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Harmonic Analysis on Metrized Graphs

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Abstract. This paper studies the Laplacian operator on a metrized graph, and its spectral theory.

1 Introduction

A metrized graph Γ is a finite connected graph equipped with a distinguished parametrization of each of its edges. (A more rigorous definition can be found in §2). In particular, Γ is a one-dimensional manifold except at finitely many "branch points", where it looks locally like an n-pointed star. A metrized graph should be thought of as an analytic object, not just a combinatorial one. In particular, there are a Laplacian operator Δ on Γ and corresponding "Green's functions" used to invert it. In this paper, we will study the spectral theory of the Laplacian on a metrized graph.

While metrized graphs occur in fields as diverse as chemistry, physics, and mathematical biology, our motivation comes from arithmetic geometry. The eigenfunction decomposition of the Arakelov-Green's function on a Riemann surface

$$g_\mu(x, y) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} f_n(x)\overline{f_n(y)} \tag{1.1}$$

plays an important role in Arakelov theory. For example, it is the key ingredient used by Faltings and Elkies to give lower bounds for $g_\mu(x, y)$-discriminant sums (see [La, §VI, Theorem 5.1]). Estimates for these sums can then be used to prove the effectivity of certain Arakelov divisors on an arithmetic surface (see [La, §V, Theorem 4.1]).

Metrized graphs were introduced as a nonarchimedean analogue of a Riemann surface in [Ru, CR, Zh]. The Laplacian on a metrized graph and the kernel of integration that inverts it, the Arakelov-Green's function $g_\mu(x, y)$ associated to a measure μ on Γ, were studied in [CR, Zh]. Those papers made it natural to ask if there is an eigenfunction decomposition of $g_\mu(x, y)$ analogous to (1.1) on a metrized graph, keeping in mind that for applications to nonarchimedean Arakelov theory, it is necessary to allow μ to be a singular measure. This paper therefore addresses the question "What are the eigenfunctions of the Laplacian on a metrized graph Γ relative to an arbitrary bounded, signed measure μ?"

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Initially, we discovered decompositions of $g_\mu(x, y)$ for certain special examples by *ad hoc* means (see §1.8 below). This paper is the outcome of our attempt to understand those examples. For an arbitrary measure μ of total mass one on Γ, we obtain an eigenfunction expansion converging uniformly to $g_\mu(x, y)$. We give three characterizations of eigenfunctions and study their properties. We give an algorithm which computes the eigenvalues in certain cases, and report some examples for interesting measures and graphs found by students in a Research Experience for Undergraduates (REU) held at the University of Georgia in summer 2003. As applications of the general theory, we prove the positivity of an "energy pairing" on the space of measures of total mass 0, we obtain Elkies-type bounds for $g_\mu(x, y)$-discriminant sums generalizing those found for the circle graph in [HS], and we show that $\mu = dx$ is the unique measure of total mass one for which the integral operator associated to $g_\mu(x, y)$ has minimal trace. We also raise several questions for future investigation.

In the course of writing this paper, and from the referee's report, we learned of an extensive mathematical literature concerning spectral analysis and differential equations on metrized graphs, which are also known in the literature as "networks", "metric graphs", and "quantum graphs". In particular we note the papers of J. P. Roth [Ro], C. Cattaneo [Ca], S. Nicaise [N1, N2], J. von Below [Be1, Be2], F. A. Mehmeti [AM1]–[AM3], B. Gaveau, M. Okada, and T. Okada [GO], T. Okada [Ok], M. Solomyak [So], and L. Friedlander [Fr, Fr2]. These works all define the Laplacian somewhat differently than we have done in the present paper. Typically, they deal with Laplacians of functions f satisfying certain natural boundary conditions at the nodes (vertices) which make the "discrete part" of the Laplacian (as defined in the present paper) equal to zero. They then study (in our terminology) the spectral theory of the graph relative to the measure $\mu = dx$. The results in the papers cited above, while perhaps not entirely superseding ours in the case $\mu = dx$, go much further in a number of directions. In particular, most of these papers treat infinite networks, whereas we have restricted our attention to finite ones. So to a large extent, it is the emphasis on eigenfunctions with respect to general measures μ (of total mass one) which is new in this paper. We have included Section 14 below partly to illustrate why one might be interested in the theory for measures other than dx. The spectral theory of more general measures is at least partly treated in [AM2], but we are unable to draw a precise comparison because the terminology and definitions are so different.

For more discussion of the relationship between these works and the present paper, see §1.9 below.

We now give an overview of the contents of this paper. First, we will recall the notions of Laplacian operators on Γ given in [CR, Zh].

1.1 The Laplacian on the Space $\mathrm{CPA}(\Gamma)$

Let $\mathrm{CPA}(\Gamma)$ be the space of continuous, piecewise affine complex-valued functions on Γ.

For each point $p \in \Gamma$, consider the set $\mathrm{Vec}(p)$ of "formal unit vectors $\vec{v}$ emanating from p". (A rigorous definition is given in §2). The cardinality of $\mathrm{Vec}(p)$ is just the "valence" of p (which is equal to 2 for all but finitely many points $p \in \Gamma$). If

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$\vec{v} \in \mathrm{Vec}(p)$, we define the directional derivative $d_{\vec{v}} f(p)$ of f at p (in the direction $\vec{v}$) to be

$$d_{\vec{v}} f(p) = \lim_{t \to 0^+} \frac{f(p + t\vec{v}) - f(p)}{t}.$$

For $f \in \mathrm{CPA}(\Gamma)$, we define the Laplacian Δf of f to be the discrete measure $\Delta f = -\sum_{p \in \Gamma} \sigma_p(f)\delta_p$, where $\sigma_p(f)$ is the sum of the slopes of f in all directions pointing away from p. More precisely, $\sigma_p(f) = \sum_{\vec{v} \in \mathrm{Vec}(p)} d_{\vec{v}} f(p)$. (Here δ_p is the discrete probability measure on Γ with the property that $\int_\Gamma f(x)\delta_p(x) = f(p)$ for all $f \in \mathcal{C}(\Gamma)$.)

This is essentially the definition of the Laplacian Δ_{CR} given by Chinburg and Rumely in [CR], except that our Laplacian is the negative of theirs. We also remark that the definition of Δ on $\mathrm{CPA}(\Gamma)$ is closely related to the classical definition of the Laplacian matrix for a finite weighted graph (see [BF] for details).

1.2 The Laplacian on the Zhang Space

Following Zhang [Zh], one can define the Laplacian on a more general class of functions than $\mathrm{CPA}(\Gamma)$. The resulting operator combines aspects of the "discrete" Laplacian from the previous section and the "continuous" Laplacian $-f''(x)dx$ on $\mathbb{R}$.

We define the Zhang space $\mathrm{Zh}(\Gamma)$ to be the set of all continuous functions $f : \Gamma \to \mathbb{C}$ such that f is piecewise $\mathcal{C}^2$ and $f''(x) \in L^1(\Gamma)$. (In general, when we say that f is piecewise $\mathcal{C}^k$, we mean that there is a finite set of points $X_f \subset \Gamma$ such that $\Gamma \backslash X_f$ is a finite union of open intervals, and the restriction of f to each of those intervals is $\mathcal{C}^k$.)

For $f \in \mathrm{Zh}(\Gamma)$, we define Δf to be the complex Borel measure on Γ given by

$$\Delta(f) = -f''(x)dx - \sum_{p \in X_f} \Big(\sum_{\vec{v} \in \mathrm{Vec}(p)} d_{\vec{v}} f(p) \Big) \delta_p,$$

where if $x = p + t\vec{v} \in \Gamma \backslash X_f$, we set $f''(x) = d^2/dt^2 f(p + t\vec{v})$.

If $f \in \mathrm{CPA}(\Gamma) \subset \mathrm{Zh}(\Gamma)$, then clearly the two definitions of the Laplacian which we have given agree with each other.

Let $\langle\ ,\ \rangle$ denote the L^2 inner product on $\mathrm{Zh}(\Gamma)$, *i.e.*, $\langle f, g\rangle = \int_\Gamma f(x)\overline{g(x)}\,dx$. The following result shows that the Laplacian on $\mathrm{Zh}(\Gamma)$ is "self-adjoint", and justifies the choice of sign in the definition of Δ.

Proposition 1.1 *For every $f, g \in \mathrm{Zh}(\Gamma)$, $\int_\Gamma \overline{g}\, d\Delta f = \int_\Gamma f\, d\overline{\Delta g} = \langle f', g'\rangle$.*

Proof This formula is given in Zhang [Zh, proof of Lemma a.4] and is a simple consequence of integration by parts. We repeat the proof for the convenience of the reader.

Since $f, g \in \mathrm{Zh}(\Gamma)$, there is a vertex set X for Γ such that $\Gamma \backslash X$ consists of a finite union of open intervals, $f''(x)$ and $g''(x)$ are continuous on $\Gamma \backslash X$, and $f''(x), g''(x) \in L^1(\Gamma)$. In particular f' and g' are continuous on $\Gamma \backslash X$. Using that $f''(x)$ and $g''(x)$ belong to $L^1(\Gamma)$ one sees that f' and g' have boundary limits on

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each of the open segments making up $\Gamma \backslash X$, and these limits coincide with the directional derivatives at those points.

Identifying $\Gamma \backslash X$ with a union of open intervals (a_i, b_i), $i = 1, \ldots, N$, and identifying the directional derivatives at the points in X with one-sided derivatives at the endpoints, we find by integration by parts that

$$\begin{aligned}\langle f, g \rangle_{\mathrm{Dir}} &= \sum_{i=1}^{N} \Big(-\int_{a_i}^{b_i} f(x)\overline{g''(x)}\, dx + f(b_i)\overline{g'(b_i)} - f(a_i)\overline{g'(a_i)} \Big) \\ &= \sum_{i=1}^{N} \int_{a_i}^{b_i} f'(x)\overline{g'(x)}\, dx = \int_{\Gamma} f'(x)\overline{g'(x)}\, dx. \qquad \blacksquare\end{aligned}$$

It follows that the Laplacian has total mass 0:

Corollary 1.2 *For any $f \in \mathrm{Zh}(\Gamma)$, we have $\int_\Gamma \Delta f = 0$.*

Proof $\int_\Gamma \Delta f = \int_\Gamma 1\, d\Delta f = \int_\Gamma f\, d\overline{\Delta 1} = 0.$ ■

This generalizes the fact (see [CR, Corollary 2.9]) that if $f \in \mathrm{CPA}(\Gamma)$ and $\Delta f = \sum_i c_i \delta_{q_i}$, then $\sum c_i = 0$.

1.3 The Dirichlet Inner Product and Eigenfunctions of the Laplacian

Motivated by Proposition 1.1, we define the *Dirichlet inner product* on $\mathrm{Zh}(\Gamma)$ by

$$\langle f, g \rangle_{\mathrm{Dir}} = \int_\Gamma f'(x)\overline{g'(x)}\, dx.$$

We define a corresponding *Dirichlet seminorm* by $\|f\|_{\mathrm{Dir}}^2 = \langle f, f \rangle_{\mathrm{Dir}}$.

Note that $\|f\|_{\mathrm{Dir}} \geq 0$ for all $f \in \mathrm{Zh}(\Gamma)$, with equality if and only if f is constant. Thus $\|\cdot\|_{\mathrm{Dir}}$ is indeed a seminorm on $\mathrm{Zh}(\Gamma)$. To obtain an honest *norm*, let μ be a real-valued signed Borel measure on Γ with $\mu(\Gamma) = 1$ and $|\mu|(\Gamma) < \infty$, and define $\mathrm{Zh}_\mu(\Gamma) = \{f \in \mathrm{Zh}(\Gamma) : \int_\Gamma f\, d\mu = 0\}$. Then for $f \in \mathrm{Zh}_\mu(\Gamma)$, $\|f\|_{\mathrm{Dir}} = 0$ if and only if $f = 0$, *i.e.*, $\|\cdot\|_{\mathrm{Dir}}$ defines a norm on $\mathrm{Zh}_\mu(\Gamma)$.

Let $\mathrm{Dir}_\mu(\Gamma)$ denote the Hilbert space completion of $\mathrm{Zh}_\mu(\Gamma)$ with respect to the Dirichlet norm. By Corollary 3.3 below, $\mathrm{Dir}_\mu(\Gamma)$ can be naturally identified with a subspace of $\mathcal{C}(\Gamma)$, the space of continuous complex-valued functions on Γ.

We define a nonzero function $f \in \mathrm{Zh}_\mu(\Gamma)$ to be an *eigenfunction of the Laplacian*, (with respect to μ) if there exists $\lambda \in \mathbb{C}$ such that

$$\int_\Gamma \overline{g(x)}\, d(\Delta f) = \lambda \int \overline{g(x)} f(x)\, dx,$$

or equivalently $\langle f, g \rangle_{\mathrm{Dir}} = \lambda \langle f, g \rangle$, for all $g \in \mathrm{Dir}_\mu(\Gamma)$.

It is important to note that we only test this identity for g in the space $\mathrm{Dir}_\mu(\Gamma)$, so in particular the eigenvalues and eigenfunctions depend on the choice of μ.

The condition $\langle f, g\rangle_{\mathrm{Dir}} = \lambda\langle f, g\rangle$ makes sense for $f, g \in \mathrm{Dir}_\mu(\Gamma)$ as well, so there is a natural extension to $\mathrm{Dir}_\mu(\Gamma)$ of the notion of an eigenfunction of the Laplacian. (It is irrelevant whether we require g to be in $\mathrm{Zh}_\mu(\Gamma)$ or $\mathrm{Dir}_\mu(\Gamma)$.) Later we will see that the space $\mathrm{Zh}_\mu(\Gamma)$ is not always large enough to contain a complete orthonormal set of eigenfunctions, and we will describe a space which does.

Setting $f = g$ in the definition of an eigenfunction, it is easy to see that the eigenvalues of the Laplacian on $\mathrm{Zh}_\mu(\Gamma)$ (or $\mathrm{Dir}_\mu(\Gamma)$) are real and positive.

Example 1.1 Suppose $\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ on $\Gamma = [0, 1]$. We will see later (Proposition 15.1) that every eigenfunction of the Laplacian on $\mathrm{Dir}_\mu(\Gamma)$ is piecewise $\mathcal{C}^\infty$. It follows that f is an eigenfunction of Δ with respect to μ if and only if there exists $\lambda \in \mathbb{R}$ such that

$$\langle f, g\rangle_{\mathrm{Dir}} = f'(1)\overline{g(1)} - f'(0)\overline{g(0)} - \int_0^1 f''(x)\overline{g(x)}\,dx = \lambda \int_0^1 f(x)\overline{g(x)}\,dx$$

for all $g \in \mathrm{Zh}_\mu(\Gamma)$.

As $g(0) = -g(1)$ for all $g \in \mathrm{Zh}_\mu(\Gamma)$, this simplifies to

$$\overline{g(1)}[f'(1) + f'(0)] - \int_0^1 f''(x)\overline{g(x)}\,dx = \lambda \int_0^1 f(x)\overline{g(x)}\,dx.$$

It is not hard to show that this identity holds for all $g \in \mathrm{Zh}_\mu(\Gamma)$ if and only if $f'(0) = -f'(1)$ and $f''(x) = -\lambda f(x)$. As $f \in \mathrm{Zh}_\mu(\Gamma)$, we must also have $f(0) = -f(1)$. These conditions yield the eigenfunctions $a_n \cos(n\pi x)$, $b_n \sin(n\pi x)$ for all *odd* integers $n \geq 1$, and corresponding eigenvalues $n^2\pi^2$ (each occuring with multiplicity 2), again for $n \geq 1$ odd.

Example 1.2 Suppose $\mu = dx$ is Lebesgue measure on $\Gamma = [0, 1]$. It is shown in Example 16.1 that eigenfunctions of the Laplacian in $\mathrm{Dir}_\mu(\Gamma)$ satisfy $f''(x) = -\lambda f(x)$ with the boundary conditions $f'(0) = f'(1) = 0$. These conditions yield the eigenfunctions $a_n \cos(n\pi x)$ for $n = 1, 2, 3, \ldots$, with eigenvalues $\lambda_n = n^2\pi^2$.

Comparing Examples 1.1 and 1.2 illustrates the fact that the eigenfunctions and eigenvalues depend not only on Γ, but on the choice of μ.

In §3, we will discuss how the classical Rayleigh–Ritz method can be used to obtain the following result:

Theorem 1.3 *Each eigenvalue λ of Δ on $\mathrm{Dir}_\mu(\Gamma)$ occurs with finite multiplicity. If we write the eigenvalues as $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ (with corresponding eigenfunctions $f_1, f_2, \ldots$, normalized so that $\|f_n\|_2 = 1$), then $\lim_{n\to\infty} \lambda_n = \infty$ and $\{f_n\}$ forms a basis for the Hilbert space $\mathrm{Dir}_\mu(\Gamma)$.*

It follows that each $f \in \mathrm{Dir}_\mu(\Gamma)$ has a generalized Fourier expansion

$$f = \sum_{n=1}^{\infty} c_n f_n. \tag{1.2}$$

We will see later that the series in (1.2) converges uniformly to f. However, the techniques of §3 are not strong enough to prove even pointwise convergence.

To establish stronger convergence results, we study eigenfunctions of the Laplacian via an integral operator φ_μ which (in a suitable sense) is inverse to the Laplacian on Dir_μ. Working with integral operators, rather than differential operators, is a well-known method for improving convergence.

