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MAZUR INTERSECTION PROPERTIES FOR COMPACT AND WEAKLY COMPACT CONVEX SETS

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ABSTRACT. Various authors have studied when a Banach space can be renormed so that every weakly compact convex, or less restrictively every compact convex set is an intersection of balls. We first observe that each Banach space can be renormed so that every weakly compact convex set is an intersection of balls, and then we introduce and study properties that are slightly stronger than the preceding two properties respectively.

1. **Introduction and main result.** We will say a Banach space has the *Mazur intersection property* if every closed bounded convex set is an intersection of balls. This property was introduced and studied by Mazur in [6]. In [7], Phelps showed among other things, that a finite dimensional Banach space has the Mazur intersection property if and only if the set of extreme points of the dual unit ball is dense in its sphere. In a paper of fundamental significance to this subject, Giles, Gregory and Sims [4] characterized normed linear spaces with the Mazur intersection property as those spaces for which the w^*-denting points of the dual unit ball are norm dense in its sphere. For several recent important advances in this topic, we refer the reader to the striking paper by Jiménez Sevilla and Moreno [5].

This note will focus on weaker forms of the Mazur intersection property. Before proceeding, let us mention that we work in real Banach spaces X whose dual spaces are denoted by X^*. The closed unit ball and unit sphere of X are denoted by B_X and S_X. A norm is called *locally uniformly rotund* (LUR) if $\|x_n - x\| \longrightarrow 0$ whenever $2\|x_n\|^2 + 2\|x\|^2 - \|x + x_n\|^2 \longrightarrow 0$. We will let $\mathcal{K}$ denote the collection of compact convex subsets of a Banach space and $\mathcal{W}$ the weakly compact convex subsets. We will say that X has the *$\mathcal{W}$-Mazur intersection property (resp. $\mathcal{K}$-Mazur intersection property*) if every weakly compact convex (resp. every compact convex) subset of X is an intersection of balls. The $\mathcal{W}$-Mazur intersection property was introduced by Zizler in [14] where he showed that every weakly compactly generated Banach space can be renormed to possess it. The $\mathcal{K}$-Mazur intersection property was introduced by Whitfield and Zizler in [13] where they show that a wide class of Banach spaces can be renormed to have this property. A nice characterization of this property as well as some further renorming results in this direction were discovered by Sersouri in [10]. Nevertheless, in a subsequent paper [11], Sersouri asks if every Banach space can be renormed so that every compact convex set of affine dimension 1 is an intersection of balls.

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We begin by observing that Jiménez Sevilla's and Moreno's recent renorming result [5, Lemma 2.3] in conjunction with results of Plička [9] and Zizler [14] provides a strong positive answer to Sersouri's question, and in fact gives a complete answer as to when a Banach space can be renormed to have the $\mathcal{K}$- or $\mathcal{W}$-Mazur intersection property.

THEOREM 1.1. *Every Banach space X can be renormed to have the $\mathcal{W}$-Mazur intersection property.*

PROOF. By a result of Plička (see [12, Theorem 20.2, p. 683] or [9, Theorem 1]), there is a biorthogonal system $\{x_i, f_i\}_{i\in I} \subset X \times X^*$ such that $\operatorname{span}(\{f_i\})$ is weak* dense in X^*. By building on Zizler's modification [14] of Troyanski's renorming technique, Jiménez-Sevilla and Moreno [5, Lemma 2.3] showed that given any biorthogonal system $\{y_\alpha, y^*_\alpha\}_{\alpha\in A} \subset X \times X^*$, there is an equivalent norm on X whose dual norm on X^* is LUR at each point of $\operatorname{span}(\{y^*_\alpha\}_{\alpha\in A})$. Applying this result, we let $|||\cdot|||$ be an equivalent norm on X whose dual norm $|||\cdot|||^*$ is LUR at each point of $H = \operatorname{span}(\{f_i\}_{i\in I})$ where the f_i's are the dual functionals from Plička's biorthogonal system.

