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TATE DUALITY AND TRANSFER FOR SYMMETRIC ALGEBRAS OVER COMPLETE DISCRETE VALUATION RINGS

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Abstract We show that dualising transfer maps in Hochschild cohomology of symmetric algebras over complete discrete valuations rings commutes with Tate duality. This is analogous to a similar result for Tate cohomology of symmetric algebras over fields. We interpret both results in the broader context of Calabi–Yau triangulated categories.

Keywords: symmetric algebra; transfer; Tate duality

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1. Introduction

An algebra A over a commutative ring R is called *symmetric* if it is finitely generated projective as an R-module and if $A \cong A^\vee$ as A-A-bimodules, where $A^\vee = \mathrm{Hom}_R(A, R)$. In that case, the image s in $A^\vee$ of 1_A under such a bimodule isomorphism is called a *symmetrising form for* A. The form s depends on the choice of the isomorphism $A \cong A^\vee$ and is unique up to multiplication by an invertible element in $Z(A)$. There may not be a canonical choice for s. If G is a finite group, then RG is symmetric and – keeping track of the image of G in RG – does have a canonical symmetrising form, namely the map s sending 1_G to 1_R and all non-trivial group elements to 0.

For a symmetric algebra A over a field, Tate duality is a duality between the Tate-Ext spaces $\widehat{\mathrm{Ext}}_A^{n-1}(U, V)$ and $\widehat{\mathrm{Ext}}_A^{-n}(V, U)$, reviewed in § 2, for any integer n and any two finite-dimensional A-modules U, V. In particular, this yields a duality between Tate–Hochschild cohomology $\widehat{HH}^{n-1}(A)$ and $\widehat{HH}^{-n}(A)$ for all integers n. It is shown in [21] that in that case Tate duality commutes with the transfer maps introduced in [19], extending a well-known compatibility of Tate duality with restriction and transfer in finite group cohomology.

If A is instead a symmetric algebra over a complete discrete valuation ring $\mathcal{O}$ with a separable coefficient extension $K \otimes_{\mathcal{O}} A$ to the field of fractions K of $\mathcal{O}$, and if U, V are $\mathcal{O}$-free finitely generated A-modules, then Tate duality takes a different form: there is a non-degenerate bilinear form

$$\langle -, - \rangle_A : \widehat{\mathrm{Ext}}_A^n(U, V) \times \widehat{\mathrm{Ext}}_A^{-n}(V, U) \to K/\mathcal{O}$$

for any integer n, which is described explicitly in [9] and briefly reviewed in Equation (7.3). The purpose of this paper is to show that this duality commutes with the transfer maps from [19]. The proof is quite different from that in [21] due to the different description of Tate duality, as given in [9], extending the description in Thévenaz [25, § 33] for finite group algebras. Both this duality as well as the transfer maps depend on the choices of symmetrising forms. By omitting choices of symmetrising forms from the statements below we implicitly assert that these statements hold regardless of these choices.

If A, B are two symmetric $\mathcal{O}$-algebras and M is an A-B-bimodule which is finitely generated projective as a left A-module and as a right B-module, then, for any two finitely generated $\mathcal{O}$-free A-modules U, V, there is a transfer map

$$\mathrm{tr}_M(U, V) : \widehat{\mathrm{Ext}}_B^n(M^\vee \otimes_A U, M^\vee \otimes_A V) \to \widehat{\mathrm{Ext}}_A^n(U, V),$$

which we will review in § 4. In degree zero, this is the trace map defined in [24, (57)], [4] or [5, Definition 6.6]. For transfer induced by biadjoint functors between more general categories see [7], and for the graded version needed in this paper see [20, § 4, § 7] or also [21, § 5], for instance. For simplicity, we will write tr_M instead of $\mathrm{tr}_M(U, V)$ whenever U, V are clear from the context.

Theorem 1.1. *Let A, B be symmetric algebras over a complete discrete valuation ring $\mathcal{O}$ with a field of fractions K of characteristic zero. Suppose that $K \otimes_{\mathcal{O}} A$, $K \otimes_{\mathcal{O}} B$ are semisimple. Let M be an A-B-bimodule which is finitely generated projective as a left A-module and as a right B-module. Let n be an integer, and let U, V be finitely generated $\mathcal{O}$-free A-modules. For any $\alpha \in \widehat{\mathrm{Ext}}_A^n(U, V)$ and $\beta \in \widehat{\mathrm{Ext}}_B^{-n}(M^\vee \otimes_A U, M^\vee \otimes_A V)$, we have*

$$\langle \alpha, \mathrm{tr}_M(\beta) \rangle_A = \langle \mathrm{Id}_{M^\vee} \otimes \alpha, \beta \rangle_B,$$

$$\langle \mathrm{tr}_M(\beta), \alpha \rangle_A = \langle \beta, \mathrm{Id}_{M^\vee} \otimes \alpha \rangle_B.$$

Remark 1.2. One consequence of Theorem 1.1 is that Tate duality determines the transfer maps $\mathrm{tr}_M(U, V)$. This comment applies also to the transfer maps in [21, Theorem 1.2] and Tate duality for symmetric k-algebras. This has an interpretation in the context of Calabi–Yau triangulated categories, which we will describe in § 9.

Tate-Ext applied to A as a module over the symmetric algebra $A^e = A \otimes_{\mathcal{O}} A^{\mathrm{op}}$ yields Tate–Hochschild cohomology $\widehat{HH}^*(A)$ (see Equation (2.2)). Tate duality applied to this situation yields in turn a non-degenerate bilinear form

$$\langle -, - \rangle_{A^e} : \widehat{HH}^n(A) \times \widehat{HH}^{-n}(A) \to K/\mathcal{O};$$

see for instance [9, Remark 1.5]. By [19, Definition 2.9] or the more general construction principle [20, § 4, § 7] specialised to stable categories of bimodules, the A-B-bimodule M

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as above induces a transfer map

$$\mathrm{tr}_M : \widehat{HH}^*(B) \to \widehat{HH}^*(A),$$

that we will review in § 4. The dual $M^\vee$ with respect to the base ring is a B-A-bimodule which is finitely generated projective as a left B-module and as a right A-module, hence induces a transfer map $\mathrm{tr}_{M^\vee} : \widehat{HH}^*(A) \to \widehat{HH}^*(B)$. There is some abuse of notation: the transfer map tr_M in Tate–Hochschild cohomology is not quite a special case of the transfer maps $\mathrm{tr}_M(U, V)$ defined previously; their precise relationship is described in Remark 4.13. The compatibility between transfer and Tate duality for Tate–Hochschild cohomology takes the following form.

Theorem 1.3. *Let A, B be symmetric algebras over a complete discrete valuation ring $\mathcal{O}$ with a field of fractions K of characteristic zero. Suppose that $K \otimes_{\mathcal{O}} A$, $K \otimes_{\mathcal{O}} B$ are semisimple. Let M be an A-B-bimodule which is finitely generated projective as a left A-module and as a right B-module. Let n be an integer. For $\zeta \in \widehat{HH}^n(A)$ and $\tau \in \widehat{HH}^{-n}(B)$ we have*

$$\langle \zeta, \mathrm{tr}_M(\tau) \rangle_{A^e} = \langle \mathrm{tr}_{M^\vee}(\zeta), \tau \rangle_{B^e},$$

$$\langle \mathrm{tr}_M(\tau), \zeta \rangle_{A^e} = \langle \tau, \mathrm{tr}_{M^\vee}(\zeta) \rangle_{B^e}.$$

Remark 1.4. In view of the interpretation of Theorem 1.1 in terms of Calabi–Yau triangulated categories (which we will describe in § 9), it is worth noting that there are finite-dimensional algebras which are selfinjective, not necessarily symmetric, but whose stable category is Calabi–Yau of non-negative dimension. See for instance [10], [8], [14], [15] and the references therein. We further draw attention to the appendix in [26] by M. Van den Bergh regarding signs in Serre duality. We largely ignore sign issues in § 4 (notably in Equation (4.5)) because this will not needed for the results of this paper, but would be needed for an in-depth interpretation of these results in terms of Calabi–Yau duality.

Remark 1.5. The main motivation for developing this material is to extend results on finite group cohomology to symmetric algebras, in order to provide techniques to calculate cohomological invariants, such as the Castelnuovo–Mumford regularity, that might distinguish classes of symmetric algebras from being block algebras of finite groups (see [17] for calculations in this context). One such distinguishing feature (and necessary tool for calculations of the regularity) is the existence of a local cohomology spectral sequence for Hochschild cohomology, analogous to Greenlees' local cohomology spectral sequence in [13]. Benson's approach to this spectral sequence in [1] makes use of the compatibility of restriction and transfer in group cohomology with respect to Tate duality over a field. We expect that this compatibility at the level of Hochschild cohomology, both in [21] for algebras over fields, and the present paper for algebras over complete discrete valuation rings, will be one of the technical ingredients towards this programme.

Remark 1.6. Unlike in the Tate duality for symmetric algebras over fields, there is no degree shift in the Tate duality for symmetric algebras over a complete discrete valuation

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ring $\mathcal{O}$ with field of fractions K. This is due to the fact that we have replaced duality with respect to the base ring $\mathcal{O}$ by duality with respect to the injective syzygy $K/\mathcal{O}$ of $\mathcal{O}$, noting that we have a short exact sequence $0 \to \mathcal{O} \to K \to K/\mathcal{O} \to 0$ in which both K and $K/\mathcal{O}$ are injective $\mathcal{O}$-modules, with $K/\mathcal{O}$ in degree 1 of this injective resolution of $\mathcal{O}$. For Tate cohomology over more general rings, see Buchweitz [6]. Further extensions of Tate cohomology can be found, for instance, in [11], [12].

2. Preliminaries

We will use without further reference well-known basic material on stable module categories of symmetric algebras, see e.g. [22, § 2.13]. We briefly review the main properties of shift functors on stable module categories for symmetric algebra, mainly to adopt some notational abuse for simplicity of exposition later on.

Let R be a commutative Noetherian ring with unit element. Let A be an R-algebra. An A-module U is called *relatively R-projective* if the canonical surjection of A-modules $A \otimes_R U \to U$ sending $a \otimes u$ to au is split, and U is called *relatively R-injective* if the canonical injection of A-modules $U \to \mathrm{Hom}_R(A, U)$, $u \mapsto (a \mapsto au)$ is split (where $u \in U$ and $a \in A$), see [22, § 2.6] for details.