1.4 The General Measure-Valued Laplacian

An important observation is that the Laplacian exists as a measure-valued operator on a larger space than $\mathrm{Zh}(\Gamma)$. In §4, we will define a space $\mathrm{BDV}(\Gamma) \subset \mathcal{C}(\Gamma)$ which is in a precise sense the largest space of continuous functions f for which Δf exists as a complex Borel measure. (The abbreviation BDV stands for "bounded differential variation". We refer to §4 for the definition of $\mathrm{BDV}(\Gamma)$ and of Δf for $f \in \mathrm{BDV}(\Gamma)$.) We also define $\mathrm{BDV}_\mu(\Gamma) = \{f \in \mathrm{BDV}(\Gamma) : \int_\Gamma f\, d\mu = 0\}$.

Consideration of the measure-valued Laplacian on $\mathrm{BDV}(\Gamma)$ is important for our integral operator approach to the spectral theory of Δ, and the space $\mathrm{BDV}(\Gamma)$ plays a role in our theory similar to that of a Sobolev space in classical analysis.

1.5 The j-Function and the Arakelov-Green's Function g_μ

In [CR] (see also [BF]), a kernel $j_\zeta(x, y)$ giving a fundamental solution of the Laplacian is defined and studied as a function of $x, y, \zeta \in \Gamma$. For fixed ζ and y it has the following physical interpretation: when Γ is viewed as a resistive electric circuit with terminals at ζ and y, with the resistance in each edge given by its length, then $j_\zeta(x, y)$ is the voltage difference between x and ζ, when unit current enters at y and exits at ζ (with reference voltage 0 at ζ). For fixed y and ζ, the function $j_\zeta(x, y)$ is in $\mathrm{CPA}(\Gamma)$ as a function of x and satisfies the differential equation

$$\Delta_x(j_\zeta(x, y)) \;=\; \delta_y(x) - \delta_\zeta(x) \tag{1.3}$$

(see [CR, (2), p.13]; recall that our $\Delta = -\Delta_{CR}$).

As before, let μ be a real-valued signed Borel measure of total mass 1 on Γ. We define the Arakelov-Green's function $g_\mu(x, y)$ associated to μ to be

$$g_\mu(x, y) = \int_\Gamma j_\zeta(x, y)\, d\mu(\zeta) - C_\mu$$

where $C_\mu = \int_\Gamma j_\zeta(x, y)\, d\mu(\zeta) d\mu(x) d\mu(y)$.

One can characterize $g_\mu(x, y)$ as the unique function on $\Gamma \times \Gamma$ such that

(i) $g_\mu(x, y)$ is an element of $\mathrm{BDV}_\mu(\Gamma)$ as a function of both x and y.
(ii) For fixed y, g_μ satisfies the identity $\Delta_x g_\mu(x, y) = \delta_y(x) - \mu(x)$.
(iii) $\iint_{\Gamma\times\Gamma} g_\mu(x, y)\, d\mu(x) d\mu(y) = 0$.

1.6 The Integral Transform $\varphi_\mu(x, y)$

For $f \in L^2(\Gamma)$, define $\varphi_\mu(f) = \int_\Gamma g_\mu(x, y) f(y)\, dy$. We will see that $\varphi_\mu \colon L^2(\Gamma) \to \mathrm{BDV}_\mu(\Gamma)$ and that $\Delta(\varphi_\mu(f)) = f(x)dx - (\int_\Gamma f(x)\, dx)\mu$. In particular, if $\int_\Gamma g\, d\mu = 0$ then

$$\int_\Gamma \overline{g(x)} \Delta(\varphi_\mu(f)) = \int_\Gamma \overline{g(x)} f(x) dx.$$

We will see that the operator $\varphi_\mu \colon L^2(\Gamma) \to L^2(\Gamma)$ is compact and self-adjoint, and then an application of the spectral theorem will yield the following.

Theorem 1.4 *The map φ_μ has countably many eigenvalues α_i, each of which is real and occurs with finite multiplicity. The nonzero eigenvalues can be ordered so that $|\alpha_1| \geq |\alpha_2| \geq \cdots$ and $\lim_{i\to\infty} |\alpha_i| = 0$. Each eigenfunction corresponding to a nonzero eigenvalue belongs to $\mathrm{BDV}_\mu(\Gamma)$; $L^2(\Gamma)$ has an orthonormal basis consisting of eigenfunctions of φ_μ, and each eigenfunction corresponding to a nonzero eigenvalue belongs to $\mathrm{BDV}_\mu(\Gamma)$.*

We will also see that a nonzero function $f \in \mathrm{Dir}_\mu(\Gamma)$ is an eigenfunction for Δ on $\mathrm{Dir}_\mu(\Gamma)$ with eigenvalue λ if and only if it is an eigenfunction of φ_μ with eigenvalue $\alpha = 1/\lambda$.

This gives us an alternative way to understand the spectral theory of the Laplacian. In particular, it will allow us to prove that the generalized Fourier series expansion in (1.2) converges uniformly on Γ for every $f \in \mathrm{Dir}_\mu(\Gamma)$.

1.7 Eigenfunction Expansion of $g_\mu(x, y)$

The integral operator approach will also enable us to prove the following result concerning the eigenfunction expansion for $g_\mu(x, y)$.

Proposition 1.5 *Let the f_n be as in the statement of Theorem 1.3. Then the series $\sum_{n=1}^{\infty} \frac{f_n(x)\overline{f_n(y)}}{\lambda_n}$ converges uniformly to $g_\mu(x, y)$ for all $x, y \in \Gamma$.*

Proposition 1.5 can be used to prove the following results, which are closely related to results in [BR] and served as part of the motivation for this paper. For the statement of Theorem 1.6, let $\mathrm{Meas}(\Gamma)$ denote the space of all bounded signed Borel measures on Γ.

Theorem 1.6 *Among all $\nu \in \mathrm{Meas}(\Gamma)$ with $\nu(\Gamma) = 1$, μ is the unique measure minimizing the energy integral*

$$I_\mu(\nu) = \iint_{\Gamma\times\Gamma} g_\mu(x, y)\, d\nu(x)\overline{d\nu(y)}. \tag{1.4}$$

Proposition 1.7 *There exists a constant $C > 0$ such that if $N \geq 2$ and $x_1, \dots, x_N \in \Gamma$, then*

$$\frac{1}{N(N-1)} \sum_{i\neq j} g_\mu(x_i, x_j) \geq -\frac{C}{N}.$$

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We remark that Theorem 1.6 is motivated by classical results from potential theory, and Proposition 1.7 specializes to [HS, Lemma 2.1] when Γ is a circle.

In the latter part of the paper, we will study the regularity and boundedness of the eigenfunctions, and give an algorithm to compute them for a useful class of measures μ.

1.8 Examples of Eigenfunction Expansions

Here are some examples of eigenfunction expansions for $g_\mu(x, y)$.

1. $\Gamma = [0, 1]$, $\mu = dx$.

$$g_\mu(x, y) = \begin{cases} \frac{1}{2}x^2 + \frac{1}{2}(1-y)^2 - \frac{1}{6} & \text{if } x < y \\ \frac{1}{2}(1-x)^2 + \frac{1}{2}y^2 - \frac{1}{6} & \text{if } x \geq y \end{cases}$$
$$= 2\sum_{n\geq 1} \frac{\cos(n\pi x)\cos(n\pi y)}{\pi^2 n^2}.$$

2. $\Gamma = [0, 1]$, $\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$.

$$g_\mu(x, y) = \frac{1}{4} - \frac{1}{2}|x - y|$$
$$= \sum_{n\geq 1 \text{ odd}} \Big(\frac{\cos(n\pi x)\cos(n\pi y)}{\pi^2 n^2} + \frac{\sin(n\pi x)\sin(n\pi y)}{\pi^2 n^2} \Big).$$

3. $\Gamma = [0, 1]$, $\mu = \delta_0$.

$$g_\mu(x, y) = \min\{x, y\} = 8\sum_{n\geq 1 \text{ odd}} \frac{\sin(\frac{n\pi x}{2})\sin(\frac{n\pi y}{2})}{\pi^2 n^2}.$$

4. $\Gamma = \mathbb{R}/\mathbb{Z}$, $\mu = dx$.

$$g_\mu(x, y) = \frac{1}{2}|x - y|^2 - \frac{1}{2}|x - y| + \frac{1}{12}$$
$$= 2\sum_{n\geq 1} \Big(\frac{\cos(2\pi nx)\cos(2\pi ny)}{4\pi^2 n^2} + \frac{\sin(2\pi nx)\sin(2\pi ny)}{4\pi^2 n^2} \Big).$$

5. $\Gamma = \mathbb{R}/\mathbb{Z}$, $\mu = \delta_0$.

$$g_\mu(x, y) = \begin{cases} x(1-y) & \text{if } x < y, \\ y(1-x) & \text{if } x \geq y, \end{cases}$$
$$= 2\sum_{n\geq 1} \frac{\sin(n\pi x)\sin(n\pi y)}{\pi^2 n^2}.$$

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1.9 Related Work

The term "metrized graph" was introduced by the second author [Ru], and was further developed by Chinburg and Rumely [CR] and by Zhang [Zh]. In those papers, metrized graphs are seen to arise naturally from considerations in arithmetic geometry, and the function $g_\mu(x, y)$ is related to certain pairings which occur in Arakelov theory and arithmetic capacity theory. The present paper was originally motivated by the study of reduction graphs and Berkovich spaces associated with algebraic curves, with an eye toward proving general adelic equidistribution theorems for points of small canonical height (see [BR]). It has been further developed [RB] to yield a detailed study of the Laplacian on the Berkovich projective line over a complete and algebraically closed nonarchimedean field.

It seems clear, however, that the theory developed in the present work should have applications beyond its origins in arithmetic geometry. There is already a vast literature concerning objects which are more or less the same as metrized graphs. Other names one encounters for these objects include networks, metric graphs, and quantum graphs. We are not expert enough to survey the entire literature and comment on its relation to the present paper, so out of necessity we will restrict our comments to a few specific related works. Metrized graphs have applications to the physical sciences; for example, Nicaise's papers [N1, N2] are written in the context of mathematical biology, and deal with the spread of potential along the dendrites of a neuron. Kuchment's survey paper [Ku], on the other hand, is motivated by considerations in quantum physics. And the results from [AM1]–[AM3] and [Be1, Be2] have applications to various aspects of physics and electrical and mechanical engineering.

There is a well-developed mathematical theory dealing with differential equations and spectral problems on metrized graphs. We note especially the following works. J. P. Roth solved the heat equation on a metrized graph [Ro], and used this to show the existence of nonisomorphic isospectral metrized graphs ("Can you hear the shape of a graph?"). J. von Below studied the characteristic equation associated to the eigenvalue problem for heat transmission on a metrized graph, and in the case corresponding to our $\mu = dx$, gave a precise description of the eigenvalues and their multiplicities [Be1, p. 320]. He also studied very general problems of Sturm–Liouville type on graphs [Be2]. He showed they have discrete eigenspectra and that the eigenfunctions yield uniformly convergent series expansions of well-behaved functions. F. Ali Mehmeti studied nonlinear wave equations on CW-complexes of arbitrary dimension, together with boundary conditions coming from higher-dimensional analogues of Kirckhoff's laws [AM2]. By casting the theory in abstract functional-analytic form, he obtained very general results about asymptotics of eigenvalues. L. Friedlander showed that for a "generic" metrized graph, all eigenvalues of the Laplacian (relative to the measure $\mu = dx$) have multiplicity one [Fr], and he gave a sharp lower bound for the dx-eigenvalues of the Laplacian on a metrized graph [Fr2].

We should also mention here the recent book by Favre and Jonsson [FJ]. They develop a detailed theory of Laplacians on (not necessarily finite) metrized trees which is closely related to our approach in the special case where the graph is both finite and a tree.

Finally, we would like to point out that the present paper is more or less self-contained, and uses less sophisticated tools from analysis than most of the papers cited above. In particular, we make no use of the theory of Sobolev spaces (which are omnipresent in the literature on networks and quantum graphs), although their introduction would probably shorten some of the proofs here. For the reader seeking an even less technical introduction to the Laplacian on a metrized graph, a leisurely survey can be found in [BF].

2 Metrized Graphs

2.1 Definition of a Metrized Graph

We begin by giving a rigorous definition of a metrized graph, following [Zh]. We will then explain how this definition relates to the intuitive definition given in the introduction.

Definition 2.1 A *metrized graph* Γ is a compact, connected metric space such that for each $p \in \Gamma$, there exist a radius $r_p > 0$ and an integer $n_p \geq 1$ such that p has a neighborhood $V_p(r_p)$ isometric to the star-shaped set

$$S(n_p, r_p) = \{z \in \mathbb{C} : z = te^{k \cdot 2\pi i/n_p} \text{ for some } 0 \leq t < r_p \text{ and some } k \in \mathbb{Z}\},$$

equipped with the path metric.

There is a close connection between metrized graphs and finite weighted graphs. A *finite weighted graph* G is a connected, weighted combinatorial graph equipped with a collection $V = \{v_1, \ldots, v_n\}$ of vertices, $E = \{e_1, \ldots, e_m\}$ of edges, and $W = \{w_{ij}\}$ of nonnegative weights ($1 \leq i, j \leq n$) satisfying:

- If $w_{ij} = 0$, then there is no edge connecting v_i and v_j.
- If $w_{ij} > 0$, then there is a unique edge e_k connecting v_i and v_j.
- $w_{ij} = w_{ji}$ for all i, j.
- $w_{ii} = 0$ for all i.

We let $w(e_k)$ denote the weight of edge e_k, and define the *length* of e_k to be $L(e_k) = 1/w(e_k)$.

There is a natural partial ordering on the collection of finite weighted graphs, where $G' \leq G$ if we can refine G to G' by a sequence of length-preserving (as opposed to weight-preserving) subdivisions.

We write $G \sim G'$ if there exists a finite weighted graph G'' with $G'' \leq G$ and $G'' \leq G'$, *i.e.*, if G and G' have a common subdivision. It is easy to see that this defines an equivalence relation on the collection of finite weighted graphs.

Lemma 2.2 *There is a bijective correspondence between metrized graphs and equivalence classes of finite weighted graphs.*

Proof Suppose first that Γ is a metric space satisfying Definition 2.1. By compactness, it can be covered by a finite number of neighborhoods $V_p(r_p)$. Hence there

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are only finitely many points $p \in \Gamma$ for which $n_p \neq 2$. We will call this collection of points, or any finite set of points in Γ containing it, a vertex set for Γ. Let V be a nonempty vertex set for Γ. Then $\Gamma \backslash V$ has finitely many connected components. Each of these can be covered by a finite number of open intervals (if p is a vertex, then $V_p(r_p)\backslash\{p\}$ is isometric to a set composed of n_p disjoint open intervals), and so is isometric to an open interval $(0, L_i)$. The closure of such a component in Γ is an edge; it will be isometric to either a segment or a loop. After adjoining finitely many points to V, we can assume that each edge has 2 distinct boundary points, and hence is isometric to a closed interval. The resulting set of vertices and edges defines a finite weighted graph G with weights given by $w(e_k) = 1/L_k$. It is easy to see that the equivalence class of G is independent of all choices made.

Conversely, given an equivalence class of finite weighted graphs, let G be a representative of this class. We define a corresponding metrized graph Γ as follows. The points of Γ are just the points of G, where each edge of G is considered as a line segment of length L_i. It is clear from the definition of equivalence that this set is independent of the choice of G.

Choose a parametrization $\psi_i \colon [a_i, b_i] \to e_i$ of each edge e_i of G by a line segment of length L_i, and let the "chart" $\varphi_i \colon e_i \to [a_i, b_i]$ be the inverse to the parametrization ψ_i. We will call a function $\gamma \colon [a, b] \to \Gamma$ a *piecewise isometric path* if there is a finite partition $a = c_0 < c_1 < \cdots < c_m = b$ such that for each $j = 1, \dots, m$ there is an edge $e_{i(j)}$ such that $\gamma([c_{j-1}, c_j]) \subset e_{i(j)}$, and $\varphi_{i(j)} \circ \gamma(t)$ has constant derivative ± 1 on $[c_{j-1}, c_j]$. For any two points $p, q \in \Gamma$, put

$$d(p,q) = \inf_{\substack{\gamma:[a,b]\to\Gamma \\ \gamma(a)=p,\gamma(b)=q}} |b-a|,$$

where the infimum is taken over all piecewise isometric paths from p to q. It is easy to see that $d(p,q)$ is independent of the choice of parametrizations, that $d(p,q)$ defines a metric on Γ, and that for each p there is an $r_p > 0$ such that for any $q \in \Gamma$ with $d(p,q) < r_p$ there is a unique shortest piecewise isometric path from p to q. If p belongs to the vertex set of G, let n_p be the valence of G at p (the number of edges of G incident to p). Otherwise, let $n_p = 2$. Then the neighborhood $V_p(r_p) = \{q \in \Gamma : d(p,q) < r_p\}$ is isometric to the star $S(r_p, n_p) \subset \mathbb{C}$. ■

Henceforth, given a metrized graph Γ, we will frequently choose without comment a corresponding finite weighted graph G as in Lemma 2.2, together with distinguished parametrizations of the edges of G. The reader will easily verify that all important definitions below (including the definitions of directional derivatives and the Laplacian) are independent of these implicit choices.

By an "isometric path" in Γ, we will mean a one-to-one piecewise isometric path. Since Γ is connected, it is easy to see that for all $x, y \in \Gamma$ there exists an isometric path from x to y. We will say that an isometric path $\gamma \colon [0, L] \to \Gamma$ *emanates from* p, and *ends at* q, if $\gamma(0) = p$ and $\gamma(L) = q$. If $p \in \Gamma$, there are n_p distinguished isometric paths γ_i emanating from p such that any other isometric path emanating from p has an initial segment which factors through an initial segment of one of the distinguished paths. We will formally introduce n_p "unit vectors" $\vec{v}_i$ to describe these paths, and write $p + t\vec{v}_i$ for $\gamma_i(t)$. By abuse of language, we will refer to the distinguished paths $\gamma_i(t) = p + t\vec{v}_i$ as "edges emanating from p". If $f\colon \Gamma \to \mathbb{C}$ is a function, and $\vec{v} = \vec{v}_i$ a unit vector at p, then we will define the "(one-sided) derivative of f in the direction $\vec{v}$" to be

$$d_{\vec{v}} f(p) = \lim_{t \to 0^+} \frac{f(p + t\vec{v}) - f(p)}{t}$$

provided the limit exists as a finite number. Write $\mathrm{Vec}(p)$ for the collection of directions emanating from p.