We now outline the proof of [14, Theorem 1, p. 64–65], which shows that X endowed with the norm $|||\cdot|||$ has the property that every weakly compact convex set in X is an intersection of balls. Indeed, consider the topology $\mathcal{T}$ on X induced by pointwise convergence on H; because H is total, $\mathcal{T}$ is Hausdorff, and hence weakly compact sets are closed with respect $\mathcal{T}$. Thus given a weakly compact convex subset C with $0 \notin C$, the separation theorem ensures that there is an $h \in H$ with $|||h|||^* = 1$ such that $\inf h(C) > 0$. Because $|||\cdot|||^*$ is LUR at h, it follows that h is a weak* denting point of B_{X^*} (see *e.g.* [14, Lemma 2]). Now by [14, Lemma 3] there is a ball B in X such that $B \supset C$ and $0 \notin B$. It thus follows that C is an intersection of balls. ∎

2. **Further variants of the Mazur intersection property.** We will say a Banach space has the *$\mathcal{W}$-intersection property* (resp. *$\mathcal{K}$-intersection property*) if every set of the form $\overline{C + B_r}$ is an intersection of balls, where C is a weakly compact convex set (resp. compact convex set) and B_r is a closed ball of radius $r \geq 0$ centered at 0; we will denote these properties by $\mathcal{W}$-IP and $\mathcal{K}$-IP respectively. It is clear that these properties are variants of the usual Mazur intersection property, and that they are formally stronger than the $\mathcal{W}$- and $\mathcal{K}$-Mazur intersection properties respectively; moreover, they have a nice dual relationship with geometric properties of the dual ball. The remainder of this note will outline some relationships among these properties and the structure of the underlying Banach space. In the next result, for a set $S \subset X^*$ and $C \subset X$, the *C-diameter* of S is defined as $\sup\{|(f_1 - f_2)(c)| : f_1, f_2 \in S, c \in C\}$.

THEOREM 2.1. *(a) A Banach space X has the $\mathcal{W}$-IP (resp. $\mathcal{K}$-IP) if and only if for every $C \in \mathcal{W}$ (resp. $C \in \mathcal{K}$) and $\epsilon > 0$ the points in S_{X^*} that lie in a w^*-slice of B_{X^*} having C-diameter $\leq \epsilon$ is dense in S_{X^*} with respect to the topology of uniform convergence on weakly compact sets (resp. compact sets).*

(b) A Banach space X has the $\mathcal{K}$-IP if and only if the extreme points of B_{X^*} are w^*-dense in S_{X^*}.

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We will omit the proof of Theorem 2.1 because it follows from straightforward modifications of Sersouri's proof of [10, Theorem 1]. Those techniques also apply to other appropriate bornologies, for example the analog of Theorem 2.1(a) for bounded closed convex sets recaptures the main characterization of the Mazur intersection property given in [4]. Also, it is well-known that the Mazur intersection property is implied by Fréchet differentiability of the norm. Analogously, from Theorem 2.1 one can deduce that the K-IP and W-IP are respectively implied by Gateaux and weak Hadamard differentiability of the norm (see *e.g.* [2] and [3] for information on these notions of differentiability). This and the following two results show that wide classes of Banach spaces can be renormed to possess these intersection properties.

For the next result, recall that a subspace (not necessarily closed) $Y \subset X^*$ is called *norming* if there is a $\lambda > 0$ such that $\lambda\|x\| \le \sup\{\phi(x) : \phi \in Y \cap B_{X^*}\}$ for each $x \in X$. Throughout, we refer to the topology on X^* induced by uniform convergence on weakly compact subsets of X as the *Mackey topology*.

PROPOSITION 2.2. *Suppose $Y \subset X^*$ is a norming subspace. If there is a dual norm on X^* that is* LUR *(resp. strictly convex) at each point of Y, then X can be renormed to have the W*-IP *(resp. K*-IP*).*

PROOF. We prove the W-IP case only, the other case is similar. Let U_1 denote the unit ball of the dual norm on X^* that is LUR at each point of Y. Now let U_2 be the w^*-closure of $(U_1 \cap Y)$; this is a symmetric, bounded, convex and w^*-closed set. Because Y is norming, it follows that U_2 has nonempty interior and thus is the unit ball of an equivalent dual norm on X^*. Moreover, every point on the sphere of U_2 that lies in Y is a w^*-denting point of U_2, since such points are on the sphere of U_1 and are w^*-denting points of (the larger set) U_1 because its norm is LUR at those points. Now $Y \cap U_1$ is a convex set, thus its closure with respect to the Mackey topology is U_2 because the Mackey and w^*-closures are the same for convex sets. Hence the set of w^*-denting points of U_2 is Mackey dense in the sphere of U_2. Therefore Theorem 2.1(a) applies. ∎

COROLLARY 2.3. *If X is a Banach space such that $X \times X^*$ has a biorthogonal system $\{x_i, f_i\}$ for which* $\mathrm{span}(\{f_i\})$ *is norming, then X can be renormed to have the W*-IP.

