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ON HIGHER COVARIANT DERIVATIVES OF THE CURVATURE TENSORS OF KÄHLERIAN C-SPACES

Dedicated to Professor I. Mogi on his 60th birthday

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A compact simply connected complex homogeneous manifold is said briefly a C-space, which was completely classified by H. C. Wang [12]. A C-space is called to be Kählerian if it admits a Kählerian metric such that a group of isometries acts transitively on it. Hermitian symmetric spaces of compact type are typical examples of a Kählerian C-space. Let M be an arbitrary Kählerian C-space and R its curvature tensor. M. Itoh [6] expressed R in the language of Lie algebra and investigated various properties of R. In this paper, we study higher covariant derivatives of R.

First we shall show that for each M there exists a positive integer m such that

$$\underbrace{\overset{+}{\nabla}\cdots\overset{+}{\nabla}}_{m\text{ times}}R = 0\,, \qquad \underbrace{\overset{+}{\nabla}\cdots\overset{+}{\nabla}}_{(m-1)\text{times}}R \neq 0\,,$$

where $\overset{+}{\nabla}$ denotes the covariant derivative of (1, 0)-type. We call the integer m the degree of a Kählerian C-space M. Obviously, Hermitian symmetric spaces of compact type can be characterized as C-spaces with degree one.

Next we shall determine all C-spaces with degree two, which are stated as Theorems 4.1, 4.2 and 4.7. They will form a class of the "simplest" spaces among Kählerian C-spaces except for Hermitian symmetric spaces.

Our results have some applications to a theory of Kählerian submanifolds in a complex projective space. This will be discussed in a forthcoming paper [10].

§ 1. Kählerian C-spaces

In this section we recall the construction of irreducible Kählerian

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C-spaces. For details, we refer to Borel and Hirzebruch [2], Itoh [6], Nakagawa and Takagi [8], Takeuchi [11].

Let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra. The dual space of a complex vector space $\mathfrak{h}$ is denoted by $\mathfrak{h}^*$. An element α of $\mathfrak{h}^*$ is called a root of $(\mathfrak{g}, \mathfrak{h})$ if there exists a non-zero vector E_α in $\mathfrak{g}$ such that

$$[H, E_\alpha] = \alpha(H)E_\alpha \qquad \text{for } H \in \mathfrak{h} .$$

We denote by Δ the set of all non-zero roots of $(\mathfrak{g}, \mathfrak{h})$ and put $\mathfrak{g}_\alpha = \boldsymbol{C}E_\alpha$. Then we have a direct sum decomposition:

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha .$$

Since the Killing form B of $\mathfrak{g}$ is non-degenerate on $\mathfrak{h} \times \mathfrak{h}$, for each $\xi \in \mathfrak{h}^*$ we can define $H_\xi \in \mathfrak{h}$ by

$$B(H, H_\xi) = \xi(H) \qquad \text{for } H \in \mathfrak{h} .$$

The following property of B is fundamental:

$$B([X, Y], Z) + B(Y, [X, Z]) = 0 \qquad \text{for } X, Y, Z \in \mathfrak{g} .$$

Put $\mathfrak{h}_o = \sum_{\alpha \in \Delta} \boldsymbol{R}H_\alpha$ and define an inner products $(\, , \,)$ on the dual space $\mathfrak{h}_o^*$ of a real vector space $\mathfrak{h}_o$ by $(\xi, \eta) = B(H_\xi, H_\eta)$. We fix a lexicographic order $<$ on $\mathfrak{h}_o^*$. Let $\Pi = \{\alpha_1, \cdots, \alpha_\ell\}$ be the fundamental root system of Δ with respect to $<$ (so $\ell = \dim_{\boldsymbol{C}} \mathfrak{h}$). Put $\Delta^+ = \{\alpha \in \Delta \,|\, o < \alpha\}$. For each $\alpha \in \Delta$ we select a basis E_α of $\mathfrak{g}_\alpha$ in such a way that $\{H_{\alpha_1}, \cdots, H_{\alpha_\ell}, E_\alpha \ (\alpha \in \Delta)\}$ forms a Weyl's canonical basis of $\mathfrak{g}$, that is, it satisfies

$$\text{(1.1)} \qquad \begin{cases} B(E_\alpha, E_{-\alpha}) = -1 , \\ [E_\alpha, E_\beta] = N_{\alpha+\beta}E_{\alpha+\beta}, \ N_{\alpha+\beta} = N_{-\alpha-\beta} \in \boldsymbol{R} & \text{for } \alpha, \beta \in \Delta . \end{cases}$$

The first equation is equivalent to $[E_\alpha, E_{-\alpha}] = -H_\alpha$. Then the following $\mathfrak{g}_u$ is a compact real form of $\mathfrak{g}$:

$$\text{(1.2)} \qquad \mathfrak{g}_u = \sum_{\alpha \in \Delta} \boldsymbol{R}\sqrt{-1}H_\alpha + \sum_{\alpha \in \Delta} (\boldsymbol{R}A_\alpha + \boldsymbol{R}B_\alpha) ,$$

where we put $A_\alpha = E_\alpha + E_{-\alpha}$, $B_\alpha = \sqrt{-1}(E_\alpha - E_{-\alpha})$. The complex conjugation $^-$ on $\mathfrak{g}$ with respect to $\mathfrak{g}_u$ is given by

$$\text{(1.3)} \qquad \bar{E}_\alpha = E_{-\alpha}, \ \bar{E}_{-\alpha} = E_\alpha, \ \bar{H}_\alpha = -H_\alpha \qquad \text{for } \alpha \in \Delta^+ .$$

Now we choose an arbitrary non-empty subset $\Phi = \{\alpha_{i_1}, \cdots, \alpha_{i_r}\}$ of Π. Define a subset $\Delta^+(\Phi)$ of Δ^+ by

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$$\begin{aligned} \Delta^+(\Phi) &= \Delta^+(\alpha_{i_1}, \cdots, \alpha_{i_r}) \\ &= \left\{ \sum_{i=1}^{\ell} n_i \alpha_i \in \Delta^+ ;\ (n_{i_1}, \cdots, n_{i_r}) \neq 0 \right\} . \end{aligned} \tag{1.4}$$

It is clear that if $\alpha, \beta \in \Delta^+(\Phi)$, then $\alpha + \beta \in \Delta^+(\Phi)$, and that the highest root in Δ always belongs to $\Delta^+(\Phi)$. Define a complex subalgebra $\mathfrak{L}_\Phi$ of $\mathfrak{g}$ associated with Φ by

$$\mathfrak{L}_\Phi = \mathfrak{h} + \sum_{\alpha \in \Delta - \Delta^+(\Phi)} \mathfrak{g}_\alpha .$$

If we put $\mathfrak{k}_\Phi = \mathfrak{g}_u \cap \mathfrak{L}_\Phi$, then it is a subalgebra of $\mathfrak{g}_u$ expressed as

$$\mathfrak{k}_\Phi = \sum_{\alpha \in \Delta} \boldsymbol{R}\sqrt{-1}H_\alpha + \sum_{\alpha \in \Delta - \Delta^+(\Phi)} (\boldsymbol{R}A_\alpha + \boldsymbol{R}B_\alpha) . \tag{1.5}$$

Let G be the simply connected complex Lie group with Lie algebra $\mathfrak{g}$. Let L_Φ be the connected complex Lie subgroup of G with Lie algebra $\mathfrak{L}_\Phi$, and G_u, K_Φ the connected Lie subgroup of G with Lie algebras $\mathfrak{g}_u, \mathfrak{k}_\Phi$, respectively. Then we obtain an irreducible C-space $G_u/K_\Phi = G/L_\Phi$, denoted by $M(\mathfrak{g}, \Phi)$ or $M(\mathfrak{g}, \alpha_{i_1}, \cdots, \alpha_{i_r})$. Conversely, every irreducible C-space can be obtained in this way ([12]).