Assume now that A is a symmetric R-algebra. Then the two classes of relatively R-projective and relatively R-injective modules coincide (cf. [22, Theorem 2.15.1]). We denote by $\underline{\mathrm{mod}}(A)$ the relatively R-stable category of finitely generated A-modules. The objects of $\underline{\mathrm{mod}}(A)$ are the finitely generated A-modules, and morphisms in $\underline{\mathrm{mod}}(A)$ are classes of A-homomorphisms $\underline{\mathrm{Hom}}_A(U, V) = \mathrm{Hom}_A(U, V)/\mathrm{Hom}_A^{\mathrm{pr}}(U, V)$, where U, V are finitely generated A-modules and $\mathrm{Hom}_A^{\mathrm{pr}}(U, V)$ is the R-module of A-homomorphisms from U to V which factor through a finitely generated relatively R-projective A-module. Composition in $\underline{\mathrm{mod}}(A)$ is induced by that in the category of finitely generated A-modules $\mathrm{mod}(A)$. Since R is Noetherian, the category $\mathrm{mod}(A)$ is a full abelian subcategory of the category $\mathrm{Mod}(A)$ of all A-modules. The category $\underline{\mathrm{mod}}(A)$ is no longer abelian but triangulated (cf. [23, § A.3]),with shift functor Σ_A which sends a finitely generated A-module U to the cokernel of $U \to I$ for some relatively R-injective envelope I of U. As an A-module, $\Sigma_A(U)$ depends on the choice of $U \to I$ but this assignment is unique up to unique isomorphism in $\underline{\mathrm{mod}}(A)$, hence does indeed induce a functor on $\underline{\mathrm{mod}}(A)$ (cf. [22, Theorem 2.14.4]), still denoted Σ_A, which is unique up to unique isomorphism. Moreover, since A is symmetric, the functor Σ_A is an equivalence on $\underline{\mathrm{mod}}(A)$ through which $\underline{\mathrm{mod}}(A)$ becomes a triangulated category (cf. [23, Theorem A.3.2]). An inverse, denoted Σ_A^{-1}, of Σ_A is induced by the assignment sending a finitely generated A-module U to the kernel of a relatively R-projective cover $P \to U$ of U, which as before depends, as an A-module, on the choice of $P \to U$ but is unique up to unique isomorphism in $\underline{\mathrm{mod}}(A)$. For any non-negative integer n we define $\Sigma_A^n(U)$ as the nth cokernel of a relatively R-injective resolution of U, and $\Sigma_A^{-n}(U)$ as the nth kernel of a relatively R-projective resolution of U. As before, these A-modules depend on the choices of resolutions, but their images in $\underline{\mathrm{mod}}(A)$ are unique up to unique isomorphism. As functors on $\underline{\mathrm{mod}}(A)$, we have canonical isomorphisms $\Sigma_A^n \cong (\Sigma_A)^n$ and $\Sigma_A^{-n} \cong (\Sigma_A^{-1})^n$. We adopt the convention that Σ_A^0 is the identity functor on $\underline{\mathrm{mod}}(A)$. For any two integers n, m, we have canonical identifications of functors $\Sigma_A^n \circ \Sigma_A^m \cong \Sigma_A^{n+m}$ on $\underline{\mathrm{mod}}(A)$. For any integer n, we set

$$\widehat{\mathrm{Ext}}_A^n(U, V) = \underline{\mathrm{Hom}}_A(U, \Sigma_A^n(V)). \tag{2.1}$$

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Note that if U is finitely generated projective as an R-module, then a projective cover (respectively resolution) of U is also a relatively R-projective cover (respectively resolution). If U is finitely generated projective as an R-module, then a relatively R-injective envelope (respectively resolution) can be constructed by taking the R-dual of a projective cover (respectively resolution) of the R-dual of U. If U, V are two A-modules which are finitely generated projective as R-modules, then $\mathrm{Hom}_A^{\mathrm{pr}}(U,V)$ consists of the space of A-homomorphisms from U to V which factor through a finitely generated projective A-module, and for $n>0$ we have $\widehat{\mathrm{Ext}}_A^n(U,V)=\mathrm{Ext}_A^n(U,V)$. In general, U need not have an injective resolution which consists of finitely generated injective A-modules.

The functor Σ_A on $\underline{\mathrm{mod}}(A)$ lifts to an exact functor (albeit not an equivalence in general) on $\mathrm{mod}(A)$. To see this, set $A^e=A\otimes_R A^{\mathrm{op}}$, and consider A as an A^e-module. As an A-A-bimodule, $\Sigma_{A^e}^{-1}(A)$ can be chosen to be the kernel of the multiplication map $A\otimes_R A\to A$, $a\otimes b\mapsto ab$, because this is a projective cover (not necessary minimal) of A as an A^e-module. This choice ensures that $\Sigma_{A^e}^{-1}(A)$ is finitely generated projective as a left and right A-module. The R-dual of the multiplication map together with the symmetry of A yields a relatively R-injective envelope $A\to A\otimes_R A$. Again with this choice, $\Sigma_{A^e}(A)$ is an A^e-module which is finitely generated projective as a left and right A-module, and the functor $\Sigma_{A^e}(A)\otimes_A -$ on $\mathrm{mod}(A)$ is exact and induces the equivalence Σ_A on $\underline{\mathrm{mod}}(A)$. We emphasise that $\Sigma_{A^e}(A)\otimes_A -$ regarded as a functor on $\underline{\mathrm{mod}}(A)$ does not depend on this choice, but the exact lift to a functor on $\mathrm{mod}(A)$ does.

It follows from the above that choosing a symmetrising form of A uniquely determines in $\underline{\mathrm{mod}}(A^e)$ an isomorphism

$$\Sigma_{A^e}(A)\cong(\Sigma_{A^e}^{-1}(A))^\vee$$

and hence more generally, uniquely determines isomorphisms

$$\Sigma_{A^e}^n(A)\cong(\Sigma_{A^e}^{-n}(A))^\vee$$

in $\underline{\mathrm{mod}}(A^e)$, for all integers n. The Tate analogue $\widehat{HH}^*(A)$ of the Hochschild cohomology $HH^*(A)$ of A is

$$\widehat{HH}^n(A)=\widehat{\mathrm{Ext}}_{A^e}^n(A,A)=\underline{\mathrm{Hom}}_{A^e}(A,\Sigma^n(A)). \tag{2.2}$$

Since A is symmetric, hence finitely generated projective as an R-module, it follows as before that for $n>0$ we have $\widehat{HH}^n(A)=HH^n(A)$.

Let A, B, C be symmetric R-algebras. Then the R-algebras A^{op}, $A\otimes_R B$, and $A\otimes_R B^{\mathrm{op}}$ are symmetric. An A-B-bimodule, or equivalently, an $A\otimes_R B^{\mathrm{op}}$-module, is called *perfect* if it is finitely generated projective as a left A-module and as a right B-module. We denote by $\mathrm{perf}(A,B)$ the category of perfect A-B-bimodules. Note that all modules in $\mathrm{perf}(A,B)$ are finitely generated projective as R-modules. The category $\mathrm{perf}(A,B)$ is a full R-linear subcategory of $\mathrm{mod}(A\otimes_R B^{\mathrm{op}})$ which is closed under taking direct summands. We denote by $\underline{\mathrm{perf}}(A,B)$ the image of $\mathrm{perf}(A,B)$ in $\underline{\mathrm{mod}}(A\otimes_R B^{\mathrm{op}})$; this is a thick subcategory of the triangulated category $\underline{\mathrm{mod}}(A\otimes_R B^{\mathrm{op}})$. If M is a perfect A-B-bimodule and N a perfect B-C-bimodule, then $M\otimes_B N$ is a perfect A-C-bimodule. In particular, we may choose $\Sigma_{A^e}^n(A)$ in $\mathrm{perf}(A,A)$ and $\Sigma_{B^e}^n(B)$ in $\mathrm{perf}(B,B)$. With such a choice, the exact functors $\Sigma_{A^e}^n(A)\otimes_A -$ and $-\otimes_B\Sigma_{B^e}^n(B)$ on $\mathrm{mod}(A\otimes_R B^{\mathrm{op}})$ restrict to exact functors

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on $\mathrm{perf}(A,B)$, and they induce functors on $\underline{\mathrm{mod}}(A\otimes_R B^{\mathrm{op}})$ which are both canonically isomorphic to the functor $\Sigma^n_{A\otimes_R B^{\mathrm{op}}}$ on $\underline{\mathrm{mod}}(A\otimes_R B^{\mathrm{op}})$. Note that $\mathrm{perf}(A,A)$ is closed under the tensor product over A, and hence $\underline{\mathrm{perf}}(A,A)$ is a tensor triangulated category, with tensor product $-\otimes_A -$.

If the algebra under consideration is clear from the context, we will simply write Σ for the shift functor on the stable module category, and sometimes use the same letter Σ for some exact lift to the category of finitely generated modules. This is to keep notation under control, but requires some care when it comes to establishing that all constructions are well-defined.

3. Adjunction for symmetric algebras

We briefly review without proofs some formalities on bimodules over symmetric algebras; broader expositions can be found in many sources such as [4], [5], [19, § 6 Appendix], [21, § 3], [22, § 2.12]. Let R be a commutative Noetherian ring (with unit element), and let A, B be symmetric R-algebras, with symmetrising forms s and t, respectively. The functor $\mathrm{Hom}_R(-,R)$ is contravariant, and for U an R-module, we write $U^\vee = \mathrm{Hom}_A(U,R)$. Let M be a perfect A-B-bimodule. Since A, B are symmetric, the R-dual $M^\vee$ is a perfect B-A-bimodule. We have a B-A-bimodule isomorphism

$$\mathrm{Hom}_A(M,A) \cong M^\vee \tag{3.1}$$

sending $\alpha \in \mathrm{Hom}_A(M,A)$ to $s\circ\alpha$, and we have a B-A-bimodule isomorphism

$$\mathrm{Hom}_{B^{\mathrm{op}}}(M,B) \cong M^\vee \tag{3.2}$$

sending $\beta \in \mathrm{Hom}_{B^{\mathrm{op}}}(M,B)$ to $t\circ\beta$. For any A-module U, we have natural isomorphisms

$$M^\vee \otimes_A U \cong \mathrm{Hom}_A(M,A)\otimes_A U \cong \mathrm{Hom}_A(M,U) \tag{3.3}$$

where the first map is induced by the isomorphism from Equation (3.1) and the second map sends $\lambda\otimes u$ to the map $m\mapsto \lambda(m)u$, for $u\in U$, $m\in M$, $\lambda\in\mathrm{Hom}_A(M,A)$. Using that M is finitely generated projective as an A-module one sees that this is indeed an isomorphism. Combining this with the tensor-Hom adjunction shows that the functors $M\otimes_B -$ and $M^\vee\otimes_A -$ between the categories $\mathrm{mod}(A)$ and $\mathrm{mod}(B)$ of finitely generated modules over A and B, respectively, are left and right adjoint to each other. More precisely, the choices of symmetrising forms s, t determine adjunction isomorphisms

$$\mathrm{Hom}_A(M\otimes_B V, U) \cong \mathrm{Hom}_B(V, M^\vee\otimes_A U) \tag{3.4}$$

where U is a finitely generated A-module and V a finitely generated B-module. This isomorphism has the property that it sends a map of the form $\lambda_{\gamma,u}$ to the map $v\mapsto s\circ\gamma_v\otimes u$, where $\gamma\in\mathrm{Hom}_A(M\otimes_B V,A)$, $u\in U$, where $\lambda_{\gamma,u}\in\mathrm{Hom}_A(M\otimes_B V,U)$ is defined by $\lambda_{\gamma,u}(m\otimes v)=\gamma(m\otimes v)u$, and where $\gamma_v\in\mathrm{Hom}_A(M,A)$ is defined by $\gamma_v(m)=\gamma(m\otimes v)$, for all $m\in M$, $v\in V$. Maps of the form $\lambda_{\gamma,u}$ are precisely the maps which factor through A, hence span the subspace $\mathrm{Hom}^{\mathrm{pr}}_A(M\otimes_B V,U)$ of $\mathrm{Hom}_A(M\otimes_B V,U)$. The unit