3 Eigenfunctions via Comparison of Dirichlet and L^2 Norms

In this section, we discuss a "Rayleigh–Ritz"-type method for constructing eigenfunctions of the Laplacian. The method is motivated by a classical procedure for solving the Laplace–Dirichlet eigenvalue problem (see [PE, Ch. 4]).

Recall that the L^2 inner product for $f, g \in L^2(\Gamma)$ is

$$\langle f, g \rangle = \langle f, g \rangle_{L^2} = \int_\Gamma f(x)\overline{g(x)}\, dx,$$

with associated norm $\|\cdot\|_2 = \langle f, f \rangle^{1/2}$.

Let μ be a real-valued signed Borel measure on Γ with $\mu(\Gamma) = 1$ and $|\mu|(\Gamma) < \infty$. Recall from §1.3 that $\mathrm{Zh}(\Gamma)$ denotes the space of all continuous functions $f\colon \Gamma \to \mathbb{C}$ such that f is piecewise $\mathcal{C}^2$ and $f'' \in L^1(\Gamma)$, and that

$$\mathrm{Zh}_\mu(\Gamma) = \left\{ f \in \mathrm{Zh}(\Gamma) : \int_\Gamma f\, d\mu = 0 \right\}.$$

Also, $\mathrm{Dir}_\mu(\Gamma)$ denotes the completion of $\mathrm{Zh}_\mu(\Gamma)$ with respect to the Dirichlet norm $\|\cdot\|_{\mathrm{Dir}}$, and $L^2_\mu(\Gamma)$ denotes the completion of $\mathrm{Zh}_\mu(\Gamma)$ with respect to the L^2-norm.

Lemma 3.1 *For all $f \in \mathrm{Zh}_\mu(\Gamma)$, we have the estimates $\|f\|_\infty \leq \ell(\Gamma)^{\frac{1}{2}} \cdot |\mu|(\Gamma) \cdot \|f\|_{\mathrm{Dir}}$, and the* Poincaré Inequality $\|f\|_2 \leq \ell(\Gamma) \cdot |\mu|(\Gamma) \cdot \|f\|_{\mathrm{Dir}}$, *where $\ell(\Gamma) = \int_\Gamma dx$ is the total length of Γ.*

In particular, there exist constants $C_\infty, C_2 > 0$ (depending only on μ and Γ) such that $\|f\|_\infty \leq C_\infty \|f\|_{\mathrm{Dir}}$ and $\|f\|_2 \leq C_2 \|f\|_{\mathrm{Dir}}$ for all $f \in \mathrm{Zh}_\mu(\Gamma)$.

Proof Fix a point $x_0 \in \Gamma$, let $x \in \Gamma$ be another point, and let γ be an isometric path from x_0 to x.

Since f is continuous and piecewise $\mathcal{C}^1$, it follows from the fundamental theorem of calculus that $f(x) - f(x_0) = \int_\gamma f'(t)\, dt$, so that

$$|f(x) - f(x_0)| \leq \int_\gamma |f'(t)|\, dt \leq \int_\Gamma |f'(t)|\, dt \leq \ell(\Gamma)^{1/2} \|f\|_{\mathrm{Dir}}$$

by the Cauchy–Schwarz inequality. Since $f \in \mathrm{Zh}_\mu(\Gamma)$ we have $\int_\Gamma f(x)\,d\mu(x) = 0$, so $\int_\Gamma \big(f(x) - f(x_0)\big)\ d\mu(x) = -f(x_0)$, and

$$|f(x_0)| = \Big|\int_\Gamma (f(x) - f(x_0))\,d\mu(x)\Big| \le \int_\Gamma |f(x) - f(x_0)|\,d|\mu|(x)$$
$$\le \ell(\Gamma)^{1/2}|\mu|(\Gamma)\|f\|_{\mathrm{Dir}}.$$

This holds for each $x_0 \in \Gamma$, so $\|f\|_\infty \le \ell(\Gamma)^{\frac{1}{2}}|\mu|(\Gamma)\|f\|_{\mathrm{Dir}}$ and

$$\|f\|_2^2 = \int_\Gamma |f(x)|^2 dx \le \ell(\Gamma)^2|\mu|(\Gamma)^2\|f\|_{\mathrm{Dir}}^2$$

as desired. ∎

Corollary 3.2 *A sequence in* $\mathrm{Zh}_\mu(\Gamma)$ *which is Cauchy with respect to the Dirichlet norm is also Cauchy with respect to the* L^2*-norm. In particular, there is a natural inclusion map* $\mathrm{Dir}_\mu(\Gamma) \subseteq L^2_\mu(\Gamma)$.

Corollary 3.3 *We can uniquely identify* $\mathrm{Dir}_\mu(\Gamma)$ *with a linear subspace of* $\mathcal{C}(\Gamma)$ *containing* $\mathrm{Zh}_\mu(\Gamma)$ *in such a way that for each* $f \in \mathrm{Dir}_\mu(\Gamma)$, $\|f\|_\infty \le C_\infty\|f\|_{\mathrm{Dir}}$. *Under this identification, each* $f \in \mathrm{Dir}_\mu(\Gamma)$ *satisfies* $\int_\Gamma f(x)\,d\mu(x) = 0$.

Proof Take $f \in \mathrm{Dir}_\mu(\Gamma)$. Let $f_1, f_2, \dots \in \mathrm{Zh}_\mu(\Gamma)$ be a sequence of functions with $\|f - f_n\|_{\mathrm{Dir}} \to 0$. Then for each $\varepsilon > 0$, there is an N such that for all $n, m \ge N$, $\|f_n - f_m\|_{\mathrm{Dir}} < \varepsilon$.

By Lemma 3.1, $\|f_n - f_m\|_\infty \le C_\infty\|f_n - f_m\|_{\mathrm{Dir}}$, so the functions f_n converge uniformly to a function $F(x)$. Since each $f_n(x)$ is continuous, $F(x)$ is continuous. Furthermore, $\int_\Gamma F(x)\,d\mu(x) = 0$ since $\int_\Gamma f_n(x)\,d\mu(x) = 0$ for each n.

It is easy to see that $F(x)$ is independent of the sequence $\{f_n\}$ in $\mathrm{Zh}_\mu(\Gamma)$ converging to f. (If $\{h_n\}$ is another such sequence, apply the argument above to the sequence $f_1, h_1, f_2, h_2, \dots$.) In particular, $F(x) = 0$ if $f = 0$ in $\mathrm{Dir}_\mu(\Gamma)$. Thus, there is a natural injection $\iota_\mu\colon \mathrm{Dir}_\mu(\Gamma) \to \mathcal{C}(\Gamma)$. We use this to identify f with $F(x) = \iota_\mu(f)$.

We claim that $\|F\|_\infty \le C_\infty\|f\|_{\mathrm{Dir}}$. Indeed, if $\{f_n\}$ is a sequence of functions in $\mathrm{Zh}_\mu(\Gamma)$ converging to f, then

$$\|F\|_\infty = \lim_{n\to\infty}\|f_n\|_\infty \le \lim_{n\to\infty} C_\infty\|f_n\|_{\mathrm{Dir}} \ = \ C_\infty\|f\|_{\mathrm{Dir}}.$$

Under the identification of f with $\iota_\mu(f)$, functions in $\mathrm{Zh}_\mu(\Gamma)$ are taken to themselves, since $f \in \mathrm{Zh}_\mu(\Gamma)$ is the limit of the constant sequence $\{f, f, f, \dots\}$.

We leave the uniqueness assertion to the reader. ∎

Henceforth we will identify elements of $\mathrm{Dir}_\mu(\Gamma)$ with continuous functions as in Corollary 3.3. Since $\mathrm{Zh}_\mu(\Gamma)$ is dense in $\mathrm{Dir}_\mu(\Gamma)$, we easily deduce the following stronger version of Lemma 3.1.

Corollary 3.4 *If $f \in \mathrm{Dir}_\mu(\Gamma)$, then $\|f\|_\infty \le C_\infty \|f\|_{\mathrm{Dir}}$ and $\|f\|_2 \le C_2 \|f\|_{\mathrm{Dir}}$.*

The next lemma will be used in the proof of Lemma 3.6. For the statement, recall that a sequence $\{f_n\}$ in a Hilbert space H *converges weakly* to $f \in H$ if $\langle f_n, g\rangle \to \langle f, g\rangle$ for all $g \in H$.

Lemma 3.5 *Suppose $f_n \in \mathrm{Zh}_\mu(\Gamma)$ and $f_n \to 0$ weakly in $\mathrm{Dir}_\mu(\Gamma)$. Then for every subset S of Γ isometric to a closed interval, we have $\int_S f_n'(x)\,dx \to 0$.*

Proof Assume that S is parametrized by the segment $[a, b]$. Then by the fundamental theorem of calculus and the defining relation $\Delta_x j_\zeta(x, y) = \delta_y(x) - \delta_\zeta(x)$ for $j_\zeta(x, y)$, we have

$$\begin{aligned}\int_S f_n'(x)\,dx &= f_n(b) - f_n(a)\\ &= \int_\Gamma f_n(x)\,d(\delta_b(x) - \delta_a(x))\\ &= \int_\Gamma f_n(x)\,d(\Delta_x j_a(x, b))\\ &= \int_\Gamma f_n'(x)\Big(\frac{d}{dx} j_a(x, b)\Big)\,dx,\end{aligned}$$

where the last equality follows from Proposition 1.1.

Since $j_a(x, b) + c_\mu \in \mathrm{Zh}_\mu(\Gamma)$ for some constant $c_\mu \in \mathbb{R}$, the hypothesis of weak convergence now gives us what we want. ∎

Remark When Γ contains a cycle, there need not be a function g_S in $\mathrm{Zh}_\mu(\Gamma)$ whose derivative is the characteristic function of S. When such a function g_S exists, however, one can argue more simply that

$$\int_S f_n'(x)\,dx = \int_\Gamma f_n'(x) g_S'(x)\,dx \to \int_\Gamma 0\,dx = 0.$$

The following lemma is analogous to Rellich's theorem in classical analysis:

Lemma 3.6 *If $f_n \to f$ weakly in $\mathrm{Dir}_\mu(\Gamma)$ and the sequence $\|f_n\|_{\mathrm{Dir}}$ is bounded, then $f_n \to f$ strongly in $L^2_\mu(\Gamma)$. In particular, $\|f_n\|_2 \to \|f\|_2$.*

Proof Choose a sequence g_n of functions in $\mathrm{Zh}_\mu(\Gamma)$ such that $\|g_n - f_n\|_{\mathrm{Dir}} \to 0$. If we can prove the lemma for the sequence g_n, then the corresponding result for f_n follows easily using Corollary 3.4. We may therefore assume, without loss of generality, that each $f_n \in \mathrm{Zh}_\mu(\Gamma)$.

We want to show that $\int_\Gamma |f_n(x) - f(x)|^2 dx \to 0$. By Lebesgue's dominated convergence theorem, it is enough to show that $f_n \to f$ pointwise, and that there exists

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$M > 0$ such that $|f_n(x)| \le M$ for all $x \in \Gamma, n \ge 1$. The latter assertion is clear from Lemma 3.1, since the Dirichlet norms of the f_n's are assumed to be bounded. So it remains to prove that f_n converges pointwise to f.

Let $h_n = f_n - f$. As in the proof of Lemma 3.1, fix a point $x_0 \in \Gamma$, let $x \in \Gamma$, and choose an isometric path γ from x_0 to x. Then by the fundamental theorem of calculus, we have $h_n(x) = h_n(x_0) + \int_\gamma h_n'(t)\,dt$ for all n.

As $h_n \to 0$ weakly in $\mathrm{Dir}_\mu(\Gamma)$, we see by Lemma 3.5 that

$$|h_n(x) - h_n(x_0)| = \left| \int_\gamma h_n'(t)\,dt \right| \to 0,$$

so that $h_n(x) - h_n(x_0) \to 0$ as $n \to \infty$.

On the other hand, since the functions $h_n(x)$ are uniformly bounded on Γ, it follows by Lebesgue's dominated convergence theorem that

$$\int_\Gamma (h_n(x) - h_n(x_0))\,d\mu(x) \to 0.$$

As $\int_\Gamma h_n(x)\,d\mu(x) = 0$, this shows that $h_n(x_0) \to 0$ as $n \to \infty$ as desired. ■

We now define what it means to be an eigenfunction of the Laplacian on $\mathrm{Dir}_\mu(\Gamma)$.

Definition 3.7 A nonzero function f is an *eigenfunction* for Δ acting on $\mathrm{Dir}_\mu(\Gamma)$ if there exists a constant $\lambda \in \mathbb{C}$ (the *eigenvalue*) such that for all $g \in \mathrm{Dir}_\mu(\Gamma)$

$$\langle f, g\rangle_{\mathrm{Dir}} = \lambda \langle f, g\rangle. \tag{3.1}$$

It is clear (setting $g = f$) that every eigenvalue of Δ is real and positive, and we will write $\lambda = \gamma^2 > 0$.

Theorem 3.8

(i) *Each eigenvalue λ of Δ on $\mathrm{Dir}_\mu(\Gamma)$ is nonzero and occurs with finite multiplicity. If we write the eigenvalues as $0 < \lambda_1 \le \lambda_2 \le \cdots$ (with corresponding eigenfunctions $f_1, f_2, \dots$), then $\lim_{n\to\infty} \lambda_n = \infty$.*

(ii) *Suppose the eigenfunctions f_n are normalized so that $\|f_n\|_2 = 1$. Then $\{f_n\}$ forms a basis for $\mathrm{Dir}_\mu(\Gamma)$, and an orthonormal basis for $L^2_\mu(\Gamma)$.*

We will not give a proof of Theorem 3.8, since the ideas involved are well known (see [PE, Ch. 4], [Ku]). However, it seems worthwhile to at least point out that the "Rayleigh–Ritz" approach to proving Theorem 3.8 is based on the following useful characterization of the leading eigenvalue λ_1.

Theorem 3.9 *Let $S_1 = \{f \in \mathrm{Dir}_\mu(\Gamma) : \|f\|_2 = 1\}$ be the L^2 unit sphere in $\mathrm{Dir}_\mu(\Gamma)$, let $\gamma_1 = \inf_{f\in S_1} \|f\|_{\mathrm{Dir}}$, and let $\lambda_1 = \gamma_1^2$. Then λ_1 is the smallest eigenvalue of Δ on $\mathrm{Dir}_\mu(\Gamma)$.*

The key ingredient needed to prove Theorem 3.9 along the lines of [PE, Ch. 4] is the estimate found in Lemma 3.6.

Before moving on, it may be useful to note how eigenvalues and eigenfunctions behave under scaling in the underlying graph. If Γ is a metrized graph, and $0 < \beta \in \mathbb{R}$, we can define a new metrized graph $\Gamma(\beta)$ whose underlying point set is the same as Γ, but in which all distances are multiplied by β. Write $d(x, y)$ for the metric on Γ, and $d_\beta(x, y)$ for the metric on $\Gamma(\beta)$. There is a natural isomorphism $\sigma_\beta \colon \Gamma(\beta) \to \Gamma$ such that for all $x, y \in \Gamma(\beta)$, $d_\beta(x, y) = \beta \cdot d(\sigma_\beta(x), \sigma_\beta(y))$. If μ is a measure on Γ, then $\mu_\beta = \mu \circ \sigma_\beta$ is a measure on $\Gamma(\beta)$.

Proposition 3.10 *Let Γ be a metrized graph, and let μ be a real signed Borel measure on Γ of total mass 1. Take $\beta > 0$. Then there is a one-to-one correspondence between the eigenvalues λ_n of Δ on $\mathrm{Dir}_\mu(\Gamma)$ and the eigenvalues $\lambda_n(\beta)$ of Δ on $\mathrm{Dir}_{\mu_\beta}(\Gamma(\beta))$, given by*

$$\lambda_n(\beta) = \frac{1}{\beta^2}\lambda_n.$$

Furthermore, $f \in \mathrm{Dir}_\mu(\Gamma)$ is an eigenfunction for Δ with eigenvalue λ if and only if $f \circ \sigma_\beta \in \mathrm{Dir}_{\mu_\beta}(\Gamma(\beta))$ is an eigenfunction for Δ with eigenvalue $\frac{1}{\beta^2}\lambda$.

Proof This follows immediately from the definitions, if one notes that the Dirichlet and L^2 pairings on Γ and $\Gamma(\beta)$ are related by

$$\langle f \circ \sigma_\beta, g \circ \sigma_\beta \rangle_{\mathrm{Dir}(\Gamma(\beta))} = \frac{1}{\beta} \cdot \langle f, g \rangle_{\mathrm{Dir}(\Gamma)},$$

$$\langle f \circ \sigma_\beta, g \circ \sigma_\beta \rangle_{L^2(\Gamma(\beta))} = \beta \cdot \langle f, g \rangle_{L^2(\Gamma)},$$

as follows from the chain rule and the change of variables formula. ■

4 The Classes $\mathcal{D}(\Gamma)$ and $\mathrm{BDV}(\Gamma)$

Let $\mathcal{D}(\Gamma)$ be the class of all functions on Γ whose one-sided derivatives exist everywhere, *i.e.*, $\mathcal{D}(\Gamma) = \{f \colon \Gamma \to \mathbb{C} : d_{\vec{v}} f(p) \text{ exists for each } p \in \Gamma \text{ and } \vec{v} \in \mathrm{Vec}(p)\}$. It is easy to see that each $f \in \mathcal{D}(\Gamma)$ is continuous.

