CONTINUOUS FUNCTIONS ON THE SPHERE AND ISOMETRIES

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1. Introduction. The theorem of Borsuk-Ulam states that n odd functions on the n-dimensional sphere always have a common zero. We have tried to obtain a similar theorem by "slightly" changing the conditions for the functions, but it turned out that only a very weak analogue can be expected in our case. Here we want to prove a few results and mention some of the questions which have remained unanswered.

2. Results. Let S^n denote the unit sphere in euclidian (n+1)-space, that is, the set of points $x = (x_o, \dots, x_n$ $\Sigma_o^n x_m^2 = 1$. For each integer $n \geq 2$ denote by $k(n)$ (resp. $k^o(n)$, $k^1(n)$) the greatest integer k for which the following is true: Given k continuous (resp. continuous even, continuous odd) functions $f_i : S^n \to \mathbb{R}$ $(1 \leq i \leq k)$ and k isometries $w_i : S^n \to S^n$ $(1 \leq i \leq k)$ there always exists a point $x \in S^n$ such that the equations $f_i(x) = f_i(w_i x)$ hold simultaneously for all i, $1 \leq i \leq k$. Of course we have $1 \leq k(n) \leq \min\{k^o(n), k^1(n)\}$. The theorem of Borsuk-Ulam corresponds to Proposition 1.

PROPOSITION 1. $k^1(n) = n$.

Proof. Let f_i $(1 \leq i \leq n)$ be n odd functions and w_i n isometries. Setting $h_i(x) = f_i(x) - f_i(w_i x)$ we define n odd functions h_i for which the theorem of Borsuk-Ulam guarantees the existence of a common zero $x \in S^n$. Obviously this implies $f_i(x) = f_i(w_i x)$, and we have $k^1(n) \geq n$. On the other hand consider the functions $f_i(x) = x_i$ $(0 \leq i \leq n)$ and the isometries $w_i = \sigma$, where σ is the antipodal map. If there were a

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a point $x \in S^n$ such that $f_i(x) = f_i(w_i x)$ for all i, we would find $x_i = 0$ for each coordinate x_i, which cannot be true. This shows $k^1(n) \leq n$.

The situation changes completely if the functions need not be odd.

PROPOSITION 2. $k(n) = 1$.

Proof. Set $f(x) = \Sigma_o^n mx_m^2$ and $g(x) = x_o$, and consider the following isometries u, v:

$$(ux)_m = x_{m+1} \quad (0 \leq m \leq n-1), \quad (ux)_n = x_o ;$$

$$(vx)_o = -x_o , \quad (vx)_m = x_m \quad (1 \leq m \leq n) .$$

For each point $x \in S^n$ with $f(x) = f(ux)$ we find $x_o = \pm (n+1)^{-\frac{1}{2}}$. If we had also $g(x) = g(vx)$, this would imply $x_o = 0$, a contradiction.

If the functions under consideration are periodical with respect to the corresponding isometries, more precise statements about them can be derived. Results of this kind have been established by J. Binz in [1]. In the case of even functions we have found, as a partial answer to our question, the next proposition.

PROPOSITION 3. $k^o(n) \leq 2$.

Proof. Consider the functions f, g, h defined by

$$f(x) = \Sigma_o^n mx_m^2 , \quad g(x) = \Sigma_1^n mx_m^2 , \quad h(x) = x_o x_1$$

and the isometries

$$(ux)_m = x_{m+1} \quad (0 \leq m \leq n-1), \quad (ux)_n = x_o ;$$

$$(vx)_o = x_o , \quad (vx)_m = x_{m+1} \quad (1 \leq m \leq n-1), \quad (vx)_n = x_1 ;$$

$$(wx)_o = -x_o , \quad (wx)_m = x_m \quad (1 \leq m \leq n) .$$

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If there were a point $x \in S^n$ satisfying

(1) $f(x) = f(ux)$;

(2) $g(x) = g(vx)$;

(3) $h(x) = h(wx)$;

the following equations would hold for its coordinates

(1') $x_o = \pm (n+1)^{-\frac{1}{2}}$;

(2') $x_1 = \pm (1 - x_o^2)^{\frac{1}{2}} n^{-\frac{1}{2}}$;

(3') $x_o x_1 = 0$.

This system of equations however has no solution.

We do not know whether $k^o(n) = 1$ or $k^o(n) = 2$ even in the case $n = 2$. The only value of k^o which we have been able to determine is $k^o(3) = 1$, which may be verified by the following example. Consider the two even functions on S^3,

$$f(x) = (x_o^2 - x_1^2 - x_2^2 + x_3^2)^2 + 8(x_o x_1 - x_2 x_3)^2 ,$$

$$g(x) = x_o x_2 + x_1 x_3 ,$$

and the two isometries $u, v : S^3 \to S^3$ defined by

$$(ux)_o = (x_o - x_1 - x_2 - x_3)/2, \quad (ux)_1 = (x_o + x_1 - x_2 + x_3)/2$$

$$(ux)_2 = (x_o + x_1 + x_2 - x_3)/2, \quad (ux)_3 = (x_o - x_1 + x_2 + x_3)/2$$

$$(vx)_o = -x_3, \quad (vx)_1 = -x_2, \quad (vx)_2 = x_1, \quad (vx)_3 = x_o .$$

We verify

$$f(ux) = 4(x_o x_1 - x_2 x_3)^2 + 8(x_o x_2 + x_1 x_3)^2$$

$$g(vx) = -(x_o x_2 + x_1 x_3).$$

With regard to

$$(x_o^2 - x_1^2 - x_2^2 + x_3^2)^2 + 4(x_o x_1 - x_2 x_3)^2 + 4(x_o x_2 + x_1 x_3)^2$$

$$= (x_o^2 + x_1^2 + x_2^2 + x_3^2)^2 = 1$$

we derive from $f(x) = f(ux)$ the relation $12(x_o x_2 + x_1 x_3)^2 = 1$. It is not possible to have at the same time $g(x) = g(vx)$, since this would imply $x_o x_2 + x_1 x_3 = 0$, a contradiction to the result above.

3. Some related questions. If we add further restrictions to the functions under consideration, our problem turns into a more geometric one. Let $\ell(n)$ and $\ell^o(n)$ be defined similarly to $k(n)$ and $k^o(n)$, with the additional assumption that all the functions $f_i : S^n \to \mathbb{R}$ shall be equal to one single function, say f. By $p(n)$ and $p^o(n)$ we denote the numbers which arise if we further require that $f(x)$ $(x \in S^n)$ is the length of the segment $R_x \cap F$, where $R_x = \{\lambda x : \lambda \geq 0\}$ denotes the ray issuing from the origin and containing x, and F is a convex compact subset of E^n, which contains the origin in its interior. Of course we have $k(n) \leq \ell(n) \leq p(n)$ and $k^o(n) \leq \ell^o(n) \leq p^o(n)$. There are easy examples which show $p(n) = 1$, for all n. In the case of even functions we know much less. There exist upper bounds for $\ell^o(n)$ which lie around $\frac{1}{2}n$, and $\ell^o(3)$ equals 1. But so far we do not know any non-trivial lower bounds, not even for p^o, and our simplest open question is, as described in [2]: "Are there three congruent convex bodies in E^3, symmetric with respect to the origin, such that the intersection of their boundaries is empty?"

REFERENCES

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