robart, *Sur l'intégrabilité des sous-algèbres de Lie en dimension infinie*. Canad. J. Math. **49**(1997), 820–839.
[33] W. Rudin, *Real and Complex Analysis*. McGraw-Hill, 1987.
[34] H. H. Schaefer, *Topological Vector Spaces*. Springer-Verlag, 1971.
[35] H. Schubert, *Topologie*. Teubner-Verlag, 1964.
[36] N. Tatsuuma, H. Shimomura and T. Hirai, *On group topologies and unitary representations on inductive limits of topological groups and the case of the group of diffeomorphisms*. Math. J. Kyoto Univ. **38**(1998), 551–578.
[37] E. G. F. Thomas, *Calculus on locally convex spaces*. Preprint W-9604, Groningen Univ., 1996.

Technische Universität Darmstadt
Fachbereich Mathematik, AG 5
Schlossgartenstr. 7
64289 Darmstadt
Germany
email: gloeckner@mathematik.tu-darmstadt.de