Let $\mathcal{A}$ be the Boolean algebra of subsets of Γ generated by the *connected* open sets. Each $S \in \mathcal{A}$ is a finite disjoint union of sets isometric to open, half-open, or closed intervals (we consider isolated points to be degenerate closed intervals); conversely, all such sets belong to $\mathcal{A}$. Define the set $b(S)$ of boundary points of S to be the collection of points belonging to the closures of both S and $\Gamma \backslash S$. It is easy to see that each $S \in \mathcal{A}$ has only finitely many boundary points. Note that under this definition, if $\Gamma = [0, 1]$ and $S = [0, \frac{1}{2}]$, for example, then the left endpoint 0 is not a boundary point of S.

For each $p \in b(S)$, let $\mathrm{In}(p, S)$ be the set of "inward-directed unit vectors at p": the set of all $\vec{v} \in \mathrm{Vec}(p)$ for which $p + t\vec{v}$ belongs to S for all sufficiently small $t > 0$. Let $\mathrm{Out}(p, S) = \mathrm{Vec}(p) \backslash \mathrm{In}(p, S)$ be the collection of "outward-directed unit vectors at p". If p is an isolated point of S, then $\mathrm{In}(p, S) = \varnothing$ and $\mathrm{Out}(p, S) = \mathrm{Vec}(p)$.

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If $f \in \mathcal{D}(\Gamma)$, then we can define a finitely additive set function m_f on $\mathcal{A}$ by requiring that for each $S \in \mathcal{A}$

$$m_f(S) = \sum_{\substack{b\in b(S)\\ b\notin S}} \sum_{\vec{v}\in \mathrm{In}(b,S)} d_{\vec{v}} f(p) - \sum_{\substack{b\in b(S)\\ b\in S}} \sum_{\vec{v}\in \mathrm{Out}(b,S)} d_{\vec{v}} f(p).$$

Thus, for an open set $S \in \mathcal{A}$,

$$m_f(S) \; = \sum_{b\in b(S)} \sum_{\vec{v}\in \mathrm{In}(b,S)} d_{\vec{v}} f(p),$$

for a closed set $S \in \mathcal{A}$,

$$m_f(S) \; = \; - \sum_{b\in b(S)} \sum_{\vec{v}\in \mathrm{Out}(b,S)} d_{\vec{v}} f(p),$$

and for a set $S = \{p\}$ consisting of a single point,

$$m_f(\{p\}) \; = - \sum_{\vec{v}\in \mathrm{Vec}(p)} d_{\vec{v}} f(p).$$

The finite additivity is clear if one notes that each $S \in \mathcal{A}$ can be written as a finite disjoint union of open intervals and points, and that for each boundary point p of S, the set $\mathrm{In}(p, S)$ coincides with $\mathrm{Out}(p, \Gamma\backslash S)$.

Note that $m_f(\varnothing) = m_f(\Gamma) = 0$, since by our definition, both the empty set and the entire graph Γ have no boundary points. It follows for any $S \in \mathcal{A}$

$$m_f(\Gamma\backslash S) = -m_f(S).$$

If $f_1, f_2 \in \mathcal{D}$ and $c_1, c_2 \in \mathbb{C}$, then it is easy to see that the set function corresponding to $c_1 f_1 + c_2 f_2$ is $m_{c_1 f_1 + c_2 f_2} = c_1 m_{f_1} + c_2 m_{f_2}$.

We will say that a function $f \in \mathcal{D}(\Gamma)$ is of "bounded differential variation", and write $f \in \mathrm{BDV}(\Gamma)$, if there is a constant $B > 0$ such that for any countable collection $\mathcal{F}$ of pairwise disjoint sets in $\mathcal{A}$,

$$\sum_{S_i\in\mathcal{F}} |m_f(S_i)| \le B.$$

It is easy to see that $\mathrm{BDV}(\Gamma)$ is a $\mathbb{C}$-linear subspace of $\mathcal{D}(\Gamma)$.

Proposition 4.1 *Let Γ be a metrized graph.*

(i) *If $f \in \mathrm{BDV}(\Gamma)$, then there are only countably many points $p_i \in \Gamma$ for which $m_f(p_i) \neq 0$, and $\sum_i |m_f(p_i)|$ converges.*

(ii) *In the definition of* BDV(Γ) *one can restrict to families of pairwise disjoint connected open sets, or connected closed sets. More precisely, a function $f \in \mathcal{D}(\Gamma)$ belongs to* BDV(Γ) *if and only if either*

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(a) *there is a constant* B_1 *such that for any countable family* $\mathcal{F}$ *of pairwise disjoint connected open sets* $V_i \in \mathcal{A}$,

$$\sum_{\substack{V_i \in \mathcal{F} \\ V_i \text{ connected, open}}} |m_f(V_i)| \leq B_1$$

(b) *or there is a constant* B_2 *such that for any countable family* $\mathcal{F}$ *of pairwise disjoint connected closed sets* $E_i \in \mathcal{A}$,

$$\sum_{\substack{E_i \in \mathcal{F} \\ E_i \text{ connected, closed}}} |m_f(E_i)| \leq B_2.$$

Proof For (i), note that for each $0 < n \in \mathbb{Z}$ there can be at most $nB = B/(1/n)$ points $p \in \Gamma$ with $|m_f(\{p\})| \geq 1/n$, so there are at most countably many points with $m_\nu(\{p\}) \neq 0$. Moreover,

$$\sum_{\substack{p \in \Gamma \\ m_f(\{p\}) \neq 0}} |m_f(\{p\})| < B.$$

For (ii), note that if $f \in \mathrm{BDV}(\Gamma)$ then (ii)(a)and (b) hold trivially. Conversely, first suppose (ii)(a) holds. We begin by showing that there are only countably many points $p \in \Gamma$ for which $m_f(p) \neq 0$, and that

$$\sum_{m_f(\{p\}) \neq 0} |m_f(\{p\})| < 2B_1. \tag{4.1}$$

To see this, suppose that for some $0 < n \in \mathbb{Z}$, there were more than $2nB_1$ points p with $|m_\nu(\{p\})| \geq 1/n$. Then either there would be more than nB_1 points with $m_\nu(\{p\}) \geq 1/n$, or more than nB_1 points with $m_\nu(\{p\}) \leq -1/n$. Suppose for example that $p_1, \ldots, p_r$ satisfy $m_\nu(\{p_i\}) \leq -1/n$, where $r > nB_1$. Put $K = \{p_1, \ldots, p_r\}$, and let $V_1, \ldots, V_s$ be the connected components of $\Gamma \backslash K$. Then

$$\begin{aligned} B_1 &< -(m_f(\{p_1\}) + \cdots + m_f(\{p_r\})) = -m_f(K) \\ &= m_f(V_1 \cup \cdots \cup V_s) \ = \ m_\nu(V_1) + \cdots + m_\nu(V_s) \\ &\leq |m_f(V_1)| + \cdots + |m_f(V_r)| \leq B_1. \end{aligned}$$

Hence there can be only countably many points p_i with $m_f(\{p_i\}) \neq 0$, and (4.1) holds.

Now let $\mathcal{F}$ be any countable collection of pairwise disjoint sets $S_i \in \mathcal{A}$. Each S_i can be decomposed as a finite disjoint union of connected open sets and sets consisting of isolated points, and if $S_i = V_{i1} \cup \cdots \cup V_{i,r_i} \cup \{p_{i1}\} \cup \cdots \cup \{p_{i,s_i}\}$ is such a decomposition, then

$$\sum_{S_i \in \mathcal{F}} |m_f(S_i)| \leq \sum_{i,j} |m_f(V_{ij})| + \sum_{i,k} |m_f(\{p_{ik}\})| \leq 3B_1.$$

It follows that $f \in \mathrm{BDV}(\Gamma)$.

Next suppose that (ii)(b) holds; we will show that (ii)(a) holds as well. Consider a countable collection $\mathcal{F}$ of pairwise disjoint connected open sets V_i. Decompose $\mathcal{F} = \mathcal{F}_+ \cup \mathcal{F}_-$, where $V_i \in \mathcal{F}_+$ if and only if $m_f(V_i) \geq 0$, and $V_i \in \mathcal{F}_-$ if and only if $m_f(V_i) < 0$. Relabel the sets so that $\mathcal{F}_+ = \{V_1, V_3, V_5, \dots\}$. For each n, put $K_n = \Gamma \backslash (V_1 \cup V_3 \cup \cdots \cup V_{2n+1})$ and decompose K_n as a finite disjoint union of connected closed sets $E_1, \dots, E_r$. Then

$$\sum_{i=0}^{n} m_f(V_{2i+1}) = m_f(\Gamma \backslash K_n) = |m_f(K_n)| \leq \sum_{j=1}^{r} |m_f(E_j)| \leq B_2.$$

Letting $n \to \infty$ we see that $\sum_{V_i \in \mathcal{F}_+} m_f(V_i) \leq B_2$. Similarly $\sum_{V_i \in \mathcal{F}_-} |m_f(V_i)| \leq B_2$, so $\sum_{V_i \in \mathcal{F}} |m_f(V_i)| \leq 2B_2$, and we are done. ∎

We now come to the main result of this section.

Theorem 4.2 *If $f \in \mathrm{BDV}(\Gamma)$, then the finitely additive set function m_f extends to a bounded complex measure m_f^*, with total mass 0, on the σ-algebra of Borel sets of Γ.*

Proof We begin with a reduction. It suffices to show that the restriction of m_f to each edge e_i extends to a Baire measure on e_i. Identifying e_i with its parametrizing interval, we can assume without loss of generality that $\Gamma = [a, b]$ is a closed interval and that $f\colon [a, b] \to \mathbb{C}$ is in $\mathrm{BDV}([a, b])$.

We next decompose m_f into an atomic and an atomless part.

Let $\{p_1, p_2, \dots\}$ be the points in $[a, b]$ for which $m_f(\{p_i\}) \neq 0$. For brevity, write $c_i = m_f(\{p_i\})$; by hypothesis, $\sum_i |c_i|$ converges. Define $g\colon \mathbb{R} \to \mathbb{C}$ by

$$g(t) = -\frac{1}{2} \sum_i c_i |t - p_i|,$$

and let $m_g^* = \sum_i c_i \delta_{p_i}(x)$ be the Borel measure giving the point mass c_i to each p_i.

By a direct computation, one checks that for each $x \in \mathbb{R}$, both one-sided derivatives $g'_\pm(x)$ exist, with

$$g'_-(x) = \frac{1}{2}\Big(\sum_{x \leq p_i} c_i - \sum_{x > p_i} c_i\Big), \qquad g'_+(x) = \frac{1}{2}\Big(\sum_{x < p_i} c_i - \sum_{x \geq p_i} c_i\Big).$$

For any closed subinterval $[c, d] \subset [a, b]$,

$$m_g([c, d]) = g'_-(c) - g'_+(d) = \sum_{c \leq p_i \leq d} c_i = m_g^*([c, d]).$$

In particular, the set function m_g has precisely the same point masses as m_f. Similar computations apply to open and half-open intervals. Thus the measure m_g^* extends m_g.

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Replacing $f(t)$ by $h(t) = f(t) - g(t)$, we are reduced to the situation where $h \in \mathrm{BDV}([a,b])$ has no point masses. This means that for each $p \in (a,b)$, $0 = m_h(\{p\}) = h'_-(p) - h'_+(p)$, or in other words that $h'(p)$ exists. By hypothesis, both $h'_+(a)$ and $h'_-(b)$ exist, so $h(t)$ is differentiable on $[a,b]$. The fact that $h \in \mathrm{BDV}([a,b])$ means that $h'(t)$ is of bounded total variation.

We claim that $h'(t)$ is continuous on $[a,b]$. Suppose it were discontinous at some point p. By Rudin [Rud, p. 109], the existence of $h'(t)$ for all t means that $h'(t)$ cannot have a jump discontinuity, so the discontinuity must be due to oscillation. (Rudin states the result for real-valued functions; apply it to the real and imaginary parts of $h(t)$ separately.) Hence there would be an $\varepsilon > 0$, and a sequence of points $t_1, t_2, \ldots$, either monotonically increasing or monotonically decreasing to p, such that $|h'(t_i) - h'(t_{i+1})| > \varepsilon$ for each i. Assume for convenience that the t_i are monotonically increasing. Considering the intervals $[t_1,t_2], [t_3,t_4], \ldots$, we see that

$$\sum_{i=1}^{\infty} |m_h([t_{2i-1}, t_{2i}])| = \infty,$$

contradicting the fact that $h \in \mathrm{BDV}([a,b])$.

For $a \le t \le b$, let $T(t)$ be the cumulative total variation function of $h'(t)$: letting $\mathcal{Q}$ vary over all finite partitions $a = q_0 < q_1 < \cdots < q_m = t$ of $[a,t]$,

$$T(t) = \sup_{\mathcal{Q}} \sum_{j=1}^{m} |h'(q_j) - h'(q_{j-1})|.$$

Then $T\colon [a,b] \to \mathbb{R}$ is monotone increasing, and is continuous since $h'(t)$ is continuous. By Royden [Roy, Proposition 12, p. 301], there is a unique bounded Borel measure ν on $[a,b]$ such that $\nu((c,d]) = T(d) - T(c)$ for each half-open interval $(c,d] \subset [a,b]$. Since $T(t)$ is continuous, ν has no point masses.

Next put

$$T_1(t) = T(t) - \mathrm{Re}(h'(t)), \qquad T_2(t) = T(t) - \mathrm{Im}(h'(t))\ .$$

Then $T_1(t)$ and $T_2(t)$ are also monotone increasing and continuous, so by the same proposition there are bounded Borel measures ν_1, ν_2 on $[a,b]$ such that for each half-open interval $(c,d] \subset [a,b]$,

$$\nu_1((c,d]) = T_1(d) - T_1(c), \qquad \nu_2((c,d]) = T_2(d) - T_2(c).$$

As before, ν_1 and ν_2 have no point masses.

Now define the complex Borel measure $m_h^* = (\nu_1 - \nu) + (\nu_2 - \nu)i$. By construction, m_h^* has finite total mass. For any half-open interval $(c,d] \subset [a,b]$,

$$m_h^*((c,d]) = h'(c) - h'(d) = m_h((c,d]).$$

Since the measures ν, ν_1, and ν_2 have no point masses, $m_h^*([c,d]) = m_h^*((c,d)) = m_h^*([c,d)) = m_h^*((c,d])$. Thus m_h^* extends the finitely additive set function m_h.

Finally, adding the measures m_g^* extending m_g and m_h^* extending m_h, we obtain the desired measure $m_f^* = m_g^* + m_h^*$ extending m_f. Since $m_f(\Gamma) = 0$, also $m_f^*(\Gamma) = 0$. ∎

5 Laplacians

In this paper, for a function $f \in \mathrm{BDV}(\Gamma)$ we define the Laplacian $\Delta(f)$ to be the measure given by Theorem 4.2

$$\Delta(f) = m_f^*.$$

Sometimes we will need to apply the Laplacian to a function of several variables, fixing all but one of them. In this case we write $\Delta_x(f(x,y))$ for the Laplacian of the function $F_y(x) = f(x,y)$.

We will now show that the Laplacian on $\mathrm{BDV}(\Gamma)$ agrees (up to sign) with the Laplacians defined by Chinburg and Rumely [CR] and Zhang [Zh] on their more restricted classes of functions. If Δ_{CR} denotes the Chinburg–Rumely Laplacian on the space $\mathrm{CPA}(\Gamma)$ of continuous, piecewise affine functions on Γ, then it is easy to verify that if $f \in \mathrm{CPA}(\Gamma)$, then $f \in \mathrm{BDV}(\Gamma)$ and $\Delta_{\mathrm{CR}}(f) = -\Delta(f)$.

Recall that the Zhang space $\mathrm{Zh}(\Gamma)$ is the set of all continuous functions $f\colon \Gamma \to \mathbb{C}$ such that

(i) there is a finite set of points $X_f \subset \Gamma$ such that $\Gamma \backslash X_f$ is a finite union of open intervals, the restriction of f to each of those intervals is $\mathcal{C}^2$, and

(ii) $f''(x) \in L^1(\Gamma)$.

Here we write $f''(x)$ for $\frac{d^2}{dt^2} f(p + t\vec{v})$, if $x = p + t\vec{v} \in \Gamma \backslash X_f$.

Lemma 5.1 *The space* $\mathrm{Zh}(\Gamma)$ *is a subset of* $\mathrm{BDV}(\Gamma)$.

Proof We first prove that the directional derivatives $d_{\vec{v}} f(p)$ exist for each $p \in X_f$ and each $\vec{v} \in \mathrm{Vec}(p)$. Fix such a p and $\vec{v}$, and let t_0 be small enough that $p+t\vec{v} \in \Gamma \backslash X_f$ for all $t \in (0, t_0)$. By abuse of notation, we will write $f(t)$ for $f(p+t\vec{v})$. By hypothesis, $f \in \mathcal{C}^2((0,\delta))$. By the mean value theorem, $d_{\vec{v}} f(p)$ exists if and only if $\lim_{t\to 0^+} f'(t)$ exists (in which case the two are equal). Given $\varepsilon > 0$, let $0 < \delta < t_0$ be small enough that $\int_{(0,\delta)} |f''(t)|\, dt < \varepsilon$; this is possible since $f'' \in L^1(\Gamma)$. Then for all $t_1, t_2 \in (0,\delta)$,

$$|f'(t_2) - f'(t_1)| \le \int_{t_1}^{t_2} |f''(t)|\, dt < \varepsilon, \tag{5.1}$$

which proves that $\lim_{t\to 0^+} f'(t) = d_{\vec{v}} f(p)$ exists.