Next we describe a G_u-invariant Kählerian metric g on a C-space $M(\mathfrak{g}, \Phi)$. For a vector space V, the complexification is denoted by V^C. Put

$$\mathfrak{m}_\Phi = \sum_{\alpha \in \Delta^+(\Phi)} (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}) . \tag{1.6}$$

Hereafter we put $\mathfrak{m} = \mathfrak{m}_\Phi$ and $\mathfrak{k} = \mathfrak{k}^C$ for simplicity. Then we have a direct sum $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ orthogonal with respect to B. Denoting the tangent space of $M(\mathfrak{g}, \Phi)$ at the origin $o = K_\Phi$ by $T_o(M)$, we can identify $T_o(M)$ with $\mathfrak{g}_u$. So we may write $\mathfrak{m} = T_o(M)^C$. Since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \in \mathfrak{g}_{\alpha+\beta}$ for $\alpha, \beta \in \Delta$, we see $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$ by the definition of $\Delta^+(\Phi)$. The complex structure I of $M(\mathfrak{g}, \Phi)$ is given at o by

$$I(\mathrm{A}_\alpha) = B_\alpha,\ I(B_\alpha) = -A_\alpha \qquad \text{for } \alpha \in \Delta^+(\Phi) . \tag{1.7}$$

Put $\mathfrak{m}^\pm = \{X \in \mathfrak{m};\ I(X) = \pm\sqrt{-1}X\}$. Then we have

$$\mathfrak{m}^\pm = \sum_{\alpha \in \Delta^+(\Phi)} \mathfrak{g}_{\pm\alpha} , \tag{1.8}$$

and hence a direct sum $\mathfrak{m} = \mathfrak{m}^+ + \mathfrak{m}^-$. An element of $\mathfrak{m}^+$ is said to be of (1, 0)-type. Now we define a mapping p of $\Delta^+(\Phi)$ into Z^r associated with Φ as follows: For $\alpha = \sum_{i=1}^{\ell} n_i(\alpha)\alpha_i \in \Delta^+(\alpha_{i_1}, \cdots, \alpha_{i_r})$, we put

$$p_\alpha = (n_{i_1}(\alpha), \cdots, n_{i_r}(\alpha)) .$$

This mapping p plays an important role in this paper. Let $\omega^\alpha, \bar{\omega}^\alpha$ denote the dual forms of $E_\alpha, \bar{E}_\alpha$ ($\alpha \in \Delta^+(\Phi)$). Then any G_u-invariant Kählerian metric g is given at o by

$$g = 2 \sum_{\alpha \in \Delta^+(\Phi)} (c \cdot p_\alpha) \omega^\alpha \cdot \bar{\omega}^\alpha \tag{1.9}$$

for an r-tuple $c = (c_1, \cdots, c_r)$ of positive integers $c_1, \cdots, c_r$, where $c \cdot p_\alpha = \sum_{a=1}^r c_a n_{i_a}(\alpha)$ (Borel [1] or Itoh [6]). Conversely, any bilinear form on $\mathfrak{m} \times \mathfrak{m}$ of this type can be extended to a G_u-invariant Kählerian metric on $M(\mathfrak{g}, \Phi)$.

When $\Delta^+(\Phi) \ni \alpha$ and β satisfy $n_{i_a}(\alpha) \geqq n_{i_a}(\beta)$ for $a = 1, \cdots, r$ and $n_{i_a}(\alpha) > n_{i_a}(\beta)$ for some a, we write $p_\alpha > p_\beta$. Hence we have equivalences $\alpha \in \Delta^+(\Phi) \Leftrightarrow E_\alpha \in \mathfrak{m}^+ \Leftrightarrow p_\alpha > 0$.

§ 2. Covariant derivatives on Kählerian C-spaces

In this section we consider a Kählerian C-space $(M(\mathfrak{g}, \Phi), g)$ constructed in Section 1, where $\Phi = \{\alpha_{i_1}, \cdots, \alpha_{i_r}\}$ is a non-empty subset of the fundamental root system Π of a complex simple Lie algebra $\mathfrak{g}$ and g is a G_u-invariant Kählerian metric on M given by (1.8). We extend tensor fields, the connection and the connection form on M naturally over $\boldsymbol{C}$, and denote the extended ones by the same letters. Since M is homogeneous Riemannian manifold, it is sufficient to consider the degree at the origin o.

First we have from (1.1) and (1.9)

$$g(E_\alpha, E_{-\beta}) = -(c \cdot p_\alpha) B(E_\alpha, E_{-\beta}) = (c \cdot p_\alpha) \delta_{\alpha\beta} \qquad \text{for } \alpha, \beta \in \Delta^+(\Phi) . \tag{2.1}$$

When $r = 1$, that is, Φ consists of a single element, we take a Kählerian metric g such that $c = 1$.

For $X \in \mathfrak{g}$ we denote by $X_\mathfrak{m}$ (resp. $X_\mathfrak{k}$) the $\mathfrak{m}$ (resp. $\mathfrak{k}$)-component of X with respect to the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$. Define a symmetric bilinear mapping $U: \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ by

$$2g(U(X, Y), Z) = g([Z, X]_\mathfrak{m}, Y) + g(X, [Z, Y]_\mathfrak{m}) \qquad \text{for } X, Y, Z \in \mathfrak{m} . \tag{2.2}$$

Then the connection form $\Lambda: \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ associated with g is given by (Kobayashi and Nomizu [17] and Nomizu [9])

$$\Lambda(X) Y = U(X, Y) + \tfrac{1}{2}[X, Y] \qquad \text{for } X, Y \in \mathfrak{m} . \tag{2.3}$$

The curvature tensor R of g is given by (Nomizu [9])

$$R(X, Y)Z = [\Lambda(X), \Lambda(Y)]Z - \Lambda([X, Y]_{\mathfrak{m}})Z - [[X, Y]_{\mathfrak{k}}, Z] \quad \text{for } X, Y, Z \in \mathfrak{m} . \tag{2.4}$$

For $X \in \mathfrak{g}$ we denote by X^* the vector field on M induced by a one-parameter subgroup $\exp tX$ of G_u. It is easy to see that

$$\begin{cases} (X^*)_o = X_{\mathfrak{m}} \\ [X^*, Y^*] = -[X, Y]^* \qquad \text{for } X, Y \in \mathfrak{g} . \end{cases} \tag{2.5}$$

The covariant derivative $\nabla_{X^*}Y^*$ of Y^* in the direction X^* is given at o by ([7], p. 201)

$$(\nabla_{X^*}Y^*)_o = \Lambda(Y)X = U(X, Y) - \tfrac{1}{2}[X, Y]_{\mathfrak{m}} \qquad \text{for } X, Y \in \mathfrak{m} . \tag{2.6}$$

In the following, in order to simplify the notation, we *identify* a root vector E_α with α itself for $\alpha \in \Delta$, and *put* $\nabla_X Y = (\nabla_{X^*}Y^*)_o$ for $X, Y \in \mathfrak{m}$. Under this identification, a subset $\Delta^+(\Phi) \cup \overline{\Delta^+(\Phi)}$ of Δ forms a basis of the complexified tangent space $\mathfrak{m} = T_o(M)^C$ of M at o. We also call a root $\alpha \in \Delta^+(\Phi)$ a tangent vector.

M. Itoh [6] determined the connection form Λ of g, which can be stated as

Proposition 2.1. *Let $\alpha, \beta \in \Delta^+(\Phi)$. Then,*

$$\begin{cases} \Lambda(\alpha)\beta = (c\cdot p_\beta/c\cdot p_{\alpha+\beta})[\alpha, \beta] , \\ \Lambda(\bar\alpha)\beta = \begin{cases} [\bar\alpha, \beta] & \text{if } p_\alpha < p_\beta \\ 0 & \text{otherwise} , \end{cases} \\ \Lambda(\alpha)\bar\beta = \begin{cases} [\alpha, \bar\beta] & \text{if } p_\alpha < p_\beta \\ 0 & \text{otherwise} . \end{cases} \end{cases} \tag{2.7}$$

This and (2.6) give

Corollary 2.2. *Let $\alpha, \beta \in \Delta^+(\Phi)$. Then,*

$$\begin{cases} \nabla_\alpha\beta = -(c\cdot p_\alpha/c\cdot p_{\alpha+\beta})[\alpha, \beta] , \\ \nabla_\alpha\bar\beta = \begin{cases} -[\alpha, \bar\beta] & \text{if } p_\alpha > p_\beta \\ 0 & \text{otherwise} , \end{cases} \\ \nabla_{\bar\alpha}\beta = \begin{cases} -[\bar\alpha, \beta] & \text{if } p_\alpha > p_\beta \\ 0 & \text{otherwise} . \end{cases} \end{cases} \tag{2.8}$$

The curvature tensor R of g can be restated as

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$$R(\alpha, \bar{\lambda}, \beta) := R(\alpha, \bar{\lambda})\beta$$

$$(2.9) \qquad = [A(\alpha), A(\bar{\lambda})]\beta - A([\alpha, \bar{\lambda}]_{\mathfrak{m}})\beta - [[\alpha, \bar{\lambda}]_{\mathfrak{k}}, \beta] \quad \text{for } \alpha, \lambda, \beta \in \Delta^+(\Phi) .$$