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and counit of the adjunction (3.4) are represented by bimodule homomorphisms

$$\epsilon_M : B \longrightarrow M^\vee \otimes_A M\ ,\quad 1_B \mapsto \sum_{i\in I} (s\circ\alpha_i)\otimes m_i\ , \tag{3.5}$$
$$\eta_M : M\otimes_B M^\vee \longrightarrow A\ ,\quad m\otimes(s\circ\alpha)\mapsto \alpha(m)\ ,$$

where I is a finite indexing set, $\alpha_i \in \mathrm{Hom}_A(M,A)$ and $m_i \in M$ such that $\sum_{i\in I} \alpha_i(m')m_i = m'$ for all $m'\in M$. Similarly, we have a natural isomorphism

$$\mathrm{Hom}_B(M^\vee\otimes_A U, V)\cong \mathrm{Hom}_A(U, M\otimes_B V) \tag{3.6}$$

obtained from Equation (3.4) by exchanging the roles of A and B and using $M^\vee$ instead of M together with the canonical double duality $M^{\vee\vee}\cong M$. The adjunction unit and counit of this adjunction are represented by bimodule homomorphisms

$$\epsilon_{M^\vee} : A \longrightarrow M\otimes_B M^\vee\ ,\quad 1_A\mapsto \sum_{j\in J} m_j\otimes(t\circ\beta_j)\ , \tag{3.7}$$
$$\eta_{M^\vee} : M^\vee\otimes_A M\longrightarrow B\ ,\quad (t\circ\beta)\otimes m\mapsto \beta(m)\ ,$$

where J is a finite indexing set, $\beta_j \in \mathrm{Hom}_{B^{\mathrm{op}}}(M,B)$, $m_j \in M$, such that $\sum_{j\in J} m_j\beta_j(m') = m'$ for all $m'\in M$, where $m\in M$ and $\beta\in \mathrm{Hom}_{B^{\mathrm{op}}}(M,B)$. Note that $\eta_M\circ\epsilon_{M^\vee}$ is an A-A-bimodule endomorphism of A, hence given by left or right multiplication with an element in $Z(A)$. Similarly, $\eta_{M^\vee}\circ\epsilon_M$ is a B-B-bimodule endomorphism of B, hence given by left or right multiplication with an element in $Z(B)$. Following [19, Definition 3.1], we set

$$\begin{aligned}\pi_M &= (\eta_M\circ\epsilon_{M^\vee})(1_A)\\ \pi_{M^\vee} &= (\eta_{M^\vee}\circ\epsilon_M)(1_B).\end{aligned} \tag{3.8}$$

We call π_M the *relatively M-projective element of* $Z(A)$. Similarly, $\pi_{M^\vee}$ is called the relatively $M^\vee$-projective element of $Z(B)$. These elements depend on the choices of the symmetrising forms of A and B, see [19, Remark 3.2] for details.

Remark 3.9. The adjunction isomorphisms (3.4) and (3.6) and the associated adjunction units and counits in Equations (3.5) and (3.7) commute with extensions of the ring of scalars R, where we use the fact that M, $M^\vee$ are finitely generated projective as left and right modules. More precisely, if $R\to S$ is a homomorphism of commutative rings through which S is regarded as an R-module, then, writing $SU = S\otimes_R U$ and $SU^\vee = \mathrm{Hom}_S(SU,S)$ for any R-module U, we have a canonical isomorphism $S(M^\vee\otimes_B M)\cong SM^\vee\otimes_{SB} SM$ through which $\mathrm{Id}_S\otimes\epsilon_M$ becomes the adjunction unit of $SM\otimes_{SB}-$ being left adjoint to $SM^\vee\otimes_{SA}-$. Similar statements hold for the remaining adjunction unit and the counits. This will be needed in the proofs of the two main theorems for the extension from a complete discrete valuation ring to its field of fractions.

Remark 3.10. The adjunction isomorphism (3.4) is additive in M. Thus, the adjunction unit and counit in Equation (3.5) are additive in M in the following sense: given two

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perfect A-B-bimodules M, N, the adjunction unit

$$\epsilon_{M\oplus N} : B \to (M \oplus N)^\vee \otimes_A (M \oplus N)$$

is equal to the composition of

$$\epsilon_M + \epsilon_N : B \to M^\vee \otimes_A M \oplus N^\vee \otimes_A N$$

followed by the canonical inclusion of the right side into $(M\oplus N)^\vee \otimes_A (M\oplus N)$. Similarly, the adjunction counit

$$\eta_{M\oplus N} : (M \oplus N) \otimes_B (M \oplus N)^\vee \to A$$

is equal to the map

$$\eta_M + \eta_N : M \otimes_A M^\vee \oplus N \otimes_A N^\vee \to A$$

extended by zero on the mixed summands $M \otimes_A N^\vee$ and $N \otimes_A M^\vee$. The analogous statements hold for the adjunction isomorphism (3.6) and the corresponding adjunction unit and counit in Equation (3.7).

Remark 3.11. If U, V have in addition right C-module structures for some further R-algebra C, then the isomorphisms in Equations (3.1) and (3.3) are isomorphisms of right C-modules. Thus the isomorphism (3.4) induces an isomorphism

$$\mathrm{Hom}_{A\otimes_R C^{\mathrm{op}}}(M \otimes_B V, U) \cong \mathrm{Hom}_{B\otimes_R C^{\mathrm{op}}}(V, M^\vee \otimes_A U).$$

4. Transfer for symmetric algebras

Let R be a commutative Noetherian ring (with unit element), and let A, B be symmetric R-algebras, with symmetrising forms s, t, respectively. Let M be a perfect A-B-bimodule. Following [4], for finitely generated A-modules U, V, we have a transfer map

$$\mathrm{tr}_M = \mathrm{tr}_M(U, V) : \mathrm{Hom}_B(M^\vee \otimes_A U, M^\vee \otimes_A V) \to \mathrm{Hom}_A(U, V) \tag{4.1}$$

sending a B-homomorphism $\beta : M^\vee \otimes_A U \to M^\vee \otimes_A V$ to the A-homomorphism

$$\mathrm{tr}_M(\beta) = (\eta_M \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_M \otimes \beta) \circ (\epsilon_{M^\vee} \otimes \mathrm{Id}_U). \tag{4.2}$$

More explicitly, $\mathrm{tr}_M(\beta)$ is the composition of A-homomorphisms

$$U \xrightarrow{\epsilon_{M^\vee}\otimes \mathrm{Id}_U} M \otimes_B M^\vee \otimes_A U \xrightarrow{\mathrm{Id}_M\otimes\beta} M \otimes_B M^\vee \otimes_A V \xrightarrow{\eta_M\otimes \mathrm{Id}_V} V$$

with the standard identifications $A \otimes_A U = U$ and $A \otimes_A V = V$. The functors $M \otimes_B -$ and $M^\vee \otimes_A -$ are exact and preserve finitely generated projective modules. Therefore, if

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β factorises through a projective B-module, then $\mathrm{tr}_M(\beta)$ factorises through a projective A-module, and hence tr_M induces a well-defined map, still denoted

$$\mathrm{tr}_M : \underline{\mathrm{Hom}}_B(M^\vee \otimes_A U, M^\vee \otimes_A V) \to \underline{\mathrm{Hom}}_A(U, V). \tag{4.3}$$

It also follows that in the stable module category $\underline{\mathrm{mod}}(A)$, for any integer n, we have unique isomorphisms

$$\Sigma^n_A(M \otimes_A M^\vee \otimes_A U) = M \otimes_A \Sigma^n_B(M^\vee \otimes_A U) = M \otimes_B M^\vee \otimes_A \Sigma^n_A(U)$$

and through these identifications and their analogues, we have an equality of morphisms in the stable category $\underline{\mathrm{mod}}(A)$

$$\Sigma^n_A(\mathrm{tr}_M(\beta)) = \mathrm{tr}_M(\Sigma^n_B(\beta)) : \Sigma^n_A(U) \to \Sigma^n_A(V). \tag{4.4}$$

By [20, § 7.1], there are graded versions of these transfer maps for Tate and Tate–Hochschild cohomology. An element in $\widehat{\mathrm{Ext}}^n_B(M^\vee \otimes_A U, M^\vee \otimes_A V)$ is represented by a B-homomorphism $\beta : M^\vee \otimes_A U \to M^\vee \otimes_A \Sigma^n(V)$, where we identify $\Sigma^n(M^\vee \otimes_A U) = M^\vee \otimes_A \Sigma^n(V)$ and where we use the same letter Σ for either Σ_A or Σ_B. The transfer map tr_M sends β to the element $\mathrm{tr}_M(\beta)$ in $\mathrm{Ext}^n_A(U, V)$ represented by the A-homomorphism, abusively also denoted $\mathrm{tr}_M(\beta)$, given by

$$\mathrm{tr}_M(\beta) = (\eta_M \otimes \mathrm{Id}_{\Sigma^n(V)}) \circ (\mathrm{Id}_M \otimes \beta) \circ (\epsilon_{M^\vee} \otimes \mathrm{Id}_U) \tag{4.5}$$

with the standard identifications $A \otimes_A U = U$ and $A \otimes_A \Sigma^n(V) = \Sigma^n(V)$. More explicitly, $\mathrm{tr}_M(\beta)$ is obtained as the composition

$$U \xrightarrow{\epsilon_{M^\vee} \otimes \mathrm{Id}_U} M \otimes_B M^\vee \otimes_A U \xrightarrow{\mathrm{Id}_M \otimes \beta} M \otimes_B M^\vee \otimes_A \Sigma^n(V) \xrightarrow{\eta_M \otimes \mathrm{Id}_{\Sigma^n(V)}} \Sigma^n(V) \tag{4.6}$$