Equation (5.1) also implies (in the notation of §4) that for every countable family $\mathcal{F}$ of pairwise disjoint connected closed sets $E_i \in \mathcal{A}$, we have

$$\sum_{E_i \in \mathcal{F}} |m_f(E_i)| \le \sum_{p \in X_f} |m_f(\{p\})| + \int_\Gamma |f''(t)|\, dt < \infty,$$

so that $f \in \mathrm{BDV}(\Gamma)$ as desired. ∎

For $f \in \mathrm{Zh}(\Gamma)$, Zhang [Zh, Appendix] defined the Laplacian Δ_{Zh} to be the measure

$$\Delta_{\mathrm{Zh}}(f) = -f''(x)dx - \sum_{p \in X_f} \Big(\sum_{\vec{v} \in \mathrm{Vec}(p)} d_{\vec{v}} f(p) \Big) \delta_p(x).$$

We will now see that for $f \in \mathrm{Zh}(\Gamma)$, $\Delta(f) = \Delta_{\mathrm{Zh}}(f)$.

Proposition 5.2

(i) *If $f \in \mathrm{Zh}(\Gamma)$, then $\Delta_{\mathrm{Zh}}(f) = \Delta(f)$.*

(ii) *If $f \in \mathrm{BDV}(\Gamma)$ and $\Delta(f)$ has the form $\Delta(f) = g(x)dx + \sum_{p_i \in X} c_{p_i}\delta_{p_i}(x)$ for a piecewise continuous function $g \in L^1(\Gamma)$ and a finite set X, then $f \in \mathrm{Zh}(\Gamma)$. Furthermore, let X_g be a finite set of points containing X, a vertex set for Γ, and the finitely many points where $g(x)$ is not continuous; put $c_p = 0$ for each $p \in X_g\backslash\{p_1, \dots, p_m\}$. Then $f''(x) = -g(x)$ for each $x \in \Gamma\backslash X_g$, $f'(x)$ is continuous on the closure of each segment of $\Gamma\backslash X_g$ (interpreting $f'(x)$ as a one-sided derivative at each endpoint), and $\Delta_p(f) = -c_p$ for each $p \in X_g$.*

Proof (i) Suppose $f \in \mathrm{Zh}(\Gamma)$. We have $\Delta(f)(\{p\}) = \Delta_{\mathrm{Zh}}(f)(\{p\})$ for each $p \in X_f$. To show that $\Delta(f) = \Delta_{\mathrm{Zh}}(f)$ it suffices to show that $\Delta(f)((c,d)) = \Delta_{\mathrm{Zh}}(f)((c,d))$ for each open interval (c,d) contained in an edge of $\Gamma\backslash X_f$. But

$$\Delta(f)((c,d)) = m_f((c,d)) = f'(c) - f'(d) = -\int_c^d f''(x)\,dx = \Delta_{\mathrm{Zh}}((c,d)).$$

(ii) Now suppose $f \in \mathrm{BDV}(\Gamma)$, and $\Delta(f)$ has the given form. Consider an edge in $\Gamma\backslash X_g$ and identify it with an interval (a,b) by means of the distinguished parametrization. For each $x \in (a,b)$, we have $\Delta(f)(x) = 0$, so $f'(x)$ exists. If $h > 0$ is sufficiently small, then

$$f'(x+h) - f'(x) = -\Delta(f)([x, x+h]) = -\int_x^{x+h} g(t)\,dt,$$

while if $h < 0$, then

$$f'(x+h) - f'(x) = \Delta(f)([x+h, x]) = \int_{x+h}^{x} g(t)\,dt = -\int_x^{x+h} g(t)\,dt.$$

Hence

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0}\Big(-\frac{1}{h}\int_x^{x+h} g(t)\,dt\Big) = -g(x).$$

The assertion that $\Delta_p(f) = -c_p$ for each $p \in X_g$ is clear. ■

Corollary 5.3

(i) *If $f \in \mathrm{BDV}(\Gamma)$ and $\Delta(f) = \sum_{i=1}^k c_i\delta_{p_i}$ is a discrete measure, then $f \in \mathrm{CPA}(\Gamma)$.*

(ii) *If $f \in \mathrm{BDV}(\Gamma)$ and $\Delta(f) = 0$, then $f(x) = C$ is constant.*

Proof (i) Since $\Delta(f)$ is discrete, it follows by Proposition 5.2 that $f(x) \in \mathrm{Zh}(\Gamma)$ and $\Delta(f) = \Delta_{\mathrm{Zh}}(f)$. Fixing a vertex set X for Γ containing $\{p_1, \dots, p_k\}$, we see that $f''(x) = 0$ on $\Gamma\backslash X$, so $f(x)$ is affine on each segment of $\Gamma\backslash X$, which means that $f \in \mathrm{CPA}(\Gamma)$.

(ii) If $\Delta(f) = 0$, then by part (i), $f \in \mathrm{CPA}(\Gamma)$. Suppose $f(x)$ were not constant, and put $M = \max_{x\in\Gamma} f(x)$. Let $\Gamma_0 = \{x \in \Gamma : f(x) = M\}$ and let x_0 be a boundary point of Γ_0. Since $f \in \mathrm{CPA}(\Gamma)$, one sees easily that $\Delta(f)(\{x_0\}) > 0$. This contradicts $\Delta(f) = 0$. ■

6 The Kernels $j_\zeta(x, y)$ and $g_\mu(x, y)$

In [CR], there is a detailed study of a "harmonic kernel" function $j_\zeta(x, y)$ having the following properties (see [CR, Lemma 2.10], [Zh, Appendix], or the expository article [BF]):

(i) It is jointly continuous as a function of three variables.
(ii) It is non-negative, with $j_\zeta(\zeta, y) = j_\zeta(x, \zeta) = 0$ for all $x, y, \zeta \in \Gamma$.
(iii) It is symmetric in x and y: for fixed ζ, $j_\zeta(x, y) = j_\zeta(y, x)$.
(iv) For fixed ζ and y, the function $j_{\zeta,y}(x) = j_\zeta(x, y)$ is in $\mathrm{CPA}(\Gamma)$ and satisfies the Laplacian equation $\Delta_x(j_\zeta(x, y)) = \delta_y(x) - \delta_\zeta(x)$.

For any real-valued signed Borel measure μ on Γ with $\mu(\Gamma) = 1$ and $|\mu|(\Gamma) < \infty$, put $j_\mu(x, y) = \int_\Gamma j_\zeta(x, y)\, d\mu(\zeta)$. Clearly $j_\mu(x, y)$ is symmetric, and is jointly continuous in x and y. It has the following properties which are less obvious.

Proposition 6.1

(i) *There is a constant $c_\mu(\Gamma)$ such that for each $y \in \Gamma$, $\int_\Gamma j_\mu(x, y)\, d\mu(x) = c_\mu(\Gamma)$.*
(ii) *For each $y \in \Gamma$, the function $F_y(x) = j_\mu(x, y)$ belongs to the space* $\mathrm{BDV}(\Gamma)$, *and satisfies $\Delta_x(F_y) = \delta_y(x) - \mu$.*

Proof For (i) see [CR, Lemma 2.16, p. 21].

For (ii), fix y and note that the existence of the directional derivatives $d_{\vec{v}}F_y(x)$ follows from [CR, Lemma 2.13, p. 19]. Hence $F_y \in \mathcal{D}(\Gamma)$. Also, noting that $j_y(y, x) = 0$ for all x, it follows that for the measure $\omega = \mu - \delta_y(x)$ of total mass 0, we have $j_\omega(x, y) = j_\mu(x, y)$ for all x. Then, [CR, Lemma 2.14, p. 20] tells us that for any subset $D \subset \Gamma$ which is a finite union of closed intervals, the finitely additive set function m_{F_y} satisfies $m_{F_y}(D) = -\omega(D) = (\delta_y(x) - \mu)(D)$ (again recall that $\Delta = -\Delta_{\mathrm{CR}}$). For any countable collection $\mathcal{F}$ of pairwise disjoint sets D_i of the type above, it follows that

$$\sum_{i=1}^{\infty} |m_{F_y}(D_i)| \le \sum_{i=1}^{\infty} |\omega|(D_i) \le |\omega|(\Gamma),$$

so by Proposition 4.1, $F_y \in \mathrm{BDV}(\Gamma)$. The measure $\Delta(F_y) = m^*_{F_y}$ attached to F_y is determined by its values on closed intervals, so it coincides with $\delta_y(x) - \mu$. ■

We now define a kernel $g_\mu(x, y)$ which will play a key role in the rest of the paper:

$$g_\mu(x, y) = j_\mu(x, y) - c_\mu(\Gamma).$$

It follows from Proposition 6.1 that $g_\mu(x, y)$ continuous, symmetric, and for each y

$$\int_\Gamma g_\mu(x, y)\, d\mu(x) = 0. \tag{6.1}$$

7 The Operator φ_μ

Convention For the rest of this paper, μ will denote a real, signed Borel measure with $\mu(\Gamma) = 1$ and $|\mu|(\Gamma) < \infty$.

For any complex Borel measure ν on Γ with $|\nu|(\Gamma) < \infty$, define the integral transform $\varphi_\mu(\nu)(x) = \int_\Gamma g_\mu(x, y)\, d\nu(y)$. Write

$$\mathrm{BDV}_\mu(\Gamma) = \left\{ f \in \mathrm{BDV}(\Gamma) : \int_\Gamma f(x)\, d\mu(x) = 0 \right\}.$$

Proposition 7.1 *The function $F(x) = \varphi_\mu(\nu)(x)$ belongs to $\mathrm{BDV}_\mu(\Gamma)$, and satisfies*

$$\Delta(F) = \nu - \nu(\Gamma)\mu.$$

Proof First note the following identity: for each $p \in \Gamma$, $j_\zeta(x, y) = j_\zeta(x, p) - j_y(x, p) + j_y(\zeta, p)$. To see this, fix ζ, y and p; put $H(x) = j_\zeta(x, y)$, $h(x) = j_\zeta(x, p) - j_y(x, p)$. Then $\Delta_x(H) = \Delta_x(h) = \delta_y(x) - \delta_\zeta(x)$, so by [CR, Lemma 2.6, p. 11] $H(x) - h(x)$ is a constant C. To evaluate C, take $x = \zeta$; using $j_\zeta(\zeta, y) = j_\zeta(\zeta, p) = 0$ we obtain $C = j_y(\zeta, p)$.

Inserting this into the definition of $\varphi_\mu(\nu)$, we find

$$\begin{aligned}
F(x) &= \int_\Gamma g_\mu(x, y)\, d\nu(y) \\
&= \int_\Gamma \Big(\int_\Gamma \{ j_\zeta(x, p) - j_y(x, p) + j_y(\zeta, p) \}\, d\mu(\zeta) - c_\mu(\Gamma) \Big)\, d\nu(y) \\
&= \nu(\Gamma)\, j_\mu(x, p) - j_\nu(x, p) + C_p,
\end{aligned}$$

where C_p is a constant. Hence Proposition 6.1(ii) shows that $F \in \mathrm{BDV}(\Gamma)$ and $\Delta(F) = \nu - \nu(\Gamma)\mu$.

Finally, the fact that $F \in \mathrm{BDV}_\mu(\Gamma)$ follows from Fubini's theorem:

$$\int_\Gamma F(x)\, d\mu(x) = \int_\Gamma \Big(\int_\Gamma g_\mu(x, y)\, d\mu(x) \Big)\, d\nu(y) = 0. \qquad \blacksquare$$

8 Eigenfunctions of φ_μ in $L^2(\Gamma)$

Given a Borel measurable function $f\colon \Gamma \to \mathbb{C}$, write $\|f\|_1$, $\|f\|_2$, and $\|f\|_\infty$ for its L^1, L^2 and sup norms. Write $L^1(\Gamma)$, $L^2(\Gamma)$ for the spaces of L^1, L^2 functions on Γ relative to the measure dx. It follows from the Cauchy–Schwarz inequality that $L^2(\Gamma) \subset L^1(\Gamma)$. Write $\langle f, g\rangle = \int_\Gamma f(x)\overline{g(x)}\, dx$ for the inner product on $L^2(\Gamma)$.

Given $f \in L^2(\Gamma)$, let ν be the bounded Borel measure $f(x)dx$, and define

$$\varphi_\mu(f) = \varphi_\mu(\nu) = \int_\Gamma g_\mu(x, y) f(y)\, dy.$$

By Proposition 7.1, $\varphi_\mu(f) \in \mathrm{BDV}_\mu(\Gamma)$.

Equip $\mathrm{BDV}_\mu(\Gamma)$ with the L^2 norm, and view it as a subspace of $L^2(\Gamma)$. Write $L^2_\mu(\Gamma)$ for the L^2 completion of $\mathrm{BDV}_\mu(\Gamma)$.

For the next proposition, recall that a sequence ν_n of measures on a compact space X *converges weakly* to a measure ν if $\int f d\nu_n \to \int f\, d\nu$ for all continuous functions $f\colon X \to \mathbb{R}$.

Proposition 8.1 *Suppose $\nu_1, \nu_2, \nu_3, \dots$ is a bounded sequence of complex Borel measures on Γ which converge weakly to a Borel measure ν. Then the functions $\varphi_\mu(\nu_i)$ converge uniformly to $\varphi_\mu(\nu)$ on Γ.*

Proof First, note that ν, being a weak limit of bounded measures, must be bounded. Write $F_i(x) = \varphi_\mu(\nu_i)$ and $F(x) = \varphi_\mu(\nu)$. For each x, $g_\mu(x, y)$ is a continuous function of y. Hence, by the definition of weak convergence, the functions $F_i(x)$ converge pointwise to $F(x)$.

We claim that the convergence is uniform. Fix $\varepsilon > 0$, and let $d(x, y)$ be the metric on the graph Γ. Since $g_\mu(x, y)$ is continuous and Γ is compact, $g_\mu(x, y)$ is uniformly continuous. Hence there is a $\delta > 0$ such that for any $x_1, x_2 \in \Gamma$ satisfying $d(x_1, x_2) < \delta$, $|g_\mu(x_1, y) - g_\mu(x_2, y)| < \varepsilon$ for all $y \in \Gamma$. Write $B_x(\delta) = \{z \in \Gamma : d(x, z) < \delta\}$. Since Γ is compact, it can be covered by finitely many balls $B(x_j, \delta)$, $j = 1, \dots, m$. Let N be large enough so that $|F_i(x_j) - F(x_j)| < \varepsilon$ for all $j = 1, \dots, m$ and all $i \geq N$. Then for each $x \in \Gamma$, and each $i \geq N$, if x_j is such that $d(x, x_j) < \delta$ then $|F_i(x) - F(x)| \leq |F_i(x) - F_i(x_j)| + |F_i(x_j) - F(x_j)| + |F(x_j) - F(x)|$. But $|F_i(x) - F_i(x_j)| \leq \int_\Gamma |g_\mu(x, y) - g_\mu(x_j, y)|\, d|\nu_i|(y) \leq |\nu_i|(\Gamma)\varepsilon$, and similarly $|F(x) - F(x_j)| \leq |\nu|(\Gamma)\varepsilon$. Hence $|F_i(x) - F(x)| < (\sup_i |\nu_i|(\Gamma) + |\nu|(\Gamma) + 1)\varepsilon$. ■

Proposition 8.2 *The operator $\varphi_\mu\colon L^2(\Gamma) \to L^2(\Gamma)$ is compact and self-adjoint, and maps $L^2(\Gamma)$ into* $\mathrm{BDV}_\mu(\Gamma)$.

Proof Since μ is real, $g_\mu(x, y)$ is real-valued and symmetric, so φ_μ is self-adjoint.

It remains to show that φ_μ is compact. Let $f_1, f_2, \dots \in L^2(\Gamma)$ be a sequence of functions with bounded L^2 norm. By the Cauchy–Schwarz inequality, the L^1 norms of the f_i are also bounded, so the sequence of measures $f_i(x)\, dx$ is bounded and has a subsequence converging weakly to a bounded measure ν on Γ. After passing to this subsequence we can assume it is the entire sequence $\{f_i\}$. By Proposition 8.1, the functions $\varphi_\mu(f_i)$ converge uniformly to $\varphi_\mu(\nu)$, which belongs to $\mathrm{BDV}_\mu(\Gamma)$. Clearly they converge to $\varphi_\mu(\nu)$ under the L^2 norm as well. ■

Definition 8.3 A nonzero function $f \in L^2(\Gamma)$ is an eigenfunction for φ_μ, with eigenvalue α, if $\varphi_\mu(f) = \alpha \cdot f$.

Applying well-known results from spectral theory, we have

Theorem 8.4 *The map φ_μ has countably many eigenvalues α_i, each of which is real and occurs with finite multiplicity. The nonzero eigenvalues can be ordered so that $|\alpha_1| \geq |\alpha_2| \geq \cdots$ and $\lim_{i\to\infty} |\alpha_i| = 0$. Each eigenfunction corresponding to a* *nonzero eigenvalue belongs to* $\mathrm{BDV}_\mu(\Gamma)$. *Finally, the space* $L^2(\Gamma)$ *has an orthonormal basis consisting of eigenfunctions of* φ_μ.

Proof All the assertions are contained in the spectral theorem for compact hermitian operators [La, p. 165], together with [La, Theorem 10, p. 208] and [La, Theorem 11, p. 210]. ■

We now investigate the eigenfunctions of φ_μ with eigenvalue 0, that is, the kernel of φ_μ in $L^2(\Gamma)$. We begin by reformulating Proposition 7.1 in the present context.