Put $R(\alpha, \bar{\lambda}, \beta, \bar{\mu}) = g(R(\alpha, \bar{\lambda}, \beta), \bar{\mu})$. Then we obtain fundamental formulas

$$(2.10) \qquad R(\alpha, \bar{\lambda}, \beta, \bar{\mu}) = R(\beta, \bar{\lambda}, \alpha, \bar{\mu}) = R(\alpha, \bar{\mu}, \beta, \bar{\lambda}) = \overline{R(\lambda, \bar{\alpha}, \mu, \bar{\beta})} \quad \text{for } \alpha, \beta, \lambda, \mu \in \Delta^+(\Phi) .$$

Now we define the s-th covariant derivative $\nabla^s R$ of R inductively as follows:

$$(2.11) \qquad \begin{aligned} R(\alpha, \bar{\lambda}, \beta; \gamma_1, \cdots, \gamma_s) &= \sum_{\mu \in \Delta^+(\Phi)} R(\alpha, \bar{\lambda}, \beta, \bar{\mu}; \gamma_1, \cdots, \gamma_s)\mu / g(\mu, \bar{\mu}) \\ &= \nabla_{\gamma_s} R(\alpha, \bar{\lambda}, \beta; \gamma_1, \cdots, \gamma_{s-1}) \\ &\quad - R(\nabla_{\gamma_s}\alpha, \bar{\lambda}, \beta; \gamma_1, \cdots, \gamma_{s-1}) \\ &\quad - R(\alpha, \nabla_{\gamma_s}\bar{\lambda}, \beta; \gamma_1, \cdots, \gamma_{s-1}) \\ &\quad - R(\alpha, \bar{\lambda}, \nabla_{\gamma_s}\beta; \gamma_1, \cdots, \gamma_{s-1}) \\ &\quad - \sum_{a=1}^{s-1} R(\alpha, \bar{\lambda}, \beta; \gamma_1, \cdots, \nabla_{\gamma_s}\gamma_a, \cdots, \gamma_{s-1}) \\ &\qquad \text{for } \alpha, \lambda, \beta \in \Delta^+(\Phi) \text{ and } \gamma_1, \cdots, \gamma_s \in \mathfrak{m} . \end{aligned}$$

When vectors $\gamma_1, \cdots, \gamma_s$ in (2.11) belong to $\Delta^+(\Phi)$, we write $\overset{+}{\nabla}{}^s R$ instead of $\nabla^s R$, which is called the s-th covariant derivative of (1, 0)-type of R. Then we have a basic property with respect to the covariant derivative of R:

LEMMA 2.3. *For $\alpha, \lambda, \beta \in \Delta^+(\Phi)$ and $\gamma_1, \cdots, \gamma_s \in \mathfrak{m}$,*

$$R(\alpha, \bar{\lambda}, \beta; \gamma_1, \cdots, \gamma_s) \in \mathfrak{g}_{\alpha-\lambda+\beta+\gamma_1+\cdots+\gamma_s} .$$

Proof. This follows from a relation $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, (2.7), (2.8) and (2.11).

COROLLARY 2.4. *Let ν be the highest root in Δ. Put $p_\nu = (n_{i_1}, \cdots, n_{i_r})$, and $t = t(\Phi) = n_{i_1} + \cdots + n_{i_r}$. Then we have $\overset{+}{\nabla}{}^{2t-1} R = 0$.*

Proof. For $\delta \in \Delta^+(\Phi)$ we denote by $\|\delta\|$ the sum of the components of p_δ. Then clearly, $1 \leqq \|\delta\| \leqq t = \|\nu\|$. Let $\alpha, \lambda, \beta, \gamma_1, \cdots, \gamma_{2t-1} \in \Delta^+(\Phi)$. Then we have

$$\begin{aligned} &\|\alpha - \lambda + \beta + \gamma_1 + \cdots + \gamma_{2t-1}\| \\ &\quad = \|\alpha\| + \|\beta\| + \|\gamma_1\| + \cdots + \|\gamma_{2t-1}\| - \|\lambda\| \\ &\quad \geqq 1 + 1 + 2t - 1 - t = t + 1 . \end{aligned}$$

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This and the definition of t show that a form $\alpha - \lambda + \beta + \gamma_1 + \cdots + \gamma_{2t-1}$ cannot be a root. Thus by Lemma 2.3 we have $R(\alpha, \bar{\lambda}, \beta; \gamma_1, \cdots, \gamma_{2t-1}) = 0$. q.e.d.

By Corollary 2.4, there exists uniquely the integer $d = d(\mathfrak{g}, \Phi)$ such that $\overset{+}{\nabla}{}^d R = 0$ but $\overset{+}{\nabla}{}^{d-1} R \neq 0$. We shall call the integer d the degree of a C-space $M(\mathfrak{g}, \Phi)$. Hermitian symmetric spaces of compact type can be characterized as C-spaces with degree one. In fact, let Φ consist of a single element α_a such that $p_\nu = 1$, where ν denotes the highest root in Δ. Then Corollary 2.4 and (2.10) imply that a C-space $M(\mathfrak{g}, \alpha_a)$ is symmetric. Conversely, every irreducible Hermitian symmetric space of compact type can be expressed as a C-space of the form $M(\mathfrak{g}, \alpha_a)$ where $p_\nu = 1$ (Wolf [13]).

§ 3. *C*-spaces with degree two

In this section we shall determine all irreducible C-spaces with degree two. Keep the notation in Section 2. Hereafter *we denote by* $\alpha, \beta, \gamma, \delta, \lambda, \mu$ *any elements of* $\Delta^+(\Phi)$ *unless otherwise stated.*

From (2.11) we have

$$
\begin{aligned}
R(\alpha, \bar{\lambda}, \beta; \gamma, \delta) = {} & \nabla_\delta \nabla_\gamma R(\alpha, \bar{\lambda})\beta - \nabla_{\nabla_\delta \gamma} R(\alpha, \bar{\lambda})\beta - \nabla_\gamma R(\nabla_\delta \alpha, \bar{\lambda})\beta \\
& - \nabla_\gamma R(\alpha, \nabla_\delta \bar{\lambda})\beta - \nabla_\gamma R(\alpha, \bar{\lambda})\nabla_\delta \beta \\
& - \nabla_\delta R(\nabla_\gamma \alpha, \bar{\lambda})\beta + R(\nabla_{\nabla_\delta \gamma} \alpha, \bar{\lambda})\beta + R(\nabla_\gamma \nabla_\delta \alpha, \bar{\lambda})\beta \\
& + R(\nabla_\gamma \alpha, \nabla_\delta \bar{\lambda})\beta + R(\nabla_\gamma \alpha, \bar{\lambda})\nabla_\delta \beta \\
& - \nabla_\delta R(\alpha, \nabla_\gamma \bar{\lambda})\beta + R(\nabla_\delta \alpha, \nabla_\gamma \bar{\lambda})\beta + R(\alpha, \nabla_{\nabla_\delta \alpha} \bar{\lambda})\beta \\
& + R(\alpha, \nabla_\gamma \nabla_\delta \bar{\lambda})\beta + R(\alpha, \nabla_\gamma \bar{\lambda})\nabla_\delta \beta \\
& - \nabla_\delta R(\alpha, \bar{\lambda})\nabla_\gamma \beta + R(\nabla_\delta \alpha, \bar{\lambda})\nabla_\gamma \beta + R(\alpha, \nabla_\delta \bar{\lambda})\nabla_\gamma \beta \\
& + R(\alpha, \bar{\lambda})\nabla_{\nabla_\delta \gamma} \beta + R(\alpha, \bar{\lambda})\nabla_\gamma \nabla_\delta \beta \,.
\end{aligned}
\tag{3.1}
$$

The equation (3.1) is not so complicated as it looks because it contains many simple cases. For example,

LEMMA 3.1. *Unless* $p_\alpha > p_\beta$, *then* $\nabla_\alpha \bar{\beta} = 0$ *and* $\Lambda(\bar{\beta})\alpha = 0$. *In particular, if* λ *is the highest root in* Δ, *then* $\nabla_\alpha \bar{\lambda} = 0$ *and* $\Lambda(\bar{\lambda})\alpha = 0$.

Proof. This is a restatement of (2.7) and (2.8). q.e.d.