A variation of the same principle applied to bimodules yields in particular a transfer for Tate–Hochschild cohomology. We use again simply Σ instead of $\Sigma_{A \otimes_k A^{\mathrm{op}}}$ or $\Sigma_{B \otimes_k B^{\mathrm{op}}}$. An element $\zeta \in \widehat{HH}^n(B)$ is represented by a B-B-bimodule homomorphism, abusively denoted by the same letter, $\zeta : B \to \Sigma^n(B)$. We denote by $\mathrm{tr}_M(\zeta)$ the element in $\widehat{HH}^n(A)$ represented by the A-A-bimodule homomorphism

$$M \otimes_B M^\vee = M \otimes_B B \otimes_B M^\vee \xrightarrow{\mathrm{Id}_M \otimes \zeta \otimes \mathrm{Id}_{M^\vee}} M \otimes_B \Sigma^n(B) \otimes_B M^\vee = \Sigma^n(M \otimes_B M^\vee)$$

precomposed with the adjunction unit $\epsilon_{M^\vee} : A \to M \otimes_B M^\vee$ and composed with the 'shifted' adjunction counit $\Sigma^n(\eta_M) : \Sigma^n(M \otimes_B M^\vee) \to \Sigma^n(A)$. The identification $M \otimes_B \Sigma^n(B) \otimes_B M^\vee = \Sigma^n(M \otimes_B M^\vee)$ is to be understood as the canonical isomorphism in $\underline{\mathrm{mod}}(A \otimes_k A^{\mathrm{op}})$, using the fact that the functor $M \otimes_B - \otimes_B M^\vee$ sends a projective resolution of the B-B-bimodule B to a projective resolution of the A-A-bimodule $M \otimes_B$

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$M^\vee$. Modulo this identification, we thus have graded k-linear map

$$\mathrm{tr}_M : \widehat{HH}^*(B) \longrightarrow \widehat{HH}^*(A) \tag{4.7}$$

defined by

$$\mathrm{tr}_M(\zeta) = \Sigma^n(\eta_M) \circ (\mathrm{Id}_M \otimes \zeta \otimes \mathrm{Id}_{M^\vee}) \circ \epsilon_{M^\vee}. \tag{4.8}$$

Note that tr_M is not necessarily a multiplicative map from $\widehat{HH}^*(B)$ to $\widehat{HH}^*(A)$. In all the cases above, we have analogous transfer maps $\mathrm{tr}_{M^\vee}$ obtained from exchanging the roles of A and B. Using Ext instead of $\widehat{\mathrm{Ext}}$ yields the transfer maps introduced in [19]. The two are well-known to coincide for $n > 0$.

Suppose that A is R-free. Let X be an R-basis of A and $X^\vee$ the dual basis with respect to the symmetrising form s on A; that is, we have a bijection $x \mapsto x^\vee$ from X to $X^\vee$ such that $s(xx^\vee) = 1$ for $x \in X$ and $s(xy^\vee) = 0$ for $x, y \in X$ such that $x \neq y$. The element

$$z_A = \sum_{x \in X} xx^\vee \tag{4.9}$$

is called the *relative projective element with respect to s* . One easily checks that this is an element in $Z(A)$ which does not depend on the choice of the basis X, but which does depend on the choice of s. If s' is another symmetrising form, then there is a unique element $z \in Z(A)^\times$ such that $s'(a) = s(za)$ for all $a \in A$. If $X^\vee$ is as before the dual basis of X with respect to s, then $z^{-1}X^\vee$ is the dual basis of X with respect to s', and hence the relative projective element with respect to s' is equal to $z'_A = z^{-1}z_A$.

Remark 4.10. If we regard R as a symmetric algebra with the identity map as symmetrising form and take for M the A-R-bimodule A (that is, the regular bimodule A restricted to R on the right), then z_A is the relative M-projective element π_M defined in Equation (3.8). That is, z_A is the image of 1_A under the composition of bimodule homomorphisms $A \to A \otimes_R A \to A$, where the second map is given by multiplication in A and the first map is obtained by dualising the multiplication map and then using the isomorphism $A^\vee \cong A$ and $(A \otimes_R A)^\vee \cong A^\vee \otimes_R A^\vee \cong A \otimes_R A$. This definition of z_A has the advantage of not needing A to be free over R but just finitely generated projective as an R-module. For the purpose of this paper, we do not need this generality.

Tate duality for Tate–Hochschild cohomology involves bimodules, and hence we will need the following well-known description of relative projective elements for tensor products of symmetric algebras as well as their compatibility with the passage to blocks.

Lemma 4.11. *Let A, B be R-free symmetric R-algebras, with symmetrising forms s, t, respectively. Then A^{op} is symmetric algebra with s as symmetrising form, $A \otimes_R B$ is symmetric with $s \otimes t$ as symmetrising form, and $A \times B$ is symmetric with symmetrising form $s + t$. With respect to these symmetrising forms, we have*

(i) $z_{A^{\mathrm{op}}} = z_A$.

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(ii) $z_{A\otimes_B B} = z_A \otimes z_B$.
(iii) $z_{A\times B} = (z_A, z_B)$.

Proof. A trivial verification shows that s, $s\otimes t$, and $s+t$ are symmetrising forms of $A^{\rm op}$, $A\otimes_R B$, and $A\times B$, respectively. Let X be an R-basis of A, with dual basis X' and corresponding bijection $x\mapsto x'$ from X to X' as in Equation (4.9). Then X and X' are also dual to each other with respect to s as a symmetrising form of $A^{\rm op}$. This implies $z_{A^{\rm op}} = z_A$, whence (i). Let Y be an R-basis of B with dual basis Y' and corresponding bijection $y\mapsto y'$ for $y\in Y$. Then the image in $A\otimes_R B$ of $X\otimes Y$ is an R-basis, and its dual basis with respect to $s\otimes t$ is $X'\otimes Y'$, with the bijection from $X\otimes Y$ to $X'\otimes Y'$ mapping $x\otimes y$ to $x'\otimes y'$, for $x\in X$ and $y\in Y$. It follows that

$$z_{A\otimes_R B} = \sum_{x\in X, y\in Y} xx'\otimes yy' = (\sum_{x\in X} xx')\otimes(\sum_{y\in Y} yy') = z_A\otimes z_B$$

as stated in (ii). The union $(X\times\{0\})\cup(\{0\}\times Y)$ is an R-basis of $A\times B$ with dual basis $(X'\times\{0\})\cup(\{0\}\times Y')$. Statement (iii) follows. □

Remark 4.12. The additivity properties of adjunction units and counits mentioned in Remark 3.10 as well as the additivity of shift functors on stable module categories imply that the transfer maps above are additive in M. More precisely, for M, N perfect A-B-bimodules, we have $\mathrm{tr}_{M\oplus N} = \mathrm{tr}_M + \mathrm{tr}_N$ for all the variations of transfer maps tr_M considered in Equations (4.1), (4.5) and (4.8).

Remark 4.13. The transfer map in Tate–Hochschild cohomology (4.7) and (4.8) is not strictly speaking a special case of the transfer maps $\mathrm{tr}_M(U,V)$, but the two are related via a generalisation of $\mathrm{tr}_M(U,V)$. Let A, B, C be symmetric R-algebras, and let U,V be finitely generated $A\otimes_R C^{\rm op}$-modules. The transfer map $\mathrm{tr}_M = \mathrm{tr}_M(U,V)$ from Equation (4.1) induces a map, yet again denoted

$$\mathrm{tr}_M = \mathrm{tr}_M(U,V) : \mathrm{Hom}_{B\otimes_R C^{\rm op}}(M^\vee\otimes_A U, M^\vee\otimes_A V) \to \mathrm{Hom}_{A\otimes_R C^{\rm op}}(U,V)$$

sending a $B\otimes_R C^{\rm op}$-homomorphism $\beta : M^\vee\otimes_A U \to M^\vee\otimes_A V$ to the $A\otimes_R C^{\rm op}$-homomorphism

$$\mathrm{tr}_M(\beta) = (\eta_M\otimes \mathrm{Id}_V)\circ(\mathrm{Id}_M\otimes\beta)\circ(\epsilon_{M^\vee}\otimes\mathrm{Id}_U).$$

The functor $M\otimes_B -$ sends a projective $B\otimes_R C^{\rm op}$-module to a projective $A\otimes_R C^{\rm op}$-module, and hence if β factors through a projective $B\otimes_R C^{\rm op}$-module, then $\mathrm{Id}_M\otimes\beta$ factors through a projective $A\otimes_R C^{\rm op}$-module. Thus tr_M induces a well-defined map

$$\mathrm{tr}_M : \underline{\mathrm{Hom}}_{B\otimes_R C^{\rm op}}(M^\vee\otimes_A U, M^\vee\otimes_A V) \to \underline{\mathrm{Hom}}_{A\otimes_R C^{\rm op}}(U,V).$$

Applied with $C=A$, $U=A$, $V=\Sigma^n_{A^e}(A)$, this yields a map

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$$\mathrm{tr}_M : \underline{\mathrm{Hom}}_{B\otimes_R A^{\mathrm{op}}}(M^\vee, \Sigma^n(M^\vee)) \to \underline{\mathrm{Hom}}_{A^e}(A, \Sigma^n(A)) \tag{4.14}$$

where we have made use of the standard identifications $\Sigma^n(B) \otimes_B M^\vee \cong \Sigma^n(M^\vee) \cong M^\vee \otimes_A \Sigma^n(A)$ in the stable category $\underline{\mathrm{mod}}(B \otimes_R A^{\mathrm{op}})$. The functor $-\otimes_B M^\vee$ induces a graded algebra homomorphism $\widehat{HH}^*(B) = \widehat{\mathrm{Ext}}^*_{B^e}(B, B) \to \widehat{\mathrm{Ext}}^*_{B\otimes_R A^{\mathrm{op}}}(M^\vee, M^\vee)$, and composing this with the map tr_M from Equation (4.14) yields the transfer map in Hochschild cohomology $\widehat{HH}^*(B) \to \widehat{HH}^*(A)$ from Equations (4.7) and (4.8).

5. Adjunction maps for matrix algebras

We need to identify the adjunction maps and the transfer maps reviewed in the previous section in the case that A, B are matrix algebras. This is elementary linear algebra, so we just give some pointers towards verifications. Let R be a commutative ring. Let U, V be free R-modules of finite ranks over R. Set $A = \mathrm{End}_R(U)$ and $B = \mathrm{End}_R(V)$. Then A and B are symmetric R-algebras with symmetrising forms the trace maps trace_U, trace_V, sending a R-linear endomorphism of U, V to its trace, respectively. Any other symmetrising form of A, B is of the form $\rho{\cdot}\mathrm{trace}_U$, $\rho{\cdot}\mathrm{trace}_V$ for some $\rho \in R^\times$, respectively. Set $M = U \otimes_R V^\vee$. Tensoring with M and its dual is the simplest instance of a Morita equivalence; all we need to make sure in this section is that the standard maps in this context are indeed the adjunction maps with respect to the trace maps as symmetrising forms. These verifications make use of the following well-known Lemma which links traces to adjunction maps.