Lemma 8.5 *For each* $f \in L^2(\Gamma)$,

$$\Delta(\varphi_\mu(f)) = f(x)\,dx - \Big(\int_\Gamma f(x)\,dx\Big)\,\mu.$$

Proposition 8.6 *If* $\mu = g(x)dx$ *for a real-valued function* $g(x) \in L^2(\Gamma)$ *with* $\int_\Gamma g(x)\,dx = 1$, *then* $\mathrm{Ker}(\varphi_\mu) = \mathbb{C}\cdot g(x)$ *and the* L^2*-closure of* $\varphi_\mu(\mathrm{BDV}_\mu(\Gamma))$ *(and of* $\varphi_\mu(L^2(\Gamma))$*) is* $\mathbb{C}\cdot g(x)^\perp$. *Otherwise,* $\mathrm{Ker}(\varphi_\mu) = \{0\}$ *and the* L^2*-closure of the image is all of* $L^2(\Gamma)$.

In either case, $\mathrm{Ker}(\varphi_\mu) \cap \mathrm{BDV}_\mu(\Gamma) = \{0\}$.

Proof Suppose $0 \neq f \in L^2(\Gamma)$ belongs to $\mathrm{Ker}(\varphi_\mu)$. By Lemma 8.5,

$$0 = \Delta_x(\varphi_\mu(f)) = f(x)dx - \Big(\int_\Gamma f(x)\,dx\Big)\,\mu.$$

Write $C_f = \int_\Gamma f(x)\,dx$. Since $f \neq 0$, the equation above shows that $C_f \neq 0$. Hence $\mu = (1/C_f)f(x)dx = g(x)dx$, with $g \in L^2(\Gamma)$ and $\int_\Gamma g(x)\,dx = 1$, and $f \in \mathbb{C}\cdot g(x)$. Conversely if μ has this form, then

$$\varphi_\mu(g)(x) = \int_\Gamma g_\mu(x,y)g(y)\,dy = \int_\Gamma g_\mu(x,y)\,d\mu(y) = 0$$

by formula (6.1) and the symmetry of $g_\mu(x,y)$.

When $\mu = g(x)dx$, if $g(x) \notin \mathrm{BDV}(\Gamma)$, then clearly $\mathrm{Ker}(\varphi_\mu) \cap \mathrm{BDV}_\mu(\Gamma) = \{0\}$. If $g(x) \in \mathrm{BDV}(\Gamma)$, then since $g(x)$ is real-valued, $\int_\Gamma g(x)\,d\mu(x) = \int_\Gamma g(x)^2\,dx > 0$, so $g(x) \notin \mathrm{BDV}_\mu(\Gamma)$.

The assertions about the L^2-closure of the image of φ_μ follow from our description of the kernel and from Theorem 8.4. ■

9 Eigenfunctions from the Point of View of Laplacians

In this section we will give an integro-differential characterization of eigenfunctions of φ_μ.

Proposition 9.1 *A function* $0 \neq f \in \mathrm{BDV}(\Gamma)$ *is an eigenfunction of* φ_μ *with nonzero eigenvalue if and only if*

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(i) $\int_\Gamma f(x)\, d\mu(x) = 0$ *and*
(ii) *for some constants* $\lambda, C \in \mathbb{C}$, $\Delta(f) = \lambda \cdot (f(x)dx - C\mu)$.

If (i) *and* (ii) *hold, then necessarily* $\lambda \in \mathbb{R}$ *and* $C = \int_\Gamma f(x)\, dx$, *with* $\varphi_\mu(f) = \frac{1}{\lambda} \cdot f$.

Remark In Theorem 10.3 below, we will see that in fact $\lambda > 0$.

Proof If $0 \neq f \in \mathrm{BDV}(\Gamma)$ is an eigenfunction of φ_μ with nonzero eigenvalue α, then $\varphi_\mu(f) = \alpha f$, so $0 = \int \varphi_\mu(f)(x)\, d\mu(x) = \alpha \cdot \int_\Gamma f(x)\, d\mu(x)$ so $f \in \mathrm{BDV}_\mu(\Gamma)$. By Lemma 8.5

$$\Delta(f) = \frac{1}{\alpha} \cdot \Delta(\varphi_\mu(f)) = \frac{1}{\alpha} \cdot (f(x)dx - C_f\mu)$$

where $C_f = \int_\Gamma f(x)\, dx$.

Conversely, if (i) holds, then $f \in \mathrm{BDV}_\mu(\Gamma)$, and if (ii) also holds, then by Lemma 8.5,

$$\Delta(\lambda \cdot \varphi_\mu(f) - f) = \lambda \cdot \Big(\int_\Gamma f(x)\, dx - C \Big) \cdot \mu. \tag{9.1}$$

By Theorem 4.2,

$$\begin{aligned} 0 &= \int_\Gamma d\Delta(\lambda\varphi_\mu(f) - f) \\ &= \lambda \cdot \Big(\int_\Gamma f(x)\, dx - C \Big) \cdot \int_\Gamma d\mu = \lambda \cdot \Big(\int_\Gamma f(x)\, dx - C \Big). \end{aligned} \tag{9.2}$$

By (9.1) we have $\Delta(\lambda\varphi_\mu(f) - f)) = 0$, so Corollary 5.3 shows $\lambda\varphi_\mu(f) - f$ is a constant function. Since f and $\varphi_\mu(f)$ both belong to $\mathrm{BDV}_\mu(\Gamma)$, we conclude $\lambda\varphi_\mu(f) = f$. Here $f \neq 0$ by hypothesis, so $\lambda \neq 0$. It follows that f is an eigenvalue of φ_μ with nonzero eigenvalue $\alpha = 1/\lambda$. By Theorem 8.4, α and hence λ are real. Since $\lambda \neq 0$, equation (9.2) gives $C = \int_\Gamma f(x)\, dx$. ∎

Definition 9.2 A nonzero function $f \in \mathrm{BDV}(\Gamma)$ will be called an eigenfunction of Δ in $\mathrm{BDV}_\mu(\Gamma)$ if it satisfies conditions (i) and (ii) in Proposition 9.1

10 Positivity and the Dirichlet Semi-Norm

In this section we will show that the nonzero eigenvalues of φ_μ are positive. This is intimately connected with the positivity of the Dirichlet semi-norm on $\mathrm{BDV}_\mu(\Gamma)$, which we also investigate.

On the space $\mathrm{Zh}(\Gamma)$ the Dirichlet inner product $\langle f, g\rangle_{\mathrm{Dir}}$ is given by the equivalent formulas

$$\langle f, g\rangle_{\mathrm{Dir}} = \int_\Gamma f'(x)\overline{g'(x)}\, dx = \int_\Gamma f(x)\, d\overline{\Delta(g)} = \int_\Gamma \overline{g(x)}\, d\Delta(f). \tag{10.1}$$

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We have seen that $\mathrm{Zh}(\Gamma) \subset \mathrm{BDV}(\Gamma)$. We claim that for all $f, g \in \mathrm{BDV}(\Gamma)$

$$\int f(x)\, d\overline{\Delta(g)} = \int \overline{g(x)}\, d\Delta(f). \tag{10.2}$$

To see this, take $f, g \in \mathrm{BDV}(\Gamma)$ and write $\nu = \Delta(f)$ and $\omega = \Delta(g)$. Then $\nu(\Gamma) = \omega(\Gamma) = 0$. For any real Borel measure μ on Γ, the function $f_\mu(x) := \varphi_\mu(\nu) = \int_\Gamma g_\mu(x, y)\, d\nu(y)$ belongs to $\mathrm{BDV}_\mu(\Gamma)$. By Proposition 7.1, $\Delta(f_\mu) = \nu$. Hence $\Delta(f - f_\mu) = 0$, which means that $f = f_\mu + C$ for some constant C. Thus for any $g \in \mathrm{BDV}(\Gamma)$,

$$\int_\Gamma f(y)\, \overline{d\Delta(g)(y)} = \int_\Gamma f_\mu(y)\, \overline{d\omega(y)} = \iint_{\Gamma\times\Gamma} g_\mu(x, y)\, d\nu(x)\overline{d\omega(y)}.$$

In light of this we define the Dirichlet pairing on $\mathrm{BDV}(\Gamma) \times \mathrm{BDV}(\Gamma)$ by

$$\langle f, g\rangle_{\mathrm{Dir}} = \int_\Gamma f(y)\, \overline{d\Delta(g)(y)} = \int_\Gamma \overline{g(x)}\, d\Delta(f)(x) \tag{10.3}$$

$$= \iint_{\Gamma\times\Gamma} g_\mu(x, y)\, d\Delta(f)(x)\overline{d\Delta(g)(y)}. \tag{10.4}$$

Note that if $f(x)$ and $g(x)$ are replaced by $f(x) + C$ and $g(x) + D$ for constants C and D, the value of $\langle f, g\rangle_{\mathrm{Dir}}$ remains unchanged, since $\Delta(C) = \Delta(D) = 0$. Finally, observe that the choice of μ in (10.4) is arbitrary, since it does not occur in (10.3). In particular (taking $\mu = \delta_\zeta(x)$), the kernel $g_\mu(x, y)$ in (10.4) can be replaced by $j_\zeta(x, y)$ for any point ζ.

We will now show that $\langle f, g\rangle_{\mathrm{Dir}}$ defines a semi-norm on $\mathrm{BDV}(\Gamma)$ whose kernel consists of precisely the constant functions. For this, we will need some preliminary lemmas. The first is well known.

Lemma 10.1 *Let X, Y be compact Hausdorff topological spaces, and let μ_i (resp., ν_i) be a bounded sequence of signed Borel measures on X (resp., Y). If μ_i converges weakly to μ on X and ν_i converges weakly to ν on Y, then $\mu_i \times \nu_i$ converges weakly to $\mu \times \nu$ on $X \times Y$.*

Proof By hypothesis, there exist $M, N > 0$ such that $|\mu_i|(\Gamma) \le M$ and $|\nu_i(\Gamma)| \le N$ for all i. According to the Stone–Weierstrass theorem, the set

$$S = \Big\{ \sum c_i f_i(x) g_i(y) : c_i \in \mathbb{R},\ f_i \in \mathcal{C}(X), g_i \in \mathcal{C}(Y) \Big\}$$

is dense in $\mathcal{C}(X \times Y)$, and it follows easily from the hypotheses of weak convergence that

$$\int_{X\times Y} F\, d(\mu_i \times \nu_i) \to \int_{X\times Y} F\, d(\mu \times \nu)$$

for all $F \in S$. Now suppose $F \in \mathcal{C}(X \times Y)$ is arbitrary, and let $\varepsilon > 0$. Since S is dense, we may choose $H(x, y) \in S$ so that the difference $G(x, y) = F(x, y) - H(x, y)$ satisfies

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$\|G\|_\infty < \varepsilon$. Since $|\mu \times \nu| \le |\mu| \times |\nu|$ (which follows from the Hahn decomposition theorem), we obtain

$$\Big|\int G\, d(\mu_i \times \nu_i) - \int G\, d(\mu \times \nu)\Big| \le 2MN\|G\|_\infty < 2MN \cdot \varepsilon.$$

This gives what we want. ∎

Lemma 10.2 *Suppose $f, g \in \mathrm{BDV}(\Gamma)$ and that $f_1, f_2, \dots$ and $g_1, g_2, \dots$ are sequences of functions in $\mathrm{BDV}(\Gamma)$ such that*

(i) *$|\Delta(f)|(\Gamma)$, $|\Delta(g)|(\Gamma)$, and the sequences $|\Delta(f_i)|(\Gamma)$ and $|\Delta(g_j)|(\Gamma)$ are bounded;*
(ii) *$\Delta(f_1), \Delta(f_2), \dots$ converges weakly to $\Delta(f)$;*
(iii) *$\Delta(g_1), \Delta(g_2), \dots$ converges weakly to $\Delta(g)$.*

Then $\lim_{i\to\infty}\langle f_i, g_i\rangle_{\mathrm{Dir}} = \langle f, g\rangle_{\mathrm{Dir}}$.

Proof Write $\nu = \Delta(f)$, $\omega = \Delta(g)$, $\nu_i = \Delta(f_i)$, and $\omega_i = \Delta(g_i)$. By Lemma 10.1, the sequence of measures $\nu_i \times \omega_i$ converges weakly to $\nu \times \omega$ on $\Gamma \times \Gamma$.

Hence

$$\begin{aligned}\lim_{i\to\infty}\langle f_i, g_i\rangle_{\mathrm{Dir}} &= \lim_{i\to\infty}\iint_{\Gamma\times\Gamma} g_\mu(x, y)\, d\nu_i(x)\overline{d\omega_i(y)} \\ &= \iint_{\Gamma\times\Gamma} g_\mu(x, y)\, d\nu(x)\overline{d\omega(y)} = \langle f, g\rangle_{\mathrm{Dir}}.\end{aligned}$$ ∎

We now come to the main result of this section.

Theorem 10.3

(i) *For each $f \in \mathrm{BDV}(\Gamma)$ the Dirichlet pairing satisfies $\langle f, f\rangle_{\mathrm{Dir}} \ge 0$, with $\langle f, f\rangle_{\mathrm{Dir}} = 0$ if and only if $f(x) = C$ is constant. Thus $\langle f, g\rangle_{\mathrm{Dir}}$ is a semi-norm on $\mathrm{BDV}(\Gamma)$ with kernel $\mathbb{C}\cdot 1$, and its restriction to $\mathrm{BDV}_\mu(\Gamma)$ is a norm.*

(ii) *The nonzero eigenvalues α of φ_μ are positive, and for each $f \in L^2(\Gamma)$,*

$$\langle \varphi_\mu(f), f\rangle \ge 0.$$

Proof Let $f \in \mathrm{BDV}(\Gamma)$ be arbitrary, and put $\nu = \Delta(f)$. It is easy to construct a sequence of discrete measures $\nu_1, \nu_2, \dots$ which converge weakly to ν, with $\nu_i(\Gamma) = 0$ and $|\nu_i|(\Gamma) \le |\nu|(\Gamma)$ for each i. (Choose a sequence of numbers $\eta_i \to 0$. For each i, subdivide Γ into finitely many segments e_{ij} with length at most η_i. For each i, j, choose a point $p_{ij} \in e_{ij}$, and put $\nu_i = \sum_j \nu(e_{ij})\delta_{p_{ij}}(x)$.)

Put $f_i = \varphi_\mu(\nu_i)$ for each i. Then $\Delta(f_i) = \nu_i$ and $f_i \in \mathrm{CPA}(\Gamma) \subset \mathrm{Zh}(\Gamma)$. It follows that $\langle f_i, f_i\rangle_{\mathrm{Dir}} = \int |f_i'(x)|^2\, dx \ge 0$ for each i. By Lemma 10.2,

$$\langle f, f\rangle_{\mathrm{Dir}} = \lim_{i\to\infty}\langle f_i, f_i\rangle_{\mathrm{Dir}} \ge 0.$$

Thus, at least $\langle f, f\rangle_{\mathrm{Dir}}$ is non-negative. We will use this to show that the eigenvalues of Δ in $\mathrm{BDV}_\mu(\Gamma)$ are positive. Suppose $0 \ne f \in \mathrm{BDV}_\mu(\Gamma)$ is an eigenfunction of φ_μ with eigenvalue α. By Proposition 8.6, $\alpha \ne 0$.

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Since $\varphi_\mu(f) = \alpha \cdot f$, Lemma 8.5 shows that $\alpha \cdot \Delta(f) = \Delta(\varphi_\mu(f)) = f(x)dx - C_f \cdot \mu$ where $C_f = \int f(x)\,dx$. Recalling that α is real, and using that $\int f(x)\,d\mu(x) = 0$, we have

$$\begin{aligned} 0 \le \langle f, f\rangle_{\mathrm{Dir}} &= \int_\Gamma f(x)\,\overline{d\Delta(f)(x)} \\ &= \int_\Gamma f(x)\frac{1}{\alpha}\overline{f(x)}\,dx = \frac{1}{\alpha}\langle f, f\rangle_{L^2}. \end{aligned}$$

Since $\langle f, f\rangle_{L^2} > 0$, we must have $\alpha > 0$.

It follows from this that $\langle \varphi_\mu(f), f\rangle_{L^2} \ge 0$ for all $f \in L^2(\Gamma)$.

We can now show the positivity of $\langle f, f\rangle_{\mathrm{Dir}}$.

Since the Dirichlet pairing is positive semi-definite on $\mathrm{BDV}(\Gamma)$, it follows from the Cauchy–Schwarz inequality that if $\langle f, f\rangle_{\mathrm{Dir}} = 0$, then $\langle f, g\rangle_{\mathrm{Dir}} = 0$ for all $g \in \mathrm{BDV}(\Gamma)$, *i.e.*, $\int_\Gamma \overline{g(x)}\,d\Delta(f)(x) = 0$ for all $g \in \mathrm{BDV}(\Gamma)$. But then $\Delta(f) = 0$, so that f is constant by Corollary 5.3. In particular, $\langle f, g\rangle_{\mathrm{Dir}}$ is positive definite on $\mathrm{BDV}_\mu(\Gamma)$. ■

Application: Positivity of the Energy Pairing

Let $\mathrm{Meas}(\Gamma)$ be the space of bounded Borel measures on Γ, and let $\mathrm{Meas}_0(\Gamma)$ be the subspace consisting of measures with $\nu(\Gamma) = 0$. For $\nu, \omega \in \mathrm{Meas}(\Gamma)$, define the "$\mu$-energy pairing"

$$\langle \nu, \omega\rangle_\mu = \iint_{\Gamma\times\Gamma} g_\mu(x, y)\,d\nu(x)\overline{d\omega(y)}.$$

Note that if $\nu, \omega \in \mathrm{Meas}_0(\Gamma)$, then the energy pairing $\langle \nu, \omega\rangle := \langle \nu, \omega\rangle_\mu$ is independent of μ. Indeed, let $f, g \in \mathrm{BDV}(\Gamma)$ be functions with $\nu = \Delta(f)$, $\omega = \Delta(g)$; such functions exist by Proposition 7.1. By (10.4), $\langle \nu, \omega\rangle_\mu = \langle f, g\rangle_{\mathrm{Dir}}$, which is independent of μ.