LEMMA 3.2. (1) *If* $\alpha + \beta - \lambda \notin \Delta$, *then* $R(\alpha, \bar{\lambda})\beta = 0$. (2) *Let* λ *satisfy* $p_\lambda \geqq p_\gamma$ *for all* γ. *Let* α *and* β *satisfy* $p_\lambda \geqq p_\alpha + p_\beta$. *Then* $R(\alpha, \bar{\lambda})\beta = 0$.

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Proof. (1) is evident from a special case $R(\alpha, \bar{\lambda})\beta \in \mathfrak{g}_{\alpha+\beta-\lambda}$ of Lemma 2.3. To show (2) we use (2.9). From Lemma 3.1 we have $\Lambda(\bar{\lambda})\alpha = 0$, and so from (2.7)

$$\Lambda(\bar{\lambda})\Lambda(\alpha)\beta \in \Lambda(\bar{\lambda})\mathfrak{g}_{\alpha+\beta}\,.$$

But, by Lemma 3.1, $\Lambda(\bar{\lambda})\mathfrak{g}_{\alpha+\beta} = \{0\}$ whether $\alpha + \beta \in \Delta$ or not, hence $\Lambda(\bar{\lambda})\Lambda(\alpha)\beta = 0$. If $\alpha + \bar{\lambda} \notin \Delta$, then $R(\alpha, \bar{\lambda})\beta = 0$ since $[\alpha, \bar{\lambda}] = 0$. If $\alpha + \bar{\lambda} \in \Delta$, then $\lambda - \alpha \in \Delta^+(\Phi)$ since $p_{\lambda-\alpha} \geqq p_\beta > 0$. Thus $R(\alpha, \bar{\lambda})\beta = -\Lambda([\alpha, \bar{\lambda}])\beta = 0$ by Lemma 3.1. q.e.d.

COROLLARY 3.3 *Let λ, α and β satisfy at least one of the following two conditions:*

(1) $\alpha + \beta - \lambda \notin \Delta$.

(2) $p_\lambda \geqq p_\gamma$ *for all* γ, *and* $p_\lambda \geqq p_\alpha + p_\beta$. *Then,*

$$\begin{aligned} R(\alpha, \bar{\lambda}, \beta; \gamma, \delta) = &- \nabla_\gamma R(\nabla_\delta\alpha, \bar{\lambda})\beta - \nabla_\delta R(\nabla_\gamma\alpha, \bar{\lambda})\beta - \nabla_\gamma R(\alpha, \bar{\lambda})\nabla_\delta\beta \\ &- \nabla_\delta R(\alpha, \bar{\lambda})\nabla_\gamma\beta + R(\nabla_{\nabla_\delta\gamma}\alpha, \bar{\lambda})\beta + R(\alpha, \bar{\lambda})\nabla_{\nabla_\delta\gamma}\beta \\ &+ R(\nabla_\gamma\nabla_\delta\alpha, \bar{\lambda})\beta + R(\alpha, \bar{\lambda})\nabla_\gamma\nabla_\delta\beta + R(\nabla_\gamma\alpha, \bar{\lambda})\nabla_\delta\beta \\ &+ R(\nabla_\delta\alpha, \bar{\lambda})\nabla_\gamma\beta\,. \end{aligned} \tag{3.2}$$

Proof. Apply (2.8), Lemma 3.1 and Lemma 3.2 (1) to (3.1). q.e.d.

PROPOSITION 3.4. *Let five vectors $\lambda, \alpha, \beta, \gamma, \delta \in \Delta^+(\Phi)$ satisfy the following three conditions:*

(1) $\alpha + \beta - \lambda \notin \Delta$, *or* $p_\lambda \geqq p_\gamma$ *for all* γ *and* $p_\lambda \geqq p_\alpha + p_\beta$ *(the same condition as in Corollary 3.3).*

(2) *Let ε be the sum of any three of $\alpha, \beta, \gamma, \delta$. Then $\varepsilon \notin \Delta$ and $p_\varepsilon > p_\lambda$.*

(3) *Let μ be the sum of any two of $\alpha, \beta, \gamma, \delta$. Then $[\mu, \bar{\lambda}] \in \mathfrak{k}$.*

Then we have

$$\begin{aligned} 2R(\alpha, \bar{\lambda}, \beta, \bar{\lambda}; \gamma, \delta) = &\ B([\delta, [\alpha, \bar{\lambda}]],\ [\beta, [\gamma, \bar{\lambda}]]) \\ &+ B([\gamma, [\alpha, \bar{\lambda}]],\ [\beta, [\delta, \bar{\lambda}]]) \\ &- B([\alpha, [\gamma, \bar{\lambda}]],\ [\delta, [\beta, \bar{\lambda}]]) \\ &- B([\alpha, [\delta, \bar{\lambda}]],\ [\gamma, [\beta, \bar{\lambda}]])\,. \end{aligned} \tag{3.3}$$

Proof. By Corollary 3.3 the equation (3.2) holds. The assumption (2) and (2.8) imply $\nabla_{\nabla_\delta\gamma}\alpha = \nabla_{\nabla_\delta\gamma}\beta = \nabla_\gamma\nabla_\delta\alpha = \nabla_\gamma\nabla_\delta\beta = 0$. Thus,

$$\begin{aligned} \mathrm{A} := &\ R(\alpha, \bar{\lambda}, \beta; \gamma, \delta) \\ = &- \nabla_\gamma R(\nabla_\delta\alpha, \bar{\lambda})\beta - \nabla_\gamma R(\alpha, \bar{\lambda})\nabla_\delta\beta - \nabla_\delta R(\nabla_\gamma\alpha, \bar{\lambda})\beta \\ &- \nabla_\delta R(\alpha, \bar{\lambda})\nabla_\gamma\beta + R(\nabla_\gamma\alpha, \bar{\lambda})\nabla_\delta\beta + R(\nabla_\delta\alpha, \bar{\lambda})\nabla_\gamma\beta\,. \end{aligned} \tag{3.4}$$

We shall express the right hand side of (3.4) with respect to the bracket product [,]. The assumptions (2) and (3), together with (2.7) and (2.8), imply $[\nabla_\delta\alpha, \bar\lambda] \in \mathfrak{k}$, $A(\bar\lambda)\beta = 0$ and $A(\nabla_\delta\alpha)\beta = 0$. It follows from (2.8) and (2.9) that

$$4\nabla_\gamma R(\nabla_\delta\alpha, \bar\lambda)\beta = \quad [\gamma, [[[\delta, \alpha], \bar\lambda], \beta]] .$$

Applying similar argument to other five terms in (3.4), we find

$$\begin{aligned} 4A = [\gamma, [[[\delta, \alpha], \bar\lambda], \beta]] + [\delta, [[[\gamma, \alpha], \bar\lambda], \beta]] + [\gamma, [[\alpha, \bar\lambda], [\delta, \beta]]] \\ + [\delta, [[\alpha, \bar\lambda], [\gamma, \beta]]] - [[[\gamma, \alpha], \bar\lambda], [\delta, \beta]] - [[[\delta, \alpha], \bar\lambda], [\gamma, \beta]] . \end{aligned}$$

For a while, we write $\alpha\beta\gamma\delta$ for $[\alpha, [\beta, [\gamma, [\delta, \bar\lambda]]]]$. Then the Jacobi identities give

$$\begin{aligned} 4A = (\gamma\beta\alpha\delta - \gamma\beta\delta\alpha) + (\gamma\alpha\beta\delta - \gamma\alpha\delta\beta) \\ + \delta\beta\alpha\gamma - \delta\beta\gamma\alpha + \delta\alpha\beta\gamma - \delta\alpha\gamma\beta + \beta\delta\alpha\gamma - \delta\beta\alpha\gamma \\ - \beta\delta\gamma\alpha + \delta\beta\gamma\alpha + \beta\gamma\alpha\delta - \gamma\beta\alpha\delta - \beta\gamma\delta\alpha + \gamma\beta\delta\alpha . \end{aligned}$$

In view of the identity $\gamma\alpha\beta\delta - \gamma\alpha\delta\beta = \beta\delta\gamma\alpha - \delta\beta\gamma\alpha$, obtained from $[\alpha, [\beta, \delta]] = 0$, we have

$$4A = -\gamma\beta\delta\alpha - \delta\beta\gamma\alpha + \beta\delta\alpha\gamma + \beta\gamma\alpha\delta .$$

Now, by (2.1) and the subsequent comment and the formula $B([\alpha, \beta], \gamma) = -B(\beta, [\alpha, \gamma])$, we have proposition. q.e.d.