Lemma 5.1. *We have an isomorphism* $\mathrm{Hom}_A(U, A) \cong U^\vee$ *sending* $\lambda \in \mathrm{Hom}_A(U, A)$ *to* $\mathrm{trace}_U \circ \lambda$. *We have a commutative diagram of* A-A-*bimodule homomorphisms*

$$\begin{tikzcd}
 & U \otimes_R \mathrm{Hom}_A(U, A) \arrow[dl, "\sigma"'] \arrow[dr, "\rho"] & \\
U \otimes_R U^\vee \arrow[rr, "\alpha"] \arrow[dr, "\tau"'] & & A \arrow[dl, "\mathrm{trace}_U"] \\
 & R &
\end{tikzcd}$$

where α *sends* $u \otimes \mu$ *to the endomorphism* $u' \mapsto \mu(u') \otimes u$, σ *sends* $u \otimes \lambda$ *to* $u \otimes (\mathrm{trace}_U \circ \lambda)$, ρ *sends* $u \otimes \lambda$ *to* $\lambda(u)$, *and* τ *sends* $u \otimes \mu$ *to* $\mu(u)$, *for all* $u, u' \in U$, $\lambda \in \mathrm{Hom}_A(U, A)$, *and* $\mu \in U^\vee$.

Proof. The first statement is a special case of the isomorphism (3.1). The commutativity of the lower triangle is well-known, see for instance [22, Proposition 2.10.2] for a proof. The commutativity of the upper diagram is an easy verification. $\square$

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We identify $M^\vee$ with $V \otimes_R U^\vee$ via the obvious isomorphisms

$$M^\vee = (U \otimes_R V^\vee)^\vee = V^{\vee\vee} \otimes_R U^\vee = V \otimes_R U^\vee.$$

This leads to identifications

$$M \otimes_B M^\vee = U \otimes_R V^\vee \otimes_B V \otimes_R U^\vee = U \otimes_R U^\vee$$

where we identify $V^\vee \otimes_B V = R$ via the map $\nu \otimes v \to \nu(v)$, for $v \in V$ and $\nu \in V^\vee$. Similarly, we identify $M^\vee \otimes_A M = V \otimes_R V^\vee$. Let $\mathcal{B}$ be an R-basis of U, with dual basis in $U^\vee$ denoted $\mathcal{B}^\vee$. For $u \in \mathcal{B}$, we denote by $u^\vee$ the unique element in $\mathcal{B}^\vee$ satisfying $u^\vee(u) = 1$ and $u^\vee(u') = 0$ for $u' \in \mathcal{B}$, $u' \neq u$. Similarly, let $\mathcal{C}$ be an R-basis of V, with dual basis in $V^\vee$ denoted analogously $\mathcal{C}^\vee$. The adjunction units and counits from the preceding section in this case (with the choice of symmetrising forms trace_U, trace_V) are all isomorphisms, and their precise descriptions are as follows:

$$\epsilon_M : B \longrightarrow M^\vee \otimes_A M = V \otimes_R V^\vee \ , \quad 1_B \mapsto \sum_{v \in \mathcal{C}} v \otimes v^\vee, \tag{5.2}$$
$$\eta_M : U \otimes_R U^\vee = M \otimes_B M^\vee \longrightarrow A \ , \quad u \otimes \mu \mapsto (y \mapsto \mu(y)u),$$

where u, $y \in U$, $\mu \in U^\vee$.

$$\epsilon_{M^\vee} : A \longrightarrow M \otimes_B M^\vee = U \otimes_R U^\vee \ , \quad 1_A \mapsto \sum_{u \in \mathcal{B}} u \otimes u^\vee \ , \tag{5.3}$$
$$\eta_{M^\vee} : V \otimes_R V^\vee = M^\vee \otimes_A M \longrightarrow B \ , \quad v \otimes \nu \mapsto (w \mapsto \nu(w)v) \ ,$$

where v, $w \in V$ and $\nu \in V^\vee$. We further note that

$$\epsilon_{M^\vee} = (\eta_M)^{-1}, \qquad \eta_{M^\vee} = (\epsilon_M)^{-1}. \tag{5.4}$$

An easy verification shows that the relative projective elements in $Z(A)$ and $Z(B)$ with respect to the symmetrising forms trace_U and trace_V, respectively, are equal to

$$z_A = \mathrm{rk}_R(U) \cdot 1_R, \qquad z_B = \mathrm{rk}_R(V) \cdot 1_R. \tag{5.5}$$

Remark 5.6. Let s, t be symmetrising forms of A, B, respectively. Then $s = \lambda \cdot \mathrm{trace}_U$ and $t = \mu \cdot \mathrm{trace}_V$ for some λ, $\mu \in R^\times$. Denoting by ϵ'_M, η'_M, $\epsilon'_{M^\vee}$, $\eta'_{M^\vee}$ the adjunction maps from Equations (5.2) and (5.3) with respect to s, t, it follows that

$$\epsilon'_M = \lambda \epsilon_M, \qquad \eta'_M = \lambda^{-1} \eta_M, \qquad \epsilon_{M^\vee} = \mu \epsilon_{M^\vee}, \qquad \eta'_{M^\vee} = \mu^{-1} \eta_{M^\vee}.$$

Thus the trace map $\mathrm{tr}'_M : \mathrm{End}_B(M^\vee \otimes_A U) \to \mathrm{End}_A(U)$ with respect to s and t satisfies

$$\mathrm{tr}'_M = \lambda^{-1} \mu \mathrm{tr}_M.$$

The relative projective central elements z'_A and z'_B with respect to s, t are

$$z'_A = \lambda^{-1} z_A = \lambda^{-1} \mathrm{rk}_R(U), \qquad z'_B = \mu^{-1} z_B = \mu^{-1} \mathrm{rk}_R(V).$$

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6. Transfer for matrix algebras

Let K be a field of characteristic zero, and let U, V be finite-dimensional K-vector spaces. We set $A = \mathrm{End}_K(U)$ and $B = \mathrm{End}_K(V)$, regarded as symmetric algebras with symmetrising forms trace_U and trace_V, respectively. We set $M = U \otimes_K V^\vee$. We note that since $A \otimes_K A^{\mathrm{op}}$ and $A \otimes_K B^{\mathrm{op}}$ are simple algebras, it follows that every finitely generated A-A-bimodule is projective and isomorphic to a finite direct sum of copies of A, and every finitely generated A-B-bimodule is projective and isomorphic to a finite direct sum of copies of the simple A-B-bimodule M. For finitely generated A-modules U', U'' we denote by

$$\varphi_A : \mathrm{Hom}_A(U', U'') \times \mathrm{Hom}_A(U'', U') \to K$$

the bilinear map sending (α, β) to $z_A^{-1}\mathrm{trace}_{U'}(\beta \circ \alpha)$. We use the analogous notation φ_B for finitely generated B-modules. We keep the above notation throughout this section.

Proposition 6.1. *Let U', U'' be finitely generated A-modules. Then the bilinear map*

$$\varphi_A : \mathrm{Hom}_A(U', U'') \times \mathrm{Hom}_A(U'', U') \to K$$

is non-degenerate.

Proof. This is a trivial consequence of [9, Proposition 2.1], and easily checked directly. □

The following result is needed for the proof of Theorem 1.1.

Proposition 6.2. *Let U', U'' be finite-dimensional A-modules. Let $\alpha : U' \to U''$ be an A-homomorphism and let $\beta : M^\vee \otimes_A U' \to M^\vee \otimes_A U''$ be a B-homomorphism. For any choice of symmetrising forms on A and B, we have*

$$\varphi_A(\alpha, \mathrm{tr}_M(\beta)) = \varphi_B(\mathrm{Id}_{M^\vee} \otimes \alpha, \beta).$$

Proof. Assume first that the symmetrising forms are trace_U and trace_V. Note that U', U'' are isomorphic to finite direct sums of copies of U. Since φ_A, φ_B are additive in both components, we may assume that $U' = U'' = U$. Then α is a K-linear multiple of Id_U, and since both sides are bilinear, we may assume that $\alpha = \mathrm{Id}_U$. Thus stated equation is equivalent to

$$z_A^{-1}\mathrm{trace}_U(\mathrm{tr}_M(\beta)) = z_B^{-1}\mathrm{trace}_{M^\vee \otimes_A U}(\beta).$$

Note that $M^\vee \otimes_A U \cong V$, and hence $\beta = \lambda \mathrm{Id}_{M^\vee \otimes_A U}$ for some $\lambda \in K$. In particular, we have

$$\mathrm{trace}_{M^\vee \otimes_A U}(\beta) = \lambda \mathrm{dim}_K(V).$$

By Equation (4.2), we have $\mathrm{tr}_M(\beta) = \eta_M \circ (\mathrm{Id}_M \otimes \beta) \circ \epsilon_{M^\vee} = \lambda \eta_M \circ \epsilon_{M^\vee} = \lambda \mathrm{Id}_U$, and hence

$$\mathrm{trace}_U(\mathrm{tr}_M(\beta)) = \lambda \mathrm{dim}_K(U).$$

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Using $z_A = \dim_K(U) \cdot 1_K$ and $z_B = \dim_K(V) \cdot 1_K$, the result follows for the chosen symmetrising forms trace_U and trace_V. The result for arbitrary symmetrising forms follows easily from the Remark 5.6. □

The next result, which is a variation of the previous Proposition, will be needed in the proof of Theorem 1.3. Set $A^e = A \otimes_K A^{\mathrm{op}}$. By Lemma 4.11, we have $z_{A^e} = z_A \otimes z_A$. Thus, the inverse of this element acts on an A-A-bimodule by simultaneously multiplying by z_A^{-1} on the left and on the right. In particular, this element acts on A by multiplication by z_A^{-2}.