PROOF. Let $Y = \mathrm{span}(\{f_i\})$. According to [5, Lemma 2.3] there is a dual norm on X^* that is LUR at each point of Y. Thus Proposition 2.2 applies. ∎

We will say a norm on a dual space is *w^*-Kadec-Klee* (resp. *τ-Kadec-Klee*) if the weak* and norm (resp. Mackey and norm) topologies agree on its sphere.

REMARK 2.4. If X has the W-IP (resp. K-IP) and the dual norm on X^* is τ-Kadec-Klee (resp. w^*-Kadec-Klee), then X has the Mazur intersection property.

PROOF. This follows from Theorem 2.1 and the characterization of the Mazur intersection property given in [4, Theorem 2.1]. ∎

However, the following example shows that Remark 2.4 fails to be true if the norm and Mackey topologies only coincide sequentially on the dual sphere.

EXAMPLE 2.5. If X^* is a dual Asplund and nonreflexive Banach space, then there is a norm $\|\cdot\|^*$ on X^* such that the norm and Mackey topologies coincide sequentially on $S_{X^{**}}$ and so that $(X^*, \|\cdot\|^*)$ has the W-IP but does not have the Mazur intersection property.

PROOF. Because X^* is Asplund, it follows that Mackey and norm convergence agree sequentially in X^{**} (see *e.g.* [1, Theorem 5]), and hence on the sphere of any unit ball with respect to an equivalent norm. Because X^* is Asplund, we can renorm X^{**} with a LUR norm (see [3, Corollary VII.1.12]). Denote the restriction of this norm to X by $\|\cdot\|$, and let $\|\cdot\|^*$ be the dual norm of $\|\cdot\|$. It follows from [4, Theorem 4.1] that $(X^*, \|\cdot\|^*)$ has the W-IP. However, $(X^*, \|\cdot\|^*)$ fails the Mazur intersection property because [4, Corollary 2.7] says that such a Banach space must be reflexive. ∎

Our final theorem shows in a fairly strong fashion that Remark 2.4 fails with the W-Mazur intersection property, and that the W-Mazur intersection property does not generally imply the K-IP.

THEOREM 2.6. *Suppose X is not reflexive and admits a norm whose dual is* LUR. *Then X admits a norm $|||\cdot|||$ such that $(X, |||\cdot|||)$ has the W-Mazur intersection property and $|||\cdot|||^*$ is w^*-Kadec-Klee, but X does not have the K-IP.*

PROOF. Let B_{X^*} denote the unit ball of a dual LUR norm $\|\cdot\|^*$ on X^*. Let $x^{**} \in S_{X^{**}} \setminus X$, and choose $\phi \in S_{X^*}$ such that $x^{**}(\phi) \geq \frac{3}{4}$. Let $U_1 = \overline{\{\Lambda \in B_{X^*} : |x^{**}(\Lambda)| \leq \frac{1}{2}\}}^{w^*}$, and let $\|\cdot\|_1$ be the dual norm whose unit ball is U_1. Notice that $\phi \notin U_1$ since if $\Lambda_\alpha \in B_{X^*}$ and $|x^{**}(\Lambda_\alpha)| \leq \frac{1}{2}$, then $\|\phi - \Lambda_\alpha\|^* \geq \frac{1}{4}$. Thus $\Lambda_\alpha \not\to_{w^*} \phi$ because $\|\cdot\|^*$ is LUR. Now we have $\|\cdot\|^* \leq \|\cdot\|_1^*$ and $\|\phi\|_1^* \geq 1 + 2r$ for some $r > 0$. Because the dual LUR norms are dense among all dual norms, we choose a dual LUR norm $\|\cdot\|_2^*$ such that

$$\|\Lambda\|_1^* - r \leq \|\Lambda\|_2^* \leq \|\Lambda\|_1^* \quad \text{for all } \Lambda \in B_{X^*}.$$

In particular, for U_2 the unit ball of $\|\cdot\|_2$, one has $U_1 \subset U_2$ and $\phi \notin U_2$ because

$$\|\phi\|_2^* \geq \|\phi\|_1^* - r \geq 1 + r.$$

Now let $U_3 = B_{X^*} \cap U_2$. Then U_3 is the unit ball of the dual LUR norm $\|\cdot\|_3^* = \max\{\|\cdot\|^*, \|\cdot\|_2^*\}$. We now define the unit ball of our desired norm: $U = \operatorname{conv}(U_3 \cup \{\pm\phi\})$.