Here we shall specialize Proposition 3.4 in the form

LEMMA 3.5. *Assume that there exists uniquely a vector* $\lambda \in \Delta^+(\Phi)$ *admitting a decomposition of the form* $\lambda = \sigma + \tau$ *where* $\sigma, \tau \in \Delta^+(\Phi)$. *Assume that* $(\sigma, \sigma) = (\tau, \tau)$ *for all* $\sigma, \tau \in \Delta^+(\Phi)$ *such that* $\lambda = \sigma + \tau$. *Then, for all* $\alpha, \beta, \gamma, \delta$ *such that* $2\lambda = \alpha + \beta + \gamma + \delta$, *the following holds*:

(3.5) $$B([\alpha, [\beta, \bar\lambda]], [\gamma, [\delta, \bar\lambda]]) = B([\beta, [\alpha, \bar\lambda]], [\delta, [\gamma, \bar\lambda]]) .$$

Proof. From the assumption on λ we have equivalences $\alpha + \beta \in \Delta \Leftrightarrow \alpha + \beta = \lambda \Leftrightarrow \gamma + \delta = \lambda \Leftrightarrow \gamma + \delta \in \Delta$. Denote the left hand side of (3.5) by $\alpha\beta\cdot\gamma\delta$. If $\alpha + \beta \notin \Delta$, then we see $\alpha\beta\cdot\gamma\delta = \beta\alpha\cdot\delta\gamma$ since $[\alpha, [\beta, \bar\lambda]] = [\beta, [\alpha, \bar\lambda]]$ by Jacobi identity. If $\alpha + \beta \in \Delta$, then $\alpha + \beta = \gamma + \delta = \lambda$. Putting $[\alpha, \beta] = N\lambda$ and $[\gamma, \delta] = M\lambda$, we see from (1.1) $[\beta, \bar\lambda] = N\bar\alpha$, $[\bar\lambda, \alpha] = N\bar\beta$, $[\delta, \bar\lambda] = M\bar\gamma$ and $[\bar\lambda, \gamma] = M\bar\delta$. It follows that

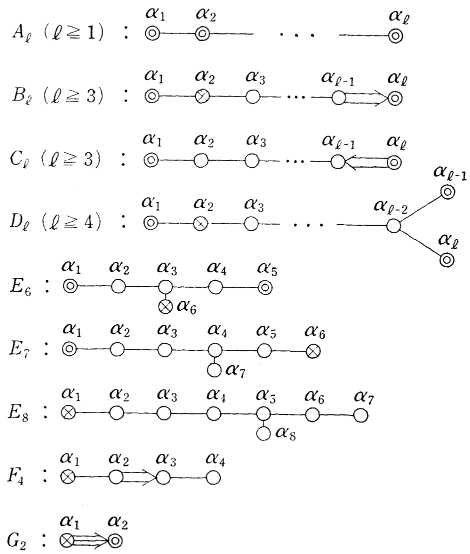
$$\begin{aligned} \alpha\beta\cdot\gamma\delta - \beta\alpha\cdot\delta\gamma &= NM(B(H_\alpha, H_\gamma) - B(H_\beta, H_\delta)) \\ &= NM((\alpha, \gamma) - (\beta, \delta)) . \end{aligned}$$

On the other hand, taking the norm of $\alpha - \gamma = \delta - \beta$, we have $(\alpha, \alpha) - 2(\alpha, \gamma) + (\gamma, \gamma) = (\delta, \delta) - 2(\delta, \beta) + (\beta, \beta)$. This and the assumption imply $(\alpha, \gamma) = (\delta, \beta)$. q.e.d.

§ 4. Theorems and proofs

In this section we shall state our results and prove them.

THEOREM 4.1. *Let α_a be any of the simple roots designed by the symbol $\otimes$ in the following Dynkin diagrams, and $\mathfrak{g}$ be a complex simple Lie algebra whose diagram contains α_a. Then the degree of the irreducible C-space $M(\mathfrak{g}, \alpha_a)$ corresponding to the pair $(\mathfrak{g}, \alpha_a)$ is equal to 2.*



(In the diagrams, the double circle $\alpha_a \circledcirc$ means that the corresponding C-space $M(\mathfrak{g}, \alpha_a)$ is Hermitian symmetric.)

In order to prove Theorem 4.1 and for later use, we shall state here a positive root system Δ^+ of each complex simple Lie algebra $\mathfrak{g}$, a fundamental root system $\alpha_1, \cdots, \alpha_\ell$ of Δ^+, and the subset $\Delta^+(\alpha_a)$ associated with a simple root α_a (cf. e.g. [3] or [4] appendix of [5]). For the five exceptional Lie algebras we omit $\Delta^+(\alpha_a)$ because the description is too complicated and we can do without them somehow.

A_ℓ $(\ell \geqq 1)$: A redundant orthonormal basis $\omega_1, \cdots, \omega_{\ell+1}$ with $\omega_1 + \cdots + \omega_{\ell+1} = 0$.

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$$\Delta^+ = \{\omega_i - \omega_j = \alpha_i + \cdots + \alpha_{j-1};\ 1 \leqq i < j \leqq \ell + 1\}\,.$$

For $1 \leqq a \leqq \ell$,

$$\Delta^+(\alpha_a) = \{\omega_i - \omega_j;\ i \leqq a \leqq j - 1\}\,.$$

B_ℓ ($\ell \geqq 2$): An orthonormal basis $\omega_1, \cdots, \omega_\ell$.

$$\begin{aligned}\Delta^+ = \{&\omega_i = \alpha_i + \cdots + \alpha_\ell;\ 1 \leqq i \leqq \ell\}\\ &\cup \{\omega_i - \omega_j = \alpha_i + \cdots + \alpha_{j-1};\ 1 \leqq i < j \leqq \ell\}\\ &\cup \{\omega_i + \omega_j = \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_\ell;\ 1 \leqq i < j \leqq \ell\}\,.\end{aligned}$$

For $1 \leqq a \leqq \ell$,

$$\begin{aligned}\Delta^+(\alpha_a) = \{\omega_i;\ i \leqq a\} &\cup \{\omega_i - \omega_j;\ i \leqq a \leqq j - 1\}\\ &\cup \{\omega_i + \omega_j;\ i \leqq a,\ i < j\}\,.\end{aligned}$$

C_ℓ ($\ell \geqq 3$): An orthonormal basis $\omega_1, \cdots, \omega_\ell$.

$$\begin{aligned}\Delta^+ = \{&\omega_i = 2\alpha_i + \cdots + 2\alpha_{\ell-1} + \alpha_\ell;\ 1 \leqq i \leqq \ell\}\\ &\cup \{\omega_i - \omega_j = \alpha_i + \cdots + \alpha_{j-1};\ 1 \leqq i < j \leqq \ell\}\\ &\cup \{\omega_i + \omega_j = \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{\ell-1} + \alpha_\ell;\\ &\qquad\qquad 1 \leqq i < j \leqq \ell\}\,.\end{aligned}$$

For $1 \leqq a \leqq \ell$,

$$\begin{aligned}\Delta^+(\alpha_a) = \{\omega_i;\ i \leqq a\} &\cup \{\omega_i - \omega_j;\ i \leqq a \leqq j - 1\}\\ &\cup \{\omega_i + \omega_j;\ i \leqq a,\ i < j\}\,.\end{aligned}$$

D_ℓ ($\ell \geqq 4$): An orthonormal basis $\omega_1, \cdots, \omega_\ell$.

$$\begin{aligned}\Delta^+ = \{&\omega_i - \omega_j = \alpha_i + \cdots + \alpha_{j-1};\ 1 \leqq i < j \leqq \ell\}\\ &\cup \{\omega_i + \omega_j = \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell;\\ &\qquad\qquad 1 \leqq i < j \leqq \ell - 1\}\\ &\cup \{\omega_i + \omega_\ell = \alpha_i + \cdots + \alpha_{\ell-2} + \alpha_\ell;\ 1 \leqq i \leqq \ell - 1\}\,.\end{aligned}$$

For $1 \leqq a \leqq \ell - 2$,

$$\begin{aligned}\Delta^+(\alpha_a) = \{&\omega_i - \omega_j;\ i \leqq a \leqq j - 1\}\\ &\cup \{\omega_i + \omega_j;\ i \leqq a,\ i < j\}\,.\end{aligned}$$

E_6: A basis $\omega_1, \cdots, \omega_6$ with $(\omega_i, \omega_i) = 8$ and $(\omega_i, \omega_j) = -1$ for $i \neq j$.