Theorem 10.4

(i) *The energy pairing $\langle \nu, \omega\rangle$ is positive definite on* $\mathrm{Meas}_0(\Gamma)$.

(ii) *For each μ, the μ-energy pairing $\langle \nu, \omega\rangle_\mu$ is positive semi-definite on* $\mathrm{Meas}(\Gamma)$, *with kernel $\mathbb{C}\mu$.*

(iii) *Among all $\nu \in \mathrm{Meas}(\Gamma)$ with $\nu(\Gamma) = 1$, μ is the unique measure which minimizes the energy integral*

$$I_\mu(\nu) = \iint_{\Gamma\times\Gamma} g_\mu(x, y)\,d\nu(x)\overline{d\nu(y)}.$$

Proof Fix μ. Given $\nu, \omega \in \mathrm{Meas}_0(\Gamma)$, put $f = \varphi_\mu(\nu)$ and $g = \varphi_\mu(\omega)$. As noted above, $\langle \nu, \omega\rangle = \langle \nu, \omega\rangle_\mu = \langle f, g\rangle_{\mathrm{Dir}}$. By Theorem 10.3, $\langle \nu, \omega\rangle$ is positive definite.

Since $\int_\Gamma g_\mu(x, y)\,d\mu(y) = 0$ for each x, clearly $I_\mu(\mu) = 0$ and $\langle \omega, \mu\rangle_\mu = \langle \mu, \omega\rangle = 0$ for each $\omega \in \mathrm{Meas}(\Gamma)$. Since $\mathrm{Meas}(\Gamma) = \mathrm{Meas}_0(\Gamma) + \mathbb{C}\mu$, it follows that $\langle \nu, \omega\rangle_\mu$ is positive semidefinite on $\mathrm{Meas}(\Gamma)$ with kernel $\mathbb{C}\mu$.

Now let ν be any measure with $\nu(\Gamma) = 1$. Then $\nu - \mu \in \mathrm{Meas}_0(\Gamma)$, so

$$\begin{aligned} I_\mu(\nu) = \langle \nu, \nu \rangle_\mu &= \langle \mu + (\nu - \mu), \mu + (\nu - \mu) \rangle_\mu \\ &= I_\mu(\mu) + \langle \nu - \mu, \nu - \mu \rangle_\mu \geq I_\mu(\mu) = 0, \end{aligned}$$

with equality if and only if $\nu = \mu$. ■

11 The Space $\mathrm{Dir}_\mu(\Gamma)$ Revisited

In this section we will use the fact that $\|f\|_{\mathrm{Dir}}$ is a norm on $\mathrm{BDV}_\mu(\Gamma)$ to show that $\mathrm{BDV}_\mu(\Gamma)$ is a subspace of $\mathrm{Dir}_\mu(\Gamma)$. This is a key step towards relating the notions of eigenfunctions in Theorem 3.8 and Theorem 8.4.

Recall that $\mathrm{Zh}(\Gamma)$ is the space of continuous functions $f\colon \Gamma \to \mathbb{C}$ which are piecewise $\mathcal{C}^2$, with $f''(x) \in L^1(\Gamma)$. Let $V(\Gamma)$ be the space of continuous functions $f\colon \Gamma \to \mathbb{C}$ such that f is piecewise $\mathcal{C}^1$ and $f' \in L^2(\Gamma)$. It easy check that $\mathrm{Zh}(\Gamma) \subset V(\Gamma)$. For $f, g \in V(\Gamma)$, we define

$$\langle f, g \rangle_{\mathrm{Dir}} = \int_\Gamma f'(x)\overline{g'(x)}\, dx. \tag{11.1}$$

It is easy to see that for $f \in V(\Gamma)$, $\langle f, f \rangle_{\mathrm{Dir}} = 0$ if and only if $f(x)$ is a constant function.

Given a real Borel measure μ of total mass 1, put

$$V_\mu(\Gamma) = \{f \in V(\Gamma) : \int f(x)\, d\mu(x) = 0\}.$$

Then the restriction of $\langle f, g \rangle_{\mathrm{Dir}}$ to $V_\mu(\Gamma)$ is an inner product.

The space $\mathrm{Zh}(\Gamma)$ is contained in $\mathrm{BDV}(\Gamma)$ and $V(\Gamma)$, and both of these spaces are contained in $\mathcal{C}(\Gamma)$, though neither is contained in the other. Putting this another way, there is a natural extension of $\langle f, g \rangle_{\mathrm{Dir}}$ on $\mathrm{Zh}(\Gamma)$ to each of $\mathrm{BDV}(\Gamma)$ and $V(\Gamma)$, and their restrictions to $\mathrm{BDV}_\mu(\Gamma)$ and $V_\mu(\Gamma)$ induce norms which coincide on $\mathrm{Zh}_\mu(\Gamma)$. Note that $\mathrm{CPA}_\mu(\Gamma)$, the space of continuous, piecewise affine functions with $\int_\Gamma f(x)\, d\mu(x) = 0$, is a subset of $\mathrm{Zh}_\mu(\Gamma)$. We will now see that both $\mathrm{BDV}_\mu(\Gamma)$ and $V_\mu(\Gamma)$ are contained in $\mathrm{Dir}_\mu(\Gamma)$, the completion of $\mathrm{Zh}_\mu(\Gamma)$ under the Dirichlet norm.

Proposition 11.1 *Under the Dirichlet norm $\|f\|_{\mathrm{Dir}} = \langle f, f \rangle_{\mathrm{Dir}}^{1/2}$,*

(i) $\mathrm{CPA}_\mu(\Gamma)$ *is dense in* $\mathrm{BDV}_\mu(\Gamma)$,
(ii) $\mathrm{CPA}_\mu(\Gamma)$ *is dense in* $V_\mu(\Gamma)$.

Before giving the proof, we need the following lemma.

Lemma 11.2 *Fix $A, B \in \mathbb{C}$, and let $[a, b]$ be a closed interval. Let $\mathcal{H}$ be the set of continuous functions $f\colon [a, b] \to \mathbb{C}$ such that $f(a) = A$, $f(b) = B$, f is differentiable on (a, b), and $f'(x) \in L^2([a, b])$. Let $h(x) \in \mathcal{H}$ be the unique affine function with $h(a) = A$, $h(b) = B$. Then for each $f \in \mathcal{H}$,*

$$\int_a^b |f'(x)|^2\, dx \geq \int_a^b |f'(x) - h'(x)|^2\, dx.$$

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Proof Put $C = (B - A)/(b - a)$. Then $h'(x) = C$ for all x. For each $f \in \mathcal{H}$,

$$\int_a^b f'(x)\overline{h'(x)}\,dx = \int_a^b f'(x)\overline{C}\,dx = (f(b) - f(a)) \cdot \overline{C} \tag{11.2}$$
$$= (B - A)\overline{C} = (b - a)|C|^2 \in \mathbb{R}.$$

In particular

$$\int_a^b h'(x)\overline{h'(x)}\,dx = (b - a)|C|^2. \tag{11.3}$$

Fix $f \in \mathcal{H}$. Using (11.2) and (11.3), we see that

$$\int_a^b |f'(x) - h'(x)|^2 dx = \int_a^b (f'(x) - h'(x))(\overline{f'(x) - h'(x)})\,dx$$
$$= \int_a^b |f'(x)|^2\,dx - (b - a)|C|^2.$$

Since $(b - a)|C|^2 \geq 0$, the lemma follows. ∎

Proof of Proposition 11.1 (i) Put $\nu = \Delta(f)$, so

$$f = \varphi_\mu(\nu) \quad \text{and} \quad \langle f, f\rangle_{\mathrm{Dir}} = \iint g_\mu(x, y)\,d\nu(x)\overline{d\nu(y)}.$$

Let $\nu_1, \nu_2, \ldots$ be a bounded sequence of discrete measures of total mass zero which converge weakly to ν. Put $f_n = \varphi_\mu(\nu_n)$. Then $f_n \in \mathrm{CPA}_\mu(\Gamma)$, and $\langle f - f_n, f - f_n\rangle_{\mathrm{Dir}}$ converges to 0 by Lemma 10.1, since the measures $\nu_n \times \nu_n$, $\nu \times \nu_n$, and $\nu_n \times \nu$ converge weakly to $\nu \times \nu$ on $\Gamma \times \Gamma$.

(ii) Take $f \in V_\mu(\Gamma)$, and fix $\varepsilon > 0$. Let $X \subset \Gamma$ be the finite set of points where $f'(x)$ is not defined. After enlarging X, we can assume that it contains a vertex set for Γ. For each $p \in X$ and $\delta > 0$, let $N(p, \delta) = \{z \in \Gamma : d(p, z) < \delta\}$ be the δ-neighborhood of p under the canonical metric on Γ.

Since $f' \in L^2(\Gamma)$, there is a $\delta_1 > 0$ such that

$$\int_{\bigcup_{x\in X} N(x,\delta_1)} f'(x)\overline{f'(x)}\,dx < \varepsilon.$$

Put $U = \bigcup_{x\in X} N(x, \delta_1)$, and put $K = \Gamma\backslash U$. Then K is compact. Since $f'(x)$ is continuous on K, it is uniformly continuous. Let $0 < \delta_2 < \delta_1$ be small enough that $|f'(x) - f'(y)| < \sqrt{\varepsilon}$ for all $x, y \in K$ with $d(x, y) < \delta_2$. Let $Y = \{y_1, \ldots, y_M\} \subset K$ be a finite set such that for each $x \in K$, there is some $y_i \in Y$ with $d(x, y_i) < \delta_2$. After enlarging Y we can assume it contains all the boundary points of K.

The set $X \cup Y$ partitions Γ into a finite union of segments. Let $F\colon \Gamma \to \mathbb{C}$ be the unique continuous function such that $F(z) = f(z)$ for each $z \in X \cup Y$, and which is affine on $\Gamma\backslash(X \cup Y)$.

Let (a,b) be one of the segments in $\Gamma\backslash(X \cup Y)$. If $(a,b) \subset K$, then necessarily $d(a,b) < \delta_2$. For each $x \in (a,b)$ we have $F'(x) = (f(b) - f(a))/(b-a)$. On the other hand, by the mean value theorem there is a point $x_0 \in (a,b)$ with $f'(x_0) = (f(b) - f(a))/(b-a)$. Since $x, x_0 \in [a,b]$ it follows that $d(x,x_0) < \delta_2$. Hence, $|f'(x) - F'(x)| = |f'(x) - f'(x_0)| < \sqrt{\varepsilon}$. It follows that

$$\int_a^b |f'(x) - F'(x)|^2 \, dx \le (b-a)\varepsilon.$$

On the other hand, if $(a,b) \subset U$, then by Lemma 11.2

$$\int_a^b |f'(x) - F'(x)|^2 \, dx \le \int_a^b |f'(x)|^2 \, dx.$$

Summing over all intervals (a,b), we see that

$$\begin{aligned}\int_\Gamma |f'(x) - F'(x)|^2 \, dx &= \int_K |f'(x) - F'(x)|^2 \, dx + \int_U |f'(x) - F'(x)|^2 \, dx \\ &\le (\ell(\Gamma) + 1) \cdot \varepsilon\end{aligned}$$

where $\ell(\Gamma)$ is the total length of Γ.

Here $F(x)$ need not lie in $\mathrm{CPA}_\mu(\Gamma)$, but $F_\mu(x) = F(x) - \frac{1}{\ell(\Gamma)} \int_\Gamma F(x)\, dx$ does, and $\langle f - F_\mu, f - F_\mu\rangle_{\mathrm{Dir}} = \langle f - F, f - F\rangle_{\mathrm{Dir}}$. Since ε is arbitrary, the proposition follows. ∎

Recall that $\mathrm{Dir}_\mu(\Gamma)$ is the completion of $\mathrm{Zh}_\mu(\Gamma)$ under $\|\cdot\|_{\mathrm{Dir}}$, with norm $\|\cdot\|_{\mathrm{Dir}}$ extending the Dirichlet norm on $\mathrm{Zh}_\mu(\Gamma)$. By Proposition 11.1, both $\mathrm{BDV}_\mu(\Gamma)$ and $V_\mu(\Gamma)$ are dense subspaces of $\mathrm{Dir}_\mu(\Gamma)$. In particular, $\mathrm{Dir}_\mu(\Gamma)$ can be defined as the completion of any of the spaces $\mathrm{CPA}_\mu(\Gamma)$, $V_\mu(\Gamma)$, $\mathrm{Zh}_\mu(\Gamma)$, or $\mathrm{BDV}_\mu(\Gamma)$ under the Dirichlet norm.

Recall from Lemma 3.1 that there is a constant C_∞ such that for each $f \in \mathrm{Zh}_\mu(\Gamma)$, $\|f\|_\infty \le C_\infty \cdot \|f\|_{\mathrm{Dir}}$. As shown in Corollary 3.3, this implies that the abstract Hilbert space $\mathrm{Dir}_\mu(\Gamma)$ can be identified with a space of continuous functions on Γ.

Corollary 11.3 *We can uniquely identify* $\mathrm{BDV}_\mu(\Gamma)$ *and* $V_\mu(\Gamma)$ *with subspaces of* $\mathrm{Dir}_\mu(\Gamma)$*, where the latter is regarded as a subset of* $\mathcal{C}(\Gamma)$ *via the embedding defined in Corollary* 3.3.

Proof We must show that each $g \in \mathrm{BDV}_\mu(\Gamma)$ (resp., $V_\mu(\Gamma)$) is taken to itself under the embedding $\iota_\mu\colon \mathrm{Dir}_\mu(\Gamma) \hookrightarrow \mathcal{C}(\Gamma)$ defined in the proof of Corollary 3.3.

If $g \in \mathrm{BDV}_\mu(\Gamma)$, let $\nu = \Delta(g)$, so $g = \varphi_\mu(\nu)$. Let $\nu_1, \nu_2, \ldots$ be a bounded sequence of discrete measures of total mass zero converging weakly to ν.

By Proposition 8.1 the functions $g_n(x) = \varphi_\mu(\nu_n)$ converge uniformly to $g(x)$. Moreover, each g_n belongs to $\mathrm{CPA}_\mu(\Gamma) \subset \mathrm{Dir}_\mu(\Gamma)$. By Corollary 3.3 $\iota_\mu(g_n) = g_n$, and by Lemma 3.1 the g_n form a Cauchy sequence under $\|\cdot\|_\infty$. By definition, the image $G = \iota_\mu(g)$ is the limit of this Cauchy sequence. Hence $G = g$.

Examination of the proof of Lemma 3.1 shows that the conclusions of that Lemma hold for $V_\mu(\Gamma)$ as well as for $\mathrm{Zh}_\mu(\Gamma)$. Hence the fact that $\iota_\mu(g) = g$ for all $g \in V_\mu(\Gamma)$ follows by the proof of Corollary 3.3. ∎

Let μ_1 and μ_2 be real, signed Borel measures on Γ with $\mu_1(\Gamma) = \mu_2(\Gamma) = 1$. There is a canonical projection $\pi_{1,2}\colon \mathrm{Dir}_{\mu_1}(\Gamma) \to \mathrm{Dir}_{\mu_2}(\Gamma)$, defined by

$$\pi_{1,2}(f) = f(x) - \int_\Gamma f(x)\, d\mu_2(x).$$

It is easy to check that $\pi_{2,1} = \pi_{1,2}^{-1}$.

Proposition 11.4 *$\pi_{1,2}$ is an isometry from $\mathrm{Dir}_{\mu_1}(\Gamma)$ onto $\mathrm{Dir}_{\mu_2}(\Gamma)$ relative to the Dirichlet norms on those spaces, which takes $V_{\mu_1}(\Gamma)$ to $V_{\mu_2}(\Gamma)$, $\mathrm{CPA}_{\mu_1}(\Gamma)$ to $\mathrm{CPA}_{\mu_2}(\Gamma)$, and $\mathrm{BDV}_{\mu_1}(\Gamma)$ to $\mathrm{BDV}_{\mu_2}(\Gamma)$.*

Proof Since $\pi_{1,2}$ simply translates each function by a constant, it is clear that it takes $V_{\mu_1}(\Gamma)$ to $V_{\mu_2}(\Gamma)$, $\mathrm{CPA}_{\mu_1}(\Gamma)$ to $\mathrm{CPA}_{\mu_2}(\Gamma)$, and $\mathrm{BDV}_{\mu_1}(\Gamma)$ to $\mathrm{BDV}_{\mu_2}(\Gamma)$. Since the Dirichlet pairings on $V(\Gamma)$ and $\mathrm{BDV}(\Gamma)$ are invariant under translation by constants, we have

$$\langle \pi_{1,2}(f), \pi_{1,2}(f)\rangle_{\mathrm{Dir}} = \langle f, g\rangle_{\mathrm{Dir}} \tag{11.4}$$

for all $f, g \in V_{\mu_1}(\Gamma)$ and all $f, g \in \mathrm{BDV}_{\mu_1}(\Gamma)$. These spaces are dense in $\mathrm{Dir}_{\mu_1}(\Gamma)$, so (11.4) holds for all $f, g \in \mathrm{Dir}_{\mu_1}(\Gamma)$. This shows that $\pi_{1,2}$ is an injective isometry from $\mathrm{Dir}_{\mu_1}(\Gamma)$ into $\mathrm{Dir}_{\mu_2}(\Gamma)$. The surjectivity follows by considering the inverse map $\pi_{2,1}$. ∎

This simple proposition has two interesting consequences. First, if $\{F_n\}$ is a complete orthonormal basis for $\mathrm{Dir}_{\mu_1}(\Gamma)$ given by eigenfunctions of Δ in $\mathrm{Dir}_{\mu_1}(\Gamma)$, then $\{\pi_{1,2}(F_n)\}$ is a complete orthonormal basis for $\mathrm{Dir}_{\mu_2}(\Gamma)$. Note however that the functions $\pi_{1,2}(F_n)$ are in general not eigenfunctions of Δ in $\mathrm{Dir}_{\mu_2}(\Gamma)$. Furthermore, although the F_n are mutually orthogonal relative to the L^2 inner product $\langle\, ,\rangle$, the projected functions $\pi_{1,2}(F_n)$ are not in general orthogonal relative to $\langle\, ,\rangle$ even though they remain orthogonal relative to $\langle\, ,\rangle_{\mathrm{Dir}}$.