Proposition 6.3. *Let X be a finitely generated A-A-bimodule and let Y be a finitely generated B-B-bimodule. Let $\zeta : A \to X$ be an A-A-bimodule homomorphism, let $\xi : X \otimes_A M \to M \otimes_B Y$ be an A-B-bimodule homomorphism, and let $\sigma : Y \to B$ be a B-B-bimodule homomorphism. For any choice of symmetrising forms on A and on B, the trace on A of the map*

$$z_A^{-2} \cdot \eta_M \circ (\mathrm{Id}_M \otimes \sigma \otimes \mathrm{Id}_{M^\vee}) \circ (\xi \otimes \mathrm{Id}_{M^\vee}) \circ (\mathrm{Id}_X \otimes \epsilon_{M^\vee}) \circ \zeta$$

is equal to the trace on B of the map

$$z_B^{-2} \cdot \sigma \circ \eta_{M^\vee} \circ (\mathrm{Id}_{M^\vee} \otimes \xi) \circ (\mathrm{Id}_{M^\vee} \otimes \zeta \otimes \mathrm{Id}_M) \circ \epsilon_M.$$

Proof. Both maps in the statement are additive in X. Since A is up to isomorphism the unique indecomposable A-A-bimodule, it follows that X is isomorphic to a finite direct sum of copies of A, and hence we may assume that $X = A$. For the same reason, we may assume that $Y = B$. Then, ξ becomes an A-B-bimodule endomorphism of the simple A-B-bimodule $M = U \otimes_K V^\vee$, hence is equal to multiplication by a scalar, which we will denote abusively again by ξ. Similarly, σ becomes a bimodule endomorphism of B, so is given by multiplication with a scalar, again denoted by σ, and ζ becomes a bimodule endomorphism of A, given by multiplication with a scalar, again denoted by ζ. Thus, the two maps in the statement take the form

$$z_A^{-2}\sigma\xi\zeta \cdot (\eta_M \circ \epsilon_{M^\vee}), \qquad\qquad z_B^{-2}\sigma\xi\zeta \cdot (\eta_{M^\vee} \circ \epsilon_M). \tag{6.4}$$

Let s, t be symmetrising forms of A, B. Then $s = \lambda \cdot \mathrm{trace}_U$ and $t = \mu \cdot \mathrm{trace}_V$ for some λ, $\mu \in K^\times$. It follows from the Remark 5.6 and 5.4 that then the adjunction units and counits with respect to these symmetrising forms satisfy

$$\eta_M \circ \epsilon_{M^\vee} = \lambda^{-1}\mu \mathrm{Id}_A,$$
$$\eta_{M^\vee} \circ \epsilon_M = \lambda\mu^{-1}\mathrm{Id}_B.$$

Also by the Remark 5.6, the relative projective elements are $z_A = \lambda^{-1}\dim_K(U)$ and $z_B = \mu^{-1}\dim_K(V)$. Thus $z_A^{-2} = \lambda^2 \dim_K(U)^{-2}$ and $z_B^{-2} = \mu^2 \dim_K(V)^{-2}$. Therefore, the two maps in Equation (6.4) are equal to the two maps

$$\dim_K(U)^{-2}\lambda\mu\sigma\xi\zeta \cdot \mathrm{Id}_A,$$

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$$\dim_K(V)^{-2}\lambda\mu\sigma\xi\zeta\cdot \mathrm{Id}_B.$$

Since $\dim_K(U)^2 = \dim_K(A)$ and $\dim_K(V)^2 = \dim_K(B)$, it follows that both maps have the same trace, equal to $\lambda\mu\sigma\xi\zeta$. □

7. Proof of Theorem 1.1

Let $\mathcal{O}$ be a complete discrete valuation ring with field of fractions K of characteristic zero. Let A, B be symmetric $\mathcal{O}$-algebras such that $K\otimes_{\mathcal{O}} A$ and $K\otimes_{\mathcal{O}} B$ are semisimple. Note that then $K\otimes_{\mathcal{O}} A$ and $K\otimes_{\mathcal{O}} B$ are separable since $\mathrm{char}(K)=0$. Fix symmetrising forms s, t of A, B, respectively. Let M be an A-B-bimodule which is finitely generated projective as a left A-module and as a right B-module. Let U, V be finitely generated $\mathcal{O}$-free A-modules. We write KA instead of $K\otimes_{\mathcal{O}} A$ and KU instead of $K\otimes_{\mathcal{O}} U$; similarly for B and V. We identify $\mathrm{Hom}_{KA}(KU,KV) = K\mathrm{Hom}_A(U,V)$ whenever convenient, and we identify $\mathrm{Hom}_A(U,V)$ with its image in this space.

In degree zero, we have $\widehat{\mathrm{Ext}}^0_A(U,V) = \underline{\mathrm{Hom}}_A(U,V)$, and Tate duality takes the following form. By [9, Proposition 2.1], we have a non-degenerate bilinear form

$$\varphi_{KA}(-,-) : K\mathrm{Hom}_A(U,V)\times K\mathrm{Hom}_A(V,U)\to K \tag{7.1}$$

which sends $(\alpha,\beta)\in K\mathrm{Hom}_A(U,V)\times K\mathrm{Hom}_A(V,U)$ to the trace on KU of the KA-endomorphism $z_A^{-1}\beta\circ\alpha$ of KU. This restricts to an $\mathcal{O}$-bilinear form

$$\varphi_A : \mathrm{Hom}_A(U,V)\times \mathrm{Hom}_A(V,U)\to K. \tag{7.2}$$

By [9, Theorem 1.3] and its proof in [9, § 2], this form sends $\mathrm{Hom}^{\mathrm{pr}}_A(U,V)\times\mathrm{Hom}_A(V,U)$ and $\mathrm{Hom}_A(U,V)\times\mathrm{Hom}^{\mathrm{pr}}_A(V,U)$ to $\mathcal{O}$, and the induced bilinear form

$$\langle -,-\rangle_A : \underline{\mathrm{Hom}}_A(U,V)\times\underline{\mathrm{Hom}}_A(V,U)\to K/\mathcal{O} \tag{7.3}$$

is non-degenerate. We note that for any $\alpha\in\mathrm{Hom}_A(U,V)$ and $\beta\in\mathrm{Hom}_A(V,U)$ we have

$$\begin{gathered}\varphi_{KA}(\alpha,\beta)=\varphi_{KA}(\beta,\alpha),\\ \langle\alpha,\beta\rangle_A=\langle\beta,\alpha\rangle_A.\end{gathered}\tag{7.4}$$

To see this, observe that left multiplication by z_A^{-1} commutes with all KA-homomorphisms. Thus, the KA-endomorphism $z_A^{-1}(\beta\circ\alpha)$ of KU is equal to $(z_A^{-1}\beta)\circ\alpha$, hence has the same trace on KU as the endomorphism $\alpha\circ(z_A^{-1}\beta)$ on KV. The latter is equal to $z_A^{-1}(\alpha\circ\beta)$.

Proof of Theorem 1.1. We start by proving Theorem 1.1 in degree zero. Tate duality in degree zero takes the form as reviewed in Equations (7.1), (7.2) and (7.3). Since the functors $M\otimes_B -$ and $M^\vee\otimes_A -$ preserve finitely generated projective modules over A and B, it follows that if $\beta\in\mathrm{Hom}^{\mathrm{pr}}_A(M^\vee\otimes_A U, M^\vee\otimes_A V)$, then $\mathrm{tr}_M(\beta)\in\mathrm{Hom}^{\mathrm{pr}}_A(U,V)$.

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Similarly, if $\alpha \in \mathrm{Hom}_A^{\mathrm{pr}}(U,V)$, then $\mathrm{Id}_{M^\vee} \otimes \alpha \in \mathrm{Hom}_B^{\mathrm{pr}}(M^\vee \otimes_A U, M^\vee \otimes_A V)$. That is, it suffices to show the equality

$$\varphi_{KA}(\alpha, \mathrm{tr}_M(\beta)) = \varphi_{KB}(\mathrm{Id}_{KM^\vee} \otimes \alpha, \beta)$$

where $\alpha \in K\mathrm{Hom}_A(U,V)$ and $\beta \in K\mathrm{Hom}_B(M^\vee \otimes_A U, M^\vee \otimes_A V)$. This equation holds if and only if it holds for field extensions of K, so we may assume that KA, KB are split semisimple. That is, KA, KB are direct products of matrix algebras. Since both sides are additive, we may in fact assume that KA, KB are matrix algebras. In that case, the equation follows from Proposition 6.2. Together with Equation (7.4), this proves Theorem 1.1 for $n=0$.

To prove Theorem 1.1 in an arbitrary degree n, we need to show that the above is compatible with the shift functors Σ_A and Σ_B on the relatively $\mathcal{O}$-stable categories $\underline{\mathrm{mod}}(A)$ and $\underline{\mathrm{mod}}(B)$. For simplicity, we denote both shift functors by Σ. We have $\widehat{\mathrm{Ext}}_A^n(U,V) = \underline{\mathrm{Hom}}_A(U,\Sigma^n(V))$, and $\widehat{\mathrm{Ext}}_A^{-n}(V,U) = \underline{\mathrm{Hom}}_A(V,\Sigma^{-n}(U)) \cong \underline{\mathrm{Hom}}_A(\Sigma^n(V),U)$, where the second isomorphism is obtained from applying the functor Σ^n. The Tate duality

$$\langle -,- \rangle_A : \widehat{\mathrm{Ext}}_A^n(U,V) \times \widehat{\mathrm{Ext}}_A^{-n}(V,U) \to K/\mathcal{O}$$

is induced by the map sending $(\alpha,\gamma) \in \mathrm{Hom}_A(U,\Sigma^n(V)) \times \mathrm{Hom}_A(V,\Sigma^{-n}(U))$ to the trace on KU of the endomorphism

$$z_A^{-1}\Sigma^n(\gamma) \circ \alpha;$$

in other words, this is induced by the degree zero duality applied to U and $\Sigma^n(V)$, combined with the shift functor Σ^n. Applying the degree zero case to U and $\Sigma^n(V)$ yields the equation

$$\langle \alpha, \mathrm{tr}_M(\Sigma^n(\beta)) \rangle_A = \langle \mathrm{Id}_{M^\vee} \otimes \alpha, \Sigma^n(\beta) \rangle_B$$

in $K/\mathcal{O}$. It remains to show that the left side is equal to $\langle \alpha, \Sigma^n(\mathrm{tr}_M(\beta)) \rangle_A$. This expression depends only on the images of the morphisms α, β in their respective stable categories. By Equation (4.4), the images in the stable category $\underline{\mathrm{mod}}(A)$ of $\Sigma^n(\mathrm{tr}_M(\beta)$ and $\mathrm{tr}_M(\Sigma^n(\beta))$ are equal. Again using Equation (7.4), the result follows. □

8. Proof of Theorem 1.3

As in the previous section, let $\mathcal{O}$ be a complete discrete valuation ring with field of fractions K of characteristic zero. Let A, B be symmetric $\mathcal{O}$-algebras such that $KA = K \otimes_{\mathcal{O}} A$ and $KB = K \otimes_{\mathcal{O}} B$ are semisimple. Fix symmetrising forms s, t of A, B, respectively. Let M be an A-B-bimodule which is finitely generated projective as a left A-module and as a right B-module. As before, we set $A^e = A \otimes_{\mathcal{O}} A^{\mathrm{op}}$ and $B^e = B \otimes_{\mathcal{O}} B^{\mathrm{op}}$. Let n be an integer. The Tate duality

$$\langle -,- \rangle_{A^e} : \widehat{HH}^n(A) \times \widehat{HH}^{-n}(A) \to K/\mathcal{O}$$

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is induced by a map

$$\varphi_{A^e} : \mathrm{Hom}_{A^e}(A, \Sigma^n(A)) \times \mathrm{Hom}_{A^e}(A, \Sigma^{-n}(A)) \to K$$

sending $(\alpha, \beta) \in \mathrm{Hom}_{A^e}(A, \Sigma^n(A)) \times \mathrm{Hom}_{A^e}(A, \Sigma^{-n}(A))$ to the trace on KA of the A^e-endomorphism

$$z_A^{-2} \cdot \Sigma^n(\beta) \circ \alpha$$

where we use that the projective element $z_{A^e} = z_A \otimes z_A$ acts as multiplication by z_A^2 on the A^e-module A. Note that φ_A depends on the choices of $\Sigma^n(A)$ and $\Sigma^n(\beta)$, but the induced map to $K/\mathcal{O}$ does not. We have the analogous description for B instead of A.