$$\begin{aligned}\Delta^+ = \{&\omega_i - \omega_j;\ 1 \leqq i < j \leqq 6\}\\ &\cup \{\omega_i + \omega_j + \omega_k;\ 1 \leqq i < j < k \leqq 6\} \cup \{\omega_1 + \cdots + \omega_6\}\,.\end{aligned}$$

$$\alpha_i = \omega_i - \omega_{i+1}\ (1 \leqq i \leqq 5) \quad \text{and} \quad \alpha_6 = \omega_4 + \omega_5 + \omega_6\,.$$

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E_7: A basis $\omega_1, \cdots, \omega_7$ with $(\omega_i, \omega_i) = 8$ and $(\omega_i, \omega_j) = -1$ for $i \neq j$.

$$\begin{aligned} \Delta^+ = &\{\omega_i - \omega_j;\ 1 \leqq i < j \leqq 7\} \\ &\cup \{\omega_i + \omega_j + \omega_k;\ 1 \leqq i < j < k \leqq 7\} \\ &\cup \{\omega_1 + \cdots + \hat{\omega}_i + \cdots + \omega_7;\ 1 \leqq i \leqq 7\} \end{aligned}$$
$$\alpha_i = \omega_i - \omega_{i+1}\ (1 \leqq i \leqq 6) \quad \text{and} \quad \alpha_7 = \omega_5 + \omega_6 + \omega_7 .$$

E_8: A basis $\omega_1, \cdots, \omega_8$ with $(\omega_i, \omega_i) = 8$ and $(\omega_i, \omega_j) = -1$ for $i \neq j$.

$$\begin{aligned} \Delta^+ = &\{\omega_i - \omega_j;\ 1 \leqq i < j \leqq 8\} \\ &\cup \{\omega_i + \omega_j + \omega_k;\ 1 \leqq i < j < k \leqq 8\} \\ &\cup \{\omega_1 + \cdots + \hat{\omega}_i + \cdots + \hat{\omega}_j + \cdots + \omega_8;\ 1 \leqq i < j \leqq 8\} \\ &\cup \{\omega_1 + \cdots + \omega_{i-1} + 2\omega_i + \omega_{i+1} + \cdots + \omega_8;\ 1 \leqq i \leqq 8\} \end{aligned}$$
$$\alpha_i = \omega_i - \omega_{i+1}\ (1 \leqq i \leqq 7) \quad \text{and} \quad \alpha_8 = \omega_6 + \omega_7 + \omega_8 .$$

F_4: An orthonormal basis $\omega_1, \omega_2, \omega_3, \omega_4$.

$$\begin{aligned} \Delta^+ = &\{\omega_i \pm \omega_j;\ 1 \leqq i < j \leqq 4\} \\ &\cup \{\omega_i;\ 1 \leqq i \leqq 4\} \\ &\cup \{\tfrac{1}{2}\omega_1 \pm \tfrac{1}{2}\omega_2 \pm \tfrac{1}{2}\omega_3 \pm \tfrac{1}{2}\omega_4;\ \text{independent signs}\} \end{aligned}$$
$$\alpha_1 = \omega_2 - \omega_3,\ \alpha_2 = \omega_3 - \omega_4,\ \alpha_3 = \omega_4 \quad \text{and} \quad \alpha_4 = \tfrac{1}{2}(\omega_1 - \omega_2 - \omega_3 - \omega_4) .$$

G_2: A redundant basis $\omega_1, \omega_2, \omega_3$ with $\omega_1 + \omega_2 + \omega_3 = 0$, $(\omega_i, \omega_i) = 2$ and $(\omega_i, \omega_j) = -1$ for $i \neq j$.

$$\Delta^+ = \{\omega_i;\ 1 \leqq i \leqq 3\} \cup \{\omega_i - \omega_j;\ 1 \leqq i < j \leqq 3\}$$
$$\alpha_1 = \omega_1 - \omega_2, \qquad \alpha_2 = -\omega_1 .$$

Proof of Theorem 4.1. We use Proposition 3.4 and Lemma 3.5. First, we assert that $\Delta^+(\alpha_a)$ satisfies the condition of Lemma 3.5. In fact, when $\mathfrak{g} = E_\ell\ (\ell = 6, 7, 8)$, it is trivial since all roots have the same length. In other cases, let λ be the highest root in Δ. Then $\lambda \in \Delta^+(\alpha_a)$. Furthermore, it can be easily checked that $p_\lambda = 2$ and $p_\varepsilon = 1$ for any $\varepsilon \in \Delta^+(\alpha_a) - \{\lambda\}$. Here, we write out all possible decompositions $\lambda = \alpha + \beta$, $\alpha, \beta \in \Delta^+(\alpha_a)$ of λ for $\mathfrak{g} = B_\ell, D_\ell, F_4$ and G_2.

$$B_\ell\ (\ell \geqq 3):\quad \lambda = \omega_1 + \omega_2 = (\omega_1 \pm \omega_j) + (\omega_2 \mp \omega_j) \qquad (3 \leqq j \leqq \ell) .$$
$$D_\ell\ (\ell \geqq 4):\quad \lambda = \omega_1 + \omega_2 = (\omega_1 \pm \omega_j) + (\omega_2 \mp \omega_j) \qquad (3 \leqq j \leqq \ell) .$$
$$\begin{aligned} F_4:\quad \lambda &= \omega_1 + \omega_2 = (\omega_1 \pm \omega_j) + (\omega_2 \mp \omega_j) \qquad (j = 3, 4) \\ &= \tfrac{1}{2}(\omega_1 + \omega_2 \pm \omega_3 \pm \omega_4) + \tfrac{1}{2}(\omega_1 + \omega_2 \mp \omega_3 \mp \omega_4) . \end{aligned}$$
$$G_2:\quad \lambda = \omega_3 - \omega_2 = (\omega_3 - \omega_1) + (\omega_1 - \omega_2) .$$

As a result, we find $(\alpha, \alpha) = (\beta, \beta)$ in each case, which proves our assertion.

Next, we assert that $R(\alpha, \bar{\varepsilon}, \beta; \gamma, \delta) = 0$ for any $\alpha, \beta, \gamma, \delta, \varepsilon \in \Delta^+(\alpha_a)$. To show this, put $\mu = \alpha + \beta + \gamma + \delta - \varepsilon$. If $\mu \notin \Delta$, then Lemma 2.3 proves our assertion. So, assume $\mu \in \Delta$. In view of the fact $p_\nu = 1$ or 2 for any ν, we have $\mu = \varepsilon = \lambda$ and $p_\alpha = p_\beta = p_\gamma = p_\delta = 1$. Then it can be easily seen that our five vectors $\lambda, \alpha, \beta, \gamma$ and δ satisfy the condition of Proposition 3.4. This, together with Lemma 3.5, completes the proof of the Theorem. q.e.d.

Along the same line, we can prove

THEOREM 4.2. *The degree of an irreducible C-space $M(A_\ell, \alpha_1, \alpha_\ell)$ $(\ell \geqq 2)$ is equal to* 2.

Proof. Since $\Delta^+(\Phi) = \{\alpha_1 + \cdots + \alpha_i; 1 \leqq i \leqq \ell\} \cup \{\alpha_j + \cdots + \alpha_\ell; 1 \leqq j \leqq \ell\}$, the highest root $\lambda = \omega_1 - \omega_{\ell+1} = \alpha_1 + \cdots + \alpha_\ell$ is the only vector in $\Delta^+(\Phi)$ admitting a decomposition of the form $\lambda = \varepsilon_1 + \varepsilon_2$ where $\varepsilon_1, \varepsilon_2 \in \Delta^+(\Phi)$. Moreover, all elements of Δ have the same length. Thus λ satisfies the condition of Lemma 3.5.

In view of the fact that $p_\lambda = (1, 1)$ and $p_\alpha = (1, 0)$ or $(0, 1)$ when $\alpha \in \Delta^+(\Phi) - \{\lambda\}$, we see that, if six vectors $\alpha, \beta, \gamma, \delta, \varepsilon, \mu \in \Delta^+(\Phi)$ satisfy $\varepsilon + \mu = \alpha + \beta + \gamma + \delta$, then $\varepsilon = \mu = \lambda$ and the sum of two of $\alpha, \beta, \gamma, \delta$ must be equal to λ, and hence $\lambda, \alpha, \beta, \gamma$ and δ satisfy the condition of Proposition 3.4. Now, Proposition 3.4 and Lemma 3.4 imply $R(\alpha, \bar{\lambda}, \beta; \gamma, \delta) = 0$. q.e.d.