First, U is w^*-closed because being the convex hull of two w^*-compact sets, it is w^*-compact. Because U is w^*-closed, bounded, convex and contains $\frac{1}{2}B_{X^*}$, it follows that U is the unit ball of an equivalent norm $|||\cdot|||^*$ on X^* that is dual to some norm $|||\cdot|||$ on X. We now make the following observations concerning this construction.

CLAIM. *(1) Let $Y = \{\Lambda \in X^* : x^{**}(\Lambda) = 0\}$. Then Y is w^*-dense in X^*.*

(2) If $\Lambda \in Y$ and $|||\Lambda|||^* = 1$, then Λ is a w^*-denting point of U.

(3) $|||\cdot|||^*$ is w^*-Kadec-Klee.

(4) If $\Lambda = ax^* + b\phi$ with $a + b = 1$, $0 < b < 1$, $x^* \in U_3$, then Λ is not an extreme point of U.

The proofs of Claims 1, 2, and 4 are very straightforward, so we only verify Claim 3. Indeed, suppose $|||\Lambda_\alpha|||^* = |||\Lambda|||^* = 1$ and $\Lambda_\alpha \longrightarrow_{w^*} \Lambda$. Writing $\Lambda_\alpha = \lambda_\alpha x_\alpha^* + (1 - \lambda_\alpha)\phi_\alpha$

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where $x^*_\alpha \in B_{X^*}$ and $\phi_\alpha \in \{\pm\phi\}$. By passing to a subnet, we may suppose $\phi_\alpha = \tilde{\phi}$ for all α, $\lambda_\alpha \longrightarrow \lambda$, $x^*_\alpha \longrightarrow_{w^*} x^*$. If $\lambda = 0$, then $x_\alpha \longrightarrow \tilde{\phi}$, and we are done. Otherwise, $\lambda > 0$, and $\|x^*\|^*_3 = 1$ (because if $\|x^*\|^*_3 < 1$, then $\lambda x^* + (1-\lambda)\tilde{\phi}$ would be in the interior of U). Because $\|\cdot\|^*_3$ is LUR, we have that $x^*_\alpha \longrightarrow x^*$. This with $(1-\lambda_\alpha)\phi_\alpha \longrightarrow (1-\lambda)\tilde{\phi}$ shows that $\Lambda_\alpha \longrightarrow \Lambda$ and so Claim 3 is valid.

Now to complete the proof of the theorem, by Claims 1 and 2, the cone of w^*-denting points of U contains a w^*-dense subspace of X^*. Therefore, as in the proof of Theorem 1.1, $(X, |||\cdot|||)$ has the W-Mazur intersection property. However, Claims 3 and 4 together imply that the extreme points of U are not w^*-dense in its sphere and so $(X, |||\cdot|||)$ fails to have the K-IP according to Theorem 2.1(b). ∎

It is probably a matter of perspective, whether one would consider the K-IP or the K-Mazur intersection property the appropriate weakening of the Mazur intersection property for compact convex sets. The definition for the K-IP may not seem as natural, but it leads to the dual characterization involving the dual sphere rather than a cone (*cf.* Theorem 2.1 and [10, Theorem 1]) which corresponds to the characterization of the usual Mazur intersection property and moreover has advantages when combined with Kadec-Klee properties (Remark 2.4). Irrespective of this, the following question is apparently open.

QUESTION 2.7. Can every Banach space be renormed to have the W-IP, or less restrictively the K-IP?

One should note that the proof of Theorem 1.1 does not appear to apply for this question, because Pličko [8] has shown that there are Banach spaces that do not possess any biorthogonal system $\{x_i, f_i\}$ such that span$(\{f_i\})$ is norming.

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