As described in § 2, the functor Σ^n on $\underline{\mathrm{mod}}(A^e)$ preserves the full subcategory $\underline{\mathrm{perf}}(A, A)$ of perfect A-A-bimodules. Slightly more generally, the functor Σ^n on $\underline{\mathrm{mod}}(A \otimes_{\mathcal{O}} B^{\mathrm{op}})$ preserves $\underline{\mathrm{perf}}(A, B)$. We may choose $X = \Sigma^n(A)$ to be in $\mathrm{perf}(A, A)$. Then, the functor Σ on $\underline{\mathrm{mod}}(A^e)$ restricted to $\underline{\mathrm{perf}}(A, A)$ is canonically isomorphic to the functor induced by any of the two exact functors $X \otimes_A -$ and $- \otimes_A X$ on $\mathrm{perf}(A, A)$, and the functor Σ^n on $\underline{\mathrm{mod}}(A \otimes_{\mathcal{O}} B^{\mathrm{op}})$ is canonically isomorphic to the functor induced by the exact functor $X \otimes_A -$ on $\mathrm{perf}(A, B)$. Similarly, choosing $Y = \Sigma^n_{B^e}(B)$ in $\mathrm{perf}(B, B)$, the functor Σ^n restricted to $\underline{\mathrm{perf}}(B, B)$ is canonically isomorphic to any of the two functors induced by the exact functors $Y \otimes_B -$ and $- \otimes_B Y$ on $\mathrm{perf}(B, B)$, and the functor Σ^n on $\underline{\mathrm{mod}}(A \otimes_{\mathcal{O}} B^{\mathrm{op}})$ is canonically isomorphic to the functor induced by the exact functor $- \otimes_B Y$ on $\mathrm{perf}(A, B)$. With this notation, we will need the identification

$$X \otimes_A M = \Sigma^n_{A \otimes_{\mathcal{O}} B^{\mathrm{op}}}(M) = M \otimes_B Y, \tag{8.1}$$

in $\underline{\mathrm{perf}}(A, B)$. Denote by

$$\xi : X \otimes_A M \longrightarrow M \otimes_B Y \tag{8.2}$$

and A-B-bimodule homomorphism which induces the identification in Equation (8.1). Since ξ induces an isomorphism in $\underline{\mathrm{perf}}(A, B)$, the kernel and cokernel of ξ are projective $A \otimes_{\mathcal{O}} B^{\mathrm{op}}$-modules.

Proof of Theorem 1.3. Let $\zeta \in \widehat{HH}^n(A)$, and $\tau \in \widehat{HH}^{-n}(B)$. Represent these classes by bimodule homomorphisms, abusively denoted by the same letters,

$$\zeta : A \to \Sigma^n(A) = X, \qquad \tau : B \to \Sigma^{-n}(B).$$

Then $\Sigma^n(\tau)$ is represented by a morphism, again denoted by the same letter,

$$\Sigma^n(\tau) : \Sigma^n(B) = Y \to B$$

where we have used the identification $\Sigma^n(\Sigma^{-n}(B)) = B$ in $\underline{\mathrm{perf}}(B, B)$. We need to show that the trace on KA of

$$z_A^{-2} \cdot (\Sigma^n(\mathrm{tr}_M(\tau)) \circ \zeta)$$

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is equal to the trace on KB of

$$z_B^{-2} \cdot (\Sigma^n(\tau) \circ \mathrm{tr}_{M^\vee}(\zeta)).$$

For simplicity, all identity homomorphisms on any of the bimodules M, $M^\vee$, X, Y are denoted Id. The A^e-homomorphism $\mathrm{tr}_M(\tau)$ is equal to the composition

$$A \xrightarrow{\epsilon_{M^\vee}} M \otimes_B B \otimes_B M^\vee \xrightarrow{\mathrm{Id}\otimes\tau\otimes\mathrm{Id}} M \otimes_B \Sigma^{-n}(B) \otimes_B M^\vee \xrightarrow{\Sigma^{-n}(\eta_M)} \Sigma^{-n}(A)$$

where we have identified $M \otimes_B \Sigma^n(B) \otimes_B M^\vee$ and $\Sigma^n(M \otimes_B M^\vee)$ along a bimodule homomorphism inducing the canonical isomorphism in the stable module category $\underline{\mathrm{mod}}(A^e)$. Thus, $\Sigma^n(\mathrm{tr}_M(\tau)) \circ \zeta$ is the composition

$$A \xrightarrow{\zeta} \Sigma^n(A) \xrightarrow{\Sigma^n(\epsilon_{M^\vee})} \Sigma^n(M \otimes_B M^\vee) \xrightarrow{\mathrm{Id}\otimes\Sigma^n(\tau)\otimes\mathrm{Id}} M \otimes_B M^\vee \xrightarrow{\eta_M} A$$

where in the third term we use the identification $\Sigma^n(M \otimes_B M^\vee) = M \otimes_B \Sigma^n(B) \otimes_B M^\vee$ in $\underline{\mathrm{mod}}(A^e)$, and in the fourth term we identify $M \otimes_B M^\vee = M \otimes_B B \otimes_B M^\vee$. In terms of the bimodules X and Y, as well as replacing Σ^n by $X \otimes_A -$ as appropriate, this shows that $\Sigma^n(\mathrm{tr}_M(\tau)) \circ \zeta$ is represented by the composition of morphisms in $\mathrm{perf}(A)$

$$A \xrightarrow{\zeta} X \xrightarrow{\mathrm{Id}\otimes\epsilon_{M^\vee}} X \otimes_A M \otimes_B M^\vee \xrightarrow{\xi\otimes\mathrm{Id}} M \otimes_B Y \otimes_B M^\vee \xrightarrow{\mathrm{Id}\otimes\Sigma^n(\tau)\otimes\mathrm{Id}} M \otimes_B M^\vee \xrightarrow{\eta_M} A \tag{8.3}$$

Similarly, the map $\Sigma^n(\tau) \circ \mathrm{tr}_{M^\vee}(\zeta)$ is represented by the composition

$$B \xrightarrow{\epsilon_M} M^\vee \otimes_A M \xrightarrow{\mathrm{Id}\otimes\zeta\otimes\mathrm{Id}} M^\vee \otimes_A X \otimes_A M \xrightarrow{\mathrm{Id}\otimes\xi} M^\vee \otimes_A M \otimes_B Y \xrightarrow{\eta_{M^\vee}} Y \xrightarrow{\Sigma^n(\tau)} B \tag{8.4}$$

Since we need to compare traces of endomorphisms of KA, KB, we may extend coefficients to any field extension of K. Relative projective elements are compatible with these coefficient extensions, and hence we may assume that $\mathcal{O} = K$ is a splitting field of A and B. Thus, we may assume that A, B are finite direct products of matrix algebras over K. What we need to show is that the traces of the two maps in 8.3 and 8.4 multiplied by z_A^{-1} and z_B^{-2}, respectively, are equal. Since traces are additive and all maps above (in particular, the adjunction maps, hence relative projective elements) are compatible with the block decompositions of A, B, we may assume that $A = \mathrm{End}_K(U)$ and $B = \mathrm{End}_K(V)$ for some finite-dimensional K-vector spaces U, V. The adjunction maps are also additive in M, so we may assume that $M = U \otimes_K V^\vee$ (this is, up to isomorphism, the unique finite-dimensional indecomposable A-B-bimodule). The result follows from Proposition 6.3 with $\Sigma^n(\tau)$ instead of σ. □

9. Remarks on transfer for Calabi–Yau triangulated categories

There are two ways to associate transfer maps to a triangle functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ of Calabi–Yau categories: either by making use of a biadjoint functor $\mathcal{G}$ (if there is such a

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functor, as described in many sources such as [7], [20]), or by making use of Serre duality. Theorem 1.1 and [21, Theorem 1.1] state that for stable categories of symmetric algebras over fields or complete discrete valuation rings these two constructions coincide.

To be more precise, let $\mathcal{C}$, $\mathcal{D}$ be triangulated categories over a field k. We use the same letter Σ for the shift functors in $\mathcal{C}$ and $\mathcal{D}$. Suppose that homomorphism spaces between objects in either category are finite-dimensional and that $\mathcal{C}$, $\mathcal{D}$ admit Serre functors $\mathbb{S}$, $\mathbb{T}$, respectively. Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a k-linear functor. Let X, Y be objects in $\mathcal{C}$. Dualising the map

$$\mathrm{Hom}_{\mathcal{C}}(Y, X) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(Y), \mathcal{F}(X))$$

induced by $\mathcal{F}$, and making use of the defining property of a Serre functor, yields a map

$$\mathrm{tr}_{\mathcal{C},\mathcal{D}} : \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(X), \mathbb{T}(\mathcal{F}(Y))) \to \mathrm{Hom}_{\mathcal{C}}(X, \mathbb{S}(Y)).$$

which makes the following diagram commutative.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(X), \mathbb{T}(\mathcal{F}(Y))) \arrow[r, "\mathrm{tr}_{\mathcal{C},\mathcal{D}}"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathcal{C}}(X, \mathbb{S}(Y)) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(Y), \mathcal{F}(X))^\vee \arrow[r, "\mathcal{F}^\vee"'] & \mathrm{Hom}_{\mathcal{C}}(Y, X)^\vee
\end{tikzcd} \tag{9.1}$$

where $\mathcal{F}^\vee$ is the dual of the map induces by $\mathcal{F}$ on morphisms, and where the vertical isomorphism are Serre duality isomorphisms.