In the following, in order to prove that the C-spaces of eight types given in Theorems 4.1 and 4.2 exhaust all irreducible C-spaces with degree two, we shall give a sufficient condition for a C-space $M(\mathfrak{g}, \Phi)$ to satisfy $\overset{+}{\nabla}{}^2 R \neq 0$.

LEMMA 4.3. *Let $\alpha, \beta, \gamma \in \Delta^+(\Phi)$ satisfy*

(1) $\alpha + \beta \in \Delta$,

(2) $\beta - \gamma \in \Delta^+$,

(3) $\beta + \gamma \notin \Delta$,

(4) $\alpha + \beta - \gamma \notin \Delta$,

(5) $\alpha + \beta + \gamma \notin \Delta$. *Then,*

$$R(\alpha, \bar{\lambda}, \beta; \gamma, \alpha) = -\nabla_\gamma R(\alpha, \bar{\lambda}, \nabla_\alpha \beta) - \nabla_\alpha R(\nabla_\gamma \alpha, \bar{\lambda}, \beta) + R(\nabla_{\nabla_\alpha \gamma} \alpha, \bar{\lambda}, \beta) + R(\nabla_\gamma \alpha, \bar{\lambda}, \nabla_\alpha \beta), \tag{4.1}$$

where we put $\lambda = \alpha + \beta$.

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Proof. Put $\delta = \alpha$ in (3.1). Then, by (2.8), all terms except for four ones in (4.1) vanish since $R(\alpha, \bar{\lambda}, \beta) = 0$ by Lemma 3.2, $\nabla_\alpha \alpha = \nabla_\alpha \bar{\lambda} = 0$ by Lemma 3.1, $\nabla_{\nabla_\alpha \gamma} \bar{\lambda} = 0$ by (2), $\nabla_\gamma \beta = 0$ by (3), $\nabla_\gamma \bar{\lambda} = 0$ by (4), and $\nabla_{\nabla_\alpha \gamma} \beta = \nabla_\gamma \nabla_\alpha \beta = 0$ by (5). q.e.d.

LEMMA 4.4. *Let $\alpha, \beta, \gamma \in \Delta^+(\Phi)$ satisfy* (1) ~ (5) *in Lemma* 4.3 *and in addition the following* (6)′, (7) *and* (8).

(6)′ $2\alpha + \gamma \notin \Delta$ *or* $\alpha + \gamma - \beta \notin \Delta^+(\Phi)$,

(7) $2\alpha + \beta \notin \Delta$,

(8) $2\alpha + \beta + \gamma \notin \Delta$. *Then,*

$$R(\alpha, \bar{\lambda}, \beta; \gamma, \alpha) = b(b+1)[\alpha, [[\alpha, \gamma], \bar{\lambda}], \beta] - b[[\nabla_{[\alpha,\gamma]}\alpha, \bar{\lambda}], \beta]\,,$$

where we put $\lambda = \alpha + \beta$ and $b = -c\cdot p_\alpha / c\cdot p_{\alpha+\gamma}$.

Proof. We shall deform the right hand side of (4.1) using (2.9) and Lemma 3.1;

$$\begin{aligned}
R(\alpha, \bar{\lambda})\nabla_\alpha \beta &= -A([\alpha, \bar{\lambda}])\nabla_\alpha \beta && \text{(by (7))} \\
&= -[[\alpha, \bar{\lambda}], \nabla_\alpha \beta] && \text{(by (2.7))}\,, \\
R(\nabla_\alpha \alpha, \bar{\lambda})\beta &= -A([\nabla_\gamma \alpha, \bar{\lambda}]_{\mathfrak{m}})\beta \\
&\quad - [[\nabla_\gamma \alpha, \bar{\lambda}]_{\mathfrak{t}}, \beta] && \text{(by (2) and (8))} \\
&= -[[\nabla_\gamma \alpha, \bar{\lambda}], \beta] && \text{(by (2.7))}\,, \\
R(\nabla_\gamma \alpha, \bar{\lambda})\nabla_\alpha \beta &= -A([\nabla_\gamma \alpha, \bar{\lambda}])\nabla_\alpha \beta \\
&\quad - [[\nabla_\gamma \alpha, \bar{\lambda}]_{\mathfrak{t}}, \beta] && \text{(by (2) and (8))} \\
&= -[[\nabla_\gamma \alpha, \bar{\lambda}], \nabla_\alpha \beta] && \text{(by (2.7))}\,, \\
R(\nabla_{\nabla_\alpha \gamma} \alpha, \bar{\lambda})\beta &= -A([\nabla_{\nabla_\alpha \gamma} \alpha, \bar{\lambda}]_{\mathfrak{m}})\beta \\
&\quad - [[\nabla_{\nabla_\alpha \gamma} \alpha, \bar{\lambda}]_{\mathfrak{t}}, \beta] && \text{(by (8))} \\
&= -[[\nabla_{\nabla_\alpha \gamma} \alpha, \bar{\lambda}], \beta] && \text{(by (6) and (2.7))}\,.
\end{aligned}$$

Here, put $a = -c\cdot p_\alpha / c\cdot p_{\alpha+\beta}$. Then $\nabla_\alpha \beta = a\,[\alpha, \beta]$ and $\nabla_\beta \alpha = (a+1)[\alpha, \beta]$ by (2.8). On the other hand, since $\nabla_\gamma \alpha = (b+1)[\alpha, \gamma]$, we have

$$\begin{aligned}
R(\alpha, \bar{\lambda}, \beta; \gamma, \alpha) = a(b+1)&[[[\alpha, \bar{\lambda}], [\alpha, \beta]], \gamma] \\
&+ b(b+1)[\alpha, [[[\alpha, \gamma], \bar{\lambda}], \beta] \\
&- a(b+1)[[[\alpha, \gamma], \bar{\lambda}], [\alpha, \beta]] \\
&- [[\nabla_{\nabla_\alpha \gamma} \alpha, \bar{\lambda}], \beta]\,.
\end{aligned}$$

The sum of the first and third terms vanishes because of (4), (5) and Jacobi identity. q.e.d.

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LEMMA 4.5. *Let* $\alpha, \beta, \gamma \in \Delta^+(\Phi)$ *satisfy* (1) ~ (5) *in Lemma* 4.3 *and* (7), (8) *in Lemma* 4.4 *and in addition the following* (6), (9)

(6) $2\alpha + \gamma \notin \Delta$,

(9) $\alpha + \gamma \in \Delta$. *Then,*

$$R(\alpha, \overline{\alpha + \beta}, \beta; \gamma, \alpha) = b(b+1)[\alpha, [[[\alpha, \gamma], \overline{\alpha + \beta}], \beta]] \neq 0 .$$

Proof. This is obtained from Lemma 4.4 and the fact that $\alpha + \gamma \in \Delta$, $\gamma - \beta \in \Delta$ and $b(b+1) \neq 0$. q.e.d.

Remark 4.6. (1) If $\alpha + \beta$ is the highest root in Δ, then the conditions (5), (7) and (8) are necessarily satisfied. (2) Let Φ' be another subset of a fundamental root system $\{\alpha_1, \cdots, \alpha_\ell\}$ of $\mathfrak{g}$, and R' denote the curvature tensor of any Kählerian metric of the form (1.9) on a C-space $M(\mathfrak{g}, \Phi')$. Assume that $\Phi \subset \Phi'$ and $\alpha, \beta, \gamma \in \Delta^+(\Phi)$ satisfy the condition of Lemma 4.5. Then Lemma 4.5 implies

$$R(\alpha, \overline{\alpha + \beta}, \beta; \gamma, \alpha) = R'(\alpha, \overline{\alpha + \beta}, \beta; \gamma, \alpha) .$$

THEOREM 4.7. *Let* $\mathfrak{g}$ *be a complex simplex Lie algebra and* Φ *be a non-empty subset of a fundamental root system of* $\mathfrak{g}$. *Assume that the C-space* $M(\mathfrak{g}, \Phi)$ *corresponding to the pair* $(\mathfrak{g}, \Phi)$ *is neither a Hermitian symmetric space nor any of the C-space of eight types given in Theorems* 4.1 *and* 4.2. *Then, the degree* d *of the C-space* $M(\mathfrak{g}, \Phi)$ *is not smaller than* 3. *If* Φ *consists of a single element* α_a *and the coefficient of* α_a *in the highest root in* Δ *is equal to* 2, *then* $d = 3$.

Proof. Throughout the proof, let α_a stand for any of the simple roots designed by ○ in the diagrams of Theorem 4.1 (thus not a simple root designed by ◎ or ⊗). We divide the proof into four parts;

(I) Case where $\Phi = \{\alpha_a\}$. We shall show $d = 3$. Let λ be the highest root in Δ. Then, since $p_\lambda = 2$, we have $d \leqq 3$ by Corollary 3.4. Thus, by Corollary 4.5 it suffices to find three roots α, β and γ in $\Delta^+(\alpha_a)$ satisfying nine conditions (1) ~ (9) in Lemmas 3.3, 3.4 and 3.5. In the following we state examples of such roots α, β and γ.

(A) For $\mathfrak{g} = B_\ell$ ($\ell \geqq 4$, $3 \leqq a < \ell - 1$) or D_ℓ ($\ell \geqq 5$, $3 \leqq a \leqq \ell - 1$), put $\alpha = \omega_1 + \omega_\ell$, $\beta = \omega_2 - \omega_\ell$ and $\gamma = \omega_a - \omega_\ell$.

(B) For $\mathfrak{g} = C_\ell$ ($\ell \geqq 3$, $2 \leqq a \leqq \ell - 1$), put $\alpha = \omega_1 + \omega_\ell$, $\beta = \omega_1 - \omega_\ell$ and $\gamma = \omega_a - \omega_\ell$.

(C) For $\mathfrak{g} = E_6$, put

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$$\alpha = \begin{pmatrix} 0 & 1 & 2 & 1 & 1 \\ & & 1 & & \end{pmatrix} = \omega_2 + \omega_3 + \omega_5 ,$$

$$\beta = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ & & 1 & & \end{pmatrix} = \omega_1 + \omega_4 + \omega_6 ,$$

and

$$\gamma = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ & & 0 & & \end{pmatrix} = \omega_1 - \omega_5 ,$$

where $\begin{pmatrix} n_1 & n_2 & n_3 & n_4 & n_5 \\ & & n_6 & & \end{pmatrix}$ means a root $n_1\alpha_1 + \cdots + n_6\alpha_6$. *Hereafter we use the similar notation.*

(D) For $\mathfrak{g} = E_7$, put

$$\alpha = \begin{pmatrix} 0 & 1 & 1 & 2 & 1 & 1 \\ & & & 1 & & \end{pmatrix} = \omega_2 + \omega_4 + \omega_6 ,$$

$$\beta = \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 1 \\ & & & 1 & & \end{pmatrix} = \omega_1 + \omega_3 + \omega_5 ,$$

and

$$\gamma = \begin{pmatrix} 1 & 1 & 2 & 2 & 1 & 0 \\ & & & 1 & & \end{pmatrix} = \omega_1 + \omega_3 + \omega_7 .$$

(E) For $\mathfrak{g} = E_8$, put

$$\alpha = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & & & & 1 & & \end{pmatrix} = \omega_1 + \omega_6 + \omega_7 ,$$

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 3 & 1 \\ & & & & 2 & & \end{pmatrix} = \omega_1 + \cdots + \omega_5 + \omega_8 ,$$

and

$$\gamma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 2 & 1 \\ & & & & 2 & & \end{pmatrix} = \omega_2 + \cdots + \omega_5 + \omega_7 + \omega_8 .$$

(F) For $\mathfrak{g} = F_4$, put

$$\alpha = (1 \quad 1 \Rightarrow 1 \quad 1) ,$$
$$\beta = (1 \quad 2 \Rightarrow 3 \quad 1) ,$$

and

$$\gamma = (0 \quad 1 \Rightarrow 2 \quad 1) .$$

(II) Case where $\alpha_a \in \Phi$. By Lemma 4.5 and Remark 4.6 (2) we have $R(\alpha, \overline{\alpha + \beta}, \beta; \gamma, \alpha) \neq 0$ for α, β and γ just given in the case (I).

(III) Case where Φ consists of only simple roots designed by ◎ or ⊗, and where $\mathfrak{g} \neq B_2$. We shall describe all possible pairs $(\mathfrak{g}, \Phi)$ and examples of three roots α, β and γ in $\Delta^+(\Phi)$ satisfying nine conditions (1) $\sim$ (9).

(A) For $(A_\ell, \alpha_r, \alpha_s)$ $(\ell \geqq 3, 1 \leqq r < s < \ell + 1)$, put $\alpha = \omega_1 - \omega_{r+1}$, $\beta = \omega_{r+1} - \omega_{\ell+1}$ and $\gamma = \omega_{r+1} - \omega_{s+1}$.

(B) For $(B_\ell, \alpha_1, \alpha_2)$ $(\ell \geqq 3)$ or $(D_\ell, \alpha_1, \alpha_2)$ $(\ell \geqq 4)$, put $\alpha = \omega_2 - \omega_3$, $\beta = \omega_1 + \omega_3$ and $\gamma = \omega_1 - \omega_2$.

(C) For $(B_\ell, \alpha_1, \alpha_\ell)$ $(\ell \geqq 3)$ or $(B_\ell, \alpha_2, \alpha_\ell)$ $(\ell \geqq 3)$, put $\alpha = \omega_1 - \omega_\ell$, $\beta = \omega_2 + \omega_\ell$ and $\gamma = \omega_\ell$.

(D) For $(D_\ell, \alpha_1, \alpha_{\ell-1})$ $(\ell \geqq 4)$ or $(D_\ell, \alpha_2, \alpha_\ell)$ $(\ell \geqq 4)$, put $\alpha = \omega_1 + \omega_\ell$, $\beta = \omega_2 - \omega_\ell$ and $\gamma = \omega_a - \omega_\ell$.

(E) For $(D_\ell, \alpha_{\ell-1}, \alpha_\ell)$ $(\ell \geqq 4)$, put $\alpha = \omega_{\ell-3} - \omega_\ell$, $\beta = \omega_{\ell-2} + \omega_\ell$ and $\gamma = \omega_{\ell-1} + \omega_\ell$.

(F) For $(C_\ell, \alpha_1, \alpha_\ell)$ $(\ell \geqq 3)$, put $\alpha = \omega_1 - \omega_\ell$, $\beta = \omega_1 + \omega_\ell$ and $\gamma = \omega_2 + \omega_\ell$.

(G) For $(E_6, \alpha_1, \alpha_5)$ or $(E_6, \alpha_1, \alpha_6)$, take the same α, β and γ as in the subcase (C) of the case (I). By the symmetry of the Dynkin diagram of E_6, we can find similar three roots for $(E_6, \alpha_5, \alpha_6)$.

(H) For $(E_7, \alpha_1, \alpha_6)$, take the same α, β and γ as in the subcase (D) of the case (I).

(IV) Case where $\mathfrak{g} = B_2$ and $\Phi = \{\alpha_1, \alpha_2\}$. Put $\alpha = \omega_2$, $\beta = \omega_1$, $\gamma = \omega_1 - \omega_2$ and $\delta = \omega_1 + \omega_2$. Then we can take them in such a way that $[\alpha, \gamma] = \beta$ and $[\alpha, \beta] = \delta$, and so $[\bar{\alpha}, \beta] = -\gamma$, $[\bar{\delta}, \beta] = \bar{\alpha}$ and $[\bar{\alpha}, \delta] = \beta$ by (1.1) (cf. e.g. [4], p. 277). On the other hand, since three roots α, β and γ satisfy the condition of Lemma 4.4, we have

$$R(\alpha, \bar{\delta}, \beta; \gamma, \alpha) = b(b+1)[\alpha, [[[\alpha, \gamma], \bar{\delta}], \beta]] - b[[\nabla_\beta \alpha, \bar{\delta}], \beta] .$$

Here, from (2.7) we have $\nabla_\beta \alpha = a[\beta, \alpha] = a\delta$, where $a = -c \cdot p_\beta / c \cdot p_{\beta+\alpha}$. Hence,

$$\begin{aligned} R(\alpha, \bar{\delta}, \beta; \gamma, \alpha) &= b(b+1)[\alpha, [[\beta, \bar{\delta}], \beta]] - ab[[\delta, \bar{\delta}], \beta] \\ &= b(a+b+1)\beta . \end{aligned}$$

But, in view of $p_\alpha = (0, 1)$, $p_\beta = (1, 1)$, $p_\gamma = (1, 0)$ and $c = (c_1, c_2)$, we have $a + b + 1 = -c_2^2/(c_1 + c_2)(c_1 + 2c_2) \neq 0$, which completes the proof of Theorem 4.7. q.e.d.

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