Assume now that $\mathcal{C}$ and $\mathcal{D}$ are d-Calabi–Yau triangulated categories for some integer d. That is, the Serre functors $\mathbb{S}$, $\mathbb{T}$ are isomorphic to Σ^d on $\mathcal{C}$, $\mathcal{D}$, respectively (see Kontsevich [18] or also Keller [16] for background material and a long list of references on Calabi–Yau triangulated categories – what we call Calabi–Yau would be called weakly Calabi–Yau in many sources). Then, the previous map $\mathrm{tr}_{\mathcal{C},\mathcal{D}}$ takes the form

$$\begin{aligned} \mathrm{tr}_{\mathcal{F}^\vee} : \mathrm{Ext}^d_{\mathcal{D}}(\mathcal{F}(X), \mathcal{F}(Y)) = \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(X), \Sigma^d(\mathcal{F}(Y))) \to \mathrm{Hom}_{\mathcal{C}}(X, \Sigma^d(Y)) \\ = \mathrm{Ext}^d_{\mathcal{C}}(X, Y). \end{aligned} \tag{9.2}$$

If $\mathcal{F}$ is a functor of triangulated categories, then the functors $\Sigma \circ \mathcal{F}$ and $\mathcal{F} \circ \Sigma$ are isomorphic, so upon replacing Y by $\Sigma^{n-d}(Y)$, where n is an integer, we get in particular a map

$$\begin{aligned} \mathrm{tr}_{\mathcal{F}^\vee} : \mathrm{Ext}^n_{\mathcal{D}}(\mathcal{F}(X), \mathcal{F}(Y)) = \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(X), \Sigma^n(\mathcal{F}(Y))) \to \mathrm{Hom}_{\mathcal{C}}(X, \Sigma^n(Y)) \\ = \mathrm{Ext}^n_{\mathcal{C}}(X, Y). \end{aligned} \tag{9.3}$$

For $n = 0$ this yields a map

$$\mathrm{tr}_{\mathcal{F}^\vee} : \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(X), \mathcal{F}(Y)) \to \mathrm{Hom}_{\mathcal{C}}(X, Y). \tag{9.4}$$

Combining the above, the map $\mathrm{tr}_{\mathcal{F}^\vee}$ makes the following diagram commutative:

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(X),\mathcal{F}(Y)) \arrow[rr, "\mathrm{tr}_{\mathcal{F}^\vee}"] \arrow[d, "\cong"'] & & \mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[dd, "\cong"] \\
\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(Y),\Sigma^d(\mathcal{F}(X)))^\vee \arrow[d, "\cong"'] & & \\
\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(Y),\mathcal{F}(\Sigma^d(X)))^\vee \arrow[rr, "\mathcal{F}^\vee"'] & & \mathrm{Hom}_{\mathcal{C}}(Y,\Sigma^d(X))^\vee
\end{tikzcd}
\tag{9.5}
$$

Here the vertical isomorphisms are given by a choice of Calabi–Yau duality together with a choice of an isomorphism $\Sigma \circ \mathcal{F} \cong \mathcal{F} \circ \Sigma$, and the bottom horizontal is the dual of the map induced by $\mathcal{F}$. (Note the dependence on choices.)

If $\mathcal{F}$ has a biadjoint functor of triangulated categories $\mathcal{G} : \mathcal{D} \to \mathcal{C}$, then (following e.g. [7] or [20, § 4]) we also have a transfer map

$$\mathrm{tr}_{\mathcal{G}} : \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(X),\mathcal{F}(Y)) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Y) \tag{9.6}$$

sending a morphism $\psi : \mathcal{F}(X) \to \mathcal{F}(Y)$ to the composition of morphisms

$$X \longrightarrow \mathcal{G}(\mathcal{F}(X)) \xrightarrow{\mathcal{G}(\psi)} \mathcal{G}(\mathcal{F}(Y)) \longrightarrow Y\ ,$$

where the first map is the adjunction unit of $\mathcal{F}$ being left adjoint to $\mathcal{G}$, and the last map is the adjunction counit of $\mathcal{F}$ being right adjoint to $\mathcal{G}$. The map $\mathrm{tr}_{\mathcal{G}}$ depends on the choice of adjunction isomorphisms.

Remark 9.7. It would be desirable to spell out the exact compatibility conditions for adjunction isomorphisms and Calabi–Yau duality that would lead to an equality of the maps $\mathrm{tr}_{\mathcal{F}^\vee} = \mathrm{tr}_{\mathcal{G}}$ in Equations (9.4) and (9.6).

One can rephrase [21, Theorem 1.2] as stating that $\mathrm{tr}_{\mathcal{F}^\vee} = \mathrm{tr}_{\mathcal{G}}$. The proof amounts to showing that the choices made for adjunction isomorphisms and Tate duality determined by the choices of symmetrising forms are compatible. Indeed, if A is a symmetric k-algebra, then Tate duality turns $\underline{\mathrm{mod}}(A)$ into a (-1)-Calabi–Yau triangulated category. Given two symmetric k-algebras A, B and an A-B-bimodule M in $\mathrm{perf}(A, B)$, Theorem [21, Theorem 1.2] can be rephrased as stating that the transfer maps $\mathrm{tr}_M(U, V)$ are special cases of the construction of $\mathrm{tr}_{\mathcal{F}^\vee}$ in the above diagram 9.5; in other words, the maps in Equations (9.4) and (9.6) coincide.

Let now A be a symmetric algebra over a complete discrete valuation ring $\mathcal{O}$ with a field of fractions K of characteristic 0 such that $K \otimes_{\mathcal{O}} A$ is semisimple. Extend the notion of Calabi–Yau triangulated categories to $\mathcal{O}$-linear triangulated categories by using duality with respect to the Matlis module $K/\mathcal{O}$. Note that $K/\mathcal{O}$ is the degree 1-term of the injective resolution $K \to K/\mathcal{O}$ of $\mathcal{O}$. Note further that there are no non-zero $\mathcal{O}$-linear maps from the torsion $\mathcal{O}$-modules $\underline{\mathrm{Hom}}_A(U, V)$ to the torsion free $\mathcal{O}$-module K (where U, V are A-lattices), and hence Matlis duality coincides with $\mathrm{RHom}(-, \mathcal{O})$ on the morphism spaces in the stable module category, except for a degree shift.

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Remark 9.8. Tate duality on the stable module category $\underline{\mathrm{latt}}(A)$ of finitely generated $\mathcal{O}$-free A-modules as described in [9, Theorem 1.3] seems to suggest that $\underline{\mathrm{latt}}(A)$ should be called 0-Calabi–Yau. If, however, one were to take into account that $K/\mathcal{O}$ is in degree 1, then the total degree of Tate duality $\underline{\mathrm{Hom}}_A(U,V) \times \underline{\mathrm{Hom}}_A(V,U) \to K/\mathcal{O}$ would again be -1.

Regardless of dimension considerations, Theorem 1.1 shows that the transfer maps $\mathrm{tr}_M(U,V)$ in Theorem 1.1 are special cases of the construction given by the diagram 9.5 with k-duality replaced by $K/\mathcal{O}$-duality. As pointed out earlier, there are choices to be made: Tate duality depends on the choices of symmetrising forms, and showing that the maps from Equations (9.4) and (9.6) are equal in Theorem 1.1 boils down to being able to make compatible choices.

Remark 9.9. In the context of stable module categories of symmetric algebras, there is always a canonical choice for the commutation with shift functors. More precisely, given two symmetric $\mathcal{O}$-algebras A, B and an A-B-bimodule M in $\mathrm{perf}(A,B)$, then the functor $M \otimes_B -$ from $\mathrm{mod}(B)$ to $\mathrm{mod}(A)$ is exact, preserves projectives, hence preserves projective resolutions, and therefore induces a canonical isomorphism $\mathcal{F} \circ \Sigma \cong \Sigma \circ \mathcal{F}$, where $\mathcal{F} : \underline{\mathrm{latt}}(B) \to \underline{\mathrm{latt}}(A)$ is the functor induced by $M \otimes_B -$.

10. On products in negative degrees of Tate cohomology

Non-zero products in negative degree in Tate cohomology have implications for the depth of the non-negative part. This phenomenon was first observed in [2] in finite group cohomology, and then generalised in [3], [21, § 8]. This arises over complete discrete valuation rings as well, with essentially the same arguments used in [2].

Let A be a symmetric algebra over a complete discrete valuation ring $\mathcal{O}$ with a field of fractions K of characteristic zero. Let U, V, W be finitely generated A-modules. Let m, n be integers. Let $\alpha \in \widehat{\mathrm{Ext}}_A^m(U,V)$, $\beta \in \widehat{\mathrm{Ext}}_A^n(V,W)$, and $\gamma \in \widehat{\mathrm{Ext}}_A^{-m-n}(W,U)$. We denote by $\beta\alpha \in \widehat{\mathrm{Ext}}_A^{m+n}(U,W)$ the Yoneda product; that is, $\beta\alpha$ is represented by $\Sigma^m(\beta) \circ \alpha$, where we use the same letters α, β for representatives of their classes in $\mathrm{Hom}_A(U,\Sigma^m(V))$, $\mathrm{Hom}_A(V,\Sigma^n(W))$. We have

$$\langle \beta\alpha, \gamma \rangle_A = \langle \alpha, \gamma\beta \rangle_A \tag{10.1}$$

because both sides are equal to the image in $K/\mathcal{O}$ of the trace on KU of the endomorphism $z_A^{-1} \cdot (\Sigma^{m+n}(\gamma) \circ \Sigma^m(\beta) \circ \alpha)$ of KU, whereas before we use abusively the same letters α, β, γ for representatives in $\mathrm{Hom}_A(U,\Sigma^m(V))$, $\mathrm{Hom}_A(V,\Sigma^n(W))$, $\mathrm{Hom}_A(W,\Sigma^{-m-n}(U))$ of their classes. Applied with $U = W$ and $\gamma = \mathrm{Id}_U$ this yields

$$\langle \beta\alpha, \mathrm{Id}_U \rangle_A = \langle \alpha, \beta \rangle_A. \tag{10.2}$$

Lemma 10.3. *Let ζ be non-zero element in $\widehat{\mathrm{Ext}}_A^n(U,V)$. Then, there is a non-zero element η in $\widehat{\mathrm{Ext}}_A^{-n}(V,U)$ such that the Yoneda product $\zeta\eta$ is non-zero in $\widehat{\mathrm{Ext}}_A^0(U,U) = \underline{\mathrm{End}}_A(U)$.*

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Proof. By Tate duality, there is $\eta \in \widehat{\mathrm{Ext}}_A^{-n}(V, U)$ such that $\langle \zeta, \eta \rangle_A \neq 0$. By Equation (10.2), we have $\langle \eta\zeta, \mathrm{Id}_U \rangle \neq 0$, and hence $\eta\zeta \neq 0$ and $\eta \neq 0$. $\square$

Remark 10.4. As in [21, § 8], we denote by $\overline{\mathrm{Ext}}^*(U, U)$ the non-negative part of $\widehat{\mathrm{Ext}}_A^*(U, U)$. Adapting the arguments from [2], as reproduced in the proof of [21, Proposition 8.3], shows that if $\widehat{\mathrm{Ext}}_A^*(U, U)$ has a non-zero product of two homogeneous elements in negative degrees, and if $\overline{\mathrm{Ext}}_A^*(U, U)$ is graded-commutative, then $\overline{\mathrm{Ext}}_A^*(U, U)$ does not have a regular sequence of length 2. In particular, if $\widehat{HH}^*(A)$ has a non-zero product of two homogeneous elements in negative degrees, then its non-negative part $\overline{HH}^*(A)$ does not have a regular sequence of length 2.

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