Second, if we define the "Dirichlet space" $\mathrm{Dir}(\Gamma)$ by

$$\mathrm{Dir}(\Gamma) = \mathrm{Dir}_{\mu_1}(\Gamma) \oplus \mathbb{C}\cdot 1,$$

then $\mathrm{Dir}(\Gamma)$ contains the spaces $\mathrm{Dir}_{\mu_2}(\Gamma)$ for all μ_2, and in particular is independent of μ_1. It appears to be an important space to study.

12 Equivalence between Notions of Eigenfunctions

We will now show that the eigenfunctions of Δ in $\mathrm{Dir}_\mu(\Gamma)$ constructed in Theorem 3.8 coincide with the eigenfunctions of φ_μ given by Theorem 8.4.

This equivalence provides new information: it implies that the eigenfunctions of Δ in $\mathrm{Dir}_\mu(\Gamma)$ are "smooth enough" to belong to $\mathrm{BDV}_\mu(\Gamma)$, and that the eigenfunctions of φ_μ, which are known to provide a Hilbert space basis for $L^2_\mu(\Gamma)$, also form a basis for $\mathrm{Dir}_\mu(\Gamma)$.

Theorem 12.1 *Given a nonzero function $f \in L^2(\Gamma)$, the following are equivalent:*

(i) *f is an eigenfunction of Δ in $\mathrm{Dir}_\mu(\Gamma)$ with eigenvalue $\lambda > 0$ (Definition 3.7),*
(ii) *f is an eigenfunction of φ_μ with eigenvalue $\alpha = 1/\lambda > 0$ (Definition 8.3),*
(iii) *f is an eigenfunction of Δ in $\mathrm{BDV}_\mu(\Gamma)$ with eigenvalue $\lambda > 0$ (Definition 9.2).*

Proof We know that (ii) $\Leftrightarrow$ (iii) by Proposition 9.1, so it suffices to show that (i) $\Leftrightarrow$ (ii).

(ii) $\Rightarrow$ (i). Suppose first that $0 \neq f$ is an eigenfunction of φ_μ with eigenvalue $\alpha \neq 0$. By Theorem 8.4, $f \in \mathrm{BDV}_\mu(\Gamma)$. Theorem 10.3 shows that $\alpha > 0$. By Lemma 8.5,

$$\alpha \cdot \Delta(f) = \Delta(\varphi_\mu(f)) = f(x)\,dx - \Big(\int_\Gamma f(x)\,dx\Big)\,\mu. \tag{12.1}$$

To see that f is an eigenfunction of Δ in $\mathrm{Dir}_\mu(\Gamma)$ with eigenvalue $1/\alpha$, we must show that $\langle f, g\rangle_{\mathrm{Dir}} = \frac{1}{\alpha}\langle f, g\rangle_{L^2}$ for each $g \in \mathrm{Dir}_\mu(\Gamma)$. Since $\mathrm{Zh}_\mu(\Gamma)$ is dense in $\mathrm{Dir}_\mu(\Gamma)$, we can assume that $g \in \mathrm{Zh}_\mu(\Gamma)$. By (12.1),

$$\alpha \int_\Gamma \overline{g(x)}\,d\Delta(f)(x) = \int_\Gamma \overline{g(x)} f(x)\,dx - \Big(\int_\Gamma f(x)\,dx\Big)\cdot\Big(\int_\Gamma \overline{g(x)}\,d\mu(x)\Big).$$

Here $g(x) \in \mathrm{Zh}_\mu(\Gamma)$ and μ is real, so $\overline{g(x)} \in \mathrm{Zh}_\mu(\Gamma)$ and $\int_\Gamma \overline{g(x)}\,d\mu(x) = 0$. Thus

$$\langle f, g\rangle_{\mathrm{Dir}} = \int_\Gamma \overline{g(x)}\,d\Delta(f)(x) = \frac{1}{\alpha}\int_\Gamma \overline{g(x)} f(x)\,dx = \frac{1}{\alpha}\langle f, g\rangle_{L^2}.$$

(i) $\Rightarrow$ (ii). Conversely, suppose that $0 \neq f$ is an eigenfunction of Δ in $\mathrm{Dir}_\mu(\Gamma)$ with eigenvalue λ. By Theorem 3.8, necessarily $\lambda > 0$. Let $g_1, g_2, \ldots$ be an orthonormal basis for $L^2(\Gamma)$ consisting of eigenfunctions of φ_μ, as given by Theorem 8.4. We can expand $f = \sum_n c_n g_n$ in $L^2(\Gamma)$. Since $\int f(x)\,d\mu = 0$, Proposition 8.6 shows that for each n with $c_n \neq 0$ the corresponding eigenfunction g_n has a nonzero eigenvalue α_n, and Theorem 8.4 says that g_n belongs to $\mathrm{BDV}_\mu(\Gamma)$.

Since $\mathrm{BDV}_\mu(\Gamma) \subset \mathrm{Dir}_\mu(\Gamma)$, for each such g_n we have

$$\langle f, g_n\rangle_{\mathrm{Dir}} = \lambda\langle f, g_n\rangle_{L^2} = \lambda \cdot c_n.$$

On the other hand, g_n is an eigenfunction of Δ with eigenvalue $1/\alpha_n$, and α_n is real by Theorem 8.4, so

$$\langle f, g_n\rangle_{\mathrm{Dir}} = \overline{\langle g_n, f\rangle}_{\mathrm{Dir}} = \frac{1}{\alpha_n}\overline{\langle g_n, f\rangle}_{L^2} = \frac{1}{\alpha_n}\langle f, g_n\rangle_{L^2} = \frac{1}{\alpha_n}\cdot c_n.$$

Thus, the only eigenfunctions g_n for which $c_n \neq 0$ are ones satisfying $1/\alpha_n = \lambda$. Theorem 8.4 says that there are only finitely many such n, so

$$f = \sum_{1/\alpha_n=\lambda} c_n g_n \in \mathrm{BDV}_\mu(\Gamma).$$

Since each g_n is an eigenfunction of φ_μ with eigenvalue $\alpha_n = 1/\lambda$, so is f. ■

Corollary 12.2 *Each eigenfunction of* Δ *in* $\mathrm{Dir}_\mu(\Gamma)$ *belongs to* $\mathrm{BDV}_\mu(\Gamma)$.

Corollary 12.3 *The eigenfunctions of* φ_μ *in* $\mathrm{BDV}_\mu(\Gamma)$ *contain a complete orthonormal basis for* $\mathrm{Dir}_\mu(\Gamma)$. *More precisely, let* $\{f_n\}$ *be a complete* L^2*-orthonormal basis of eigenfunctions for* φ_μ *in* $\mathrm{BDV}_\mu(\Gamma)$, *and put* $F_n = f_n/\sqrt{\lambda_n}$. *Then* $\{F_n\}$ *is a complete orthonormal basis for* $\mathrm{Dir}_\mu(\Gamma)$.

13 The Eigenfunction Expansion of $g_\mu(x, y)$, and Applications

Let $\{f_n\}$ be a basis for $\mathrm{BDV}_\mu(\Gamma)$ consisting of eigenfunctions of Δ, normalized so that $\|f_n\|_2 = 1$ for each n.

We now consider the expansion of $g_\mu(x, y)$ in terms of the eigenfunctions $\{f_n\}$. For each $y \in \Gamma$, put $G_y(x) = g_\mu(x, y)$. In $L^2(\Gamma)$, we can write $G_y(x) = \sum_{n=1}^{\infty} a_n f_n(x)$, with

$$a_n = \langle G_y(x), f_n(x)\rangle = \int_\Gamma g_\mu(x, y)\overline{f_n(x)}\,dx = \overline{\varphi_\mu(f_n)(y)} = \frac{1}{\lambda_n}\overline{f_n(y)}.$$

Thus

$$g_\mu(x, y) = \sum_{n=1}^{\infty} \frac{f_n(x)\overline{f_n(y)}}{\lambda_n}, \tag{13.1}$$

where for each y the series converges to $g_\mu(x, y)$ in $L^2(\Gamma)$.

Proposition 13.1 *The series* $\sum_{n=1}^{\infty} \frac{f_n(x)\overline{f_n(y)}}{\lambda_n}$ *converges uniformly to* $g_\mu(x, y)$ *for all* $x, y \in \Gamma$.

Proof This follows from a classical theorem of Mercer (see [RN, p. 245]). Mercer's theorem asserts that if K is a compact measure space and $A(x, y)$ is a continuous symmetric kernel for which the corresponding integral operator $A\colon L^2(K) \to L^2(K)$ is positive (that is, $\langle Af, f\rangle \geq 0$ for all $f \in L^2(K)$), then the L^2 eigenfunction expansion of $A(x, y)$ converges uniformly to $A(x, y)$. The proof in [RN] is given when $K = [a, b]$ is a closed interval, but the argument is general.

In our case $g_\mu(x, y)$ is continuous, real-valued, and symmetric, and the integral operator φ_μ is positive by Theorem 10.3. ■

Several important facts follow from this.

Corollary 13.2 $\sum_{n=1}^{\infty} 1/\lambda_n = \int_\Gamma g_\mu(x,x)\,dx < \infty$.

Proof By Proposition 13.1 $\sum_{n=1}^{\infty} f_n(x)\overline{f_n(x)}/\lambda_n$ converges uniformly to $g_\mu(x,x)$ on Γ. Since $\langle f_n, f_n\rangle = 1$ for each n,

$$\int_\Gamma g_\mu(x,x)\,dx = \sum_{n=1}^{\infty} \frac{1}{\lambda_n}\int_\Gamma f_n(x)\overline{f_n(x)}\,dx = \sum_{n=1}^{\infty}\frac{1}{\lambda_n}. \qquad \blacksquare$$

Corollary 13.3 *$g_\mu(x,x) \geq 0$ for all $x \in \Gamma$. Indeed, $g_\mu(x,x) > 0$ for all x unless $\mu = \delta_{x_0}(x)$ for some x_0, in which case $g_\mu(x,x) > 0$ for all $x \neq x_0$.*

Proof It follows from $g_\mu(x,x) = \sum_{n=1}^{\infty} f_n(x)\overline{f_n(x)}/\lambda_n$ that $g_\mu(x,x) \geq 0$ for all x. If $g_\mu(x_0,x_0) = 0$ for some x_0, then $f_n(x_0) = 0$ for all n. But then $g_\mu(x_0,y) = \sum_n f_n(x_0)\overline{f_n(y)}/\lambda_n = 0$ for all y, which means that $0 = \Delta_y(g_\mu(x_0,y)) = \delta_{x_0}(y) - \mu(y)$. Thus $\mu = \delta_{x_0}(y)$. If there were another point x_1 with $g_\mu(x_1,x_1) = 0$, then $\delta_{x_0}(y) = \mu = \delta_{x_1}(y)$, so $x_1 = x_0$. $\blacksquare$

Corollary 13.4 $\sum_{n=1}^{\infty} |f_n(x)\overline{f_n(y)}/\lambda_n|$ *converges uniformly for all x, y.*

Proof Note that

$$\sum_{n=1}^{\infty}\left|\frac{f_n(x)}{\sqrt{\lambda_n}}\right|^2 = \sum_{n=1}^{\infty} f_n(x)\overline{f_n(x)}/\lambda_n = g_\mu(x,x)$$

for each x, and the convergence is uniform for all $x \in \Gamma$. Hence, for any $\varepsilon > 0$ there is an N such that $\sum_{n=N}^{\infty}\left|\frac{f_n(x)}{\sqrt{\lambda_n}}\right|^2 \leq \varepsilon$.

By the Cauchy–Schwarz inequality,

$$\sum_{n=1}^{\infty} |f_n(x)\overline{f_n(y)}/\lambda_n| \leq \left(\sum_{n=1}^{\infty}\left|\frac{f_n(x)}{\sqrt{\lambda_n}}\right|^2\right)^{1/2}\left(\sum_{n=1}^{\infty}\left|\frac{f_n(y)}{\sqrt{\lambda_n}}\right|^2\right)^{1/2}$$

converges. Another application of Cauchy–Schwarz gives

$$\sum_{n=N}^{\infty} |f_n(x)\overline{f_n(y)}/\lambda_n| \leq \varepsilon,$$

so the convergence is uniform. $\blacksquare$

Corollary 13.5 *With the eigenfunctions normalized so that $\|f_n\|_2 = 1$ for all n, we have $\|f_n\|_\infty = o(\sqrt{\lambda_n})$ as $n \to \infty$.*

Proof As in the proof of Corollary 13.4, for any $\varepsilon > 0$ there is an N so that $\sum_{n=N}^{\infty} |\frac{f_n(x)}{\sqrt{\lambda_n}}|^2 \leq \varepsilon$ for all x. Hence for each $n \geq N$, $\|f_n\|_\infty \leq \sqrt{\varepsilon}\cdot\sqrt{\lambda_n}$. $\blacksquare$

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Let $\{f_n\}_{1 \le n < \infty}$ be an L^2-orthonormal basis for $\mathrm{BDV}_\mu(\Gamma)$ consisting of eigenfunctions for Δ. We will now show that the L^2-expansion of each $F \in \mathrm{Dir}_\mu(\Gamma)$ in terms of the f_n converges uniformly to $F(x)$ on Γ.

Suppose $F \in \mathrm{Dir}_\mu(\Gamma)$. If $\mu = g(x)dx$ for some $g \in L^2(\Gamma)$, then by Proposition 8.6 the eigenfunctions f_n form an orthonormal basis for

$$(\mathbb{C}g)^\perp = \left\{ f \in L^2(\Gamma) : \int_\Gamma f(x)\overline{g(x)}\, dx = 0 \right\}.$$

Since $F \in \mathcal{C}(\Gamma)$ and $\int_\Gamma F(x)\, d\mu(x) = 0$, we have $F \in (\mathbb{C}g)^\perp$. If μ is not of this form, then the f_n form an orthonormal basis for $L^2(\Gamma)$.

In either case we can expand $F(x) = \sum_{n=1}^\infty c_n f_n(x)$ in $L^2(\Gamma)$.

Corollary 13.6 *For each $F \in \mathrm{Dir}_\mu(\Gamma)$, the series $\sum_{n=1}^\infty c_n f_n(x)$ converges uniformly to $F(x)$ on Γ.*

Proof Let λ_n be the eigenvalue corresponding to f_n, and put $F_n(x) = f_n(x)/\sqrt{\lambda_n}$, so $\langle F_n, F_n\rangle_{\mathrm{Dir}} = 1$. Since the F_n are eigenfunctions of Δ belonging to $\mathrm{BDV}_\mu(\Gamma)$ by Corollary 12.3 and Theorem 12.1, they are mutually orthogonal under $\langle\ ,\ \rangle_{\mathrm{Dir}}$.

By assumption, we have

$$F(x) = \sum_n c_n f_n(x) = \sum_n \sqrt{\lambda_n} c_n F_n(x)$$

in $L^2(\Gamma)$. By Parseval's inequality in $\mathrm{Dir}_\mu(\Gamma)$,

$$\sum_n \left| \sqrt{\lambda_n} c_n \right|^2 \le \langle F, F\rangle_{\mathrm{Dir}}. \tag{13.2}$$

By Corollary 13.4, the series

$$\sum_n |F_n(x)|^2 = \sum_n |f_n(x)\overline{f_n(x)}/\lambda_n|$$

converges uniformly on Γ. Hence, for each $\varepsilon > 0$, there is an N such that $\sum_{n \ge N} |F_n(x)|^2 \le \varepsilon$. By (13.2) and the Cauchy–Schwarz inequality,

$$\sum_{n \ge N} |c_n f_n(x)| \le \left(\langle F, F\rangle_{\mathrm{Dir}}\right)^{1/2} \cdot \sqrt{\varepsilon}$$

for all $x \in \Gamma$.

Thus the series $\sum_n c_n f_n(x)$ converges uniformly and absolutely to a function $G(x) \in \mathcal{C}(\Gamma)$. Since $F(x)$ and $G(x)$ are continuous, and $F = G$ in $L^2(\Gamma)$, it follows that $F(x) = G(x)$. ■

We can use Proposition 13.1 and Corollary 13.6 to give another proof of Theorem 10.4(ii), the energy minimization principle, as follows.

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Proof of Theorem 10.4(ii) Let ν be a bounded measure of total mass 1. Define the n-th Fourier coefficient of ν to be $c_n = \int_\Gamma f_n(x)\,d\nu(x)$.

By Corollary 13.6, the linear space spanned by the functions $\{f_n\}$ (which lie in $\mathrm{BDV}_\mu(\Gamma)$, and in particular have $\int f_n\,d\mu = 0$) is dense in $\mathrm{Dir}_\mu(\Gamma)$. Together with $f_0 = 1$, the f_n span a dense subspace of $\mathrm{Dir}(\Gamma)$, which itself is dense in $\mathcal{C}(\Gamma)$. By the Riesz representation theorem, it follows that $\mu = \nu$ if and only if $c_n = 0$ for all $n \geq 1$. By uniform convergence, we find that

$$\begin{aligned}
\iint g_\mu(x,y)\,d\nu(x)\overline{d\nu(y)} &= \iint \Big(\sum_{n\geq 1} f_n(x)\overline{f_n(y)}/\lambda_n\Big)\,d\nu(x)\overline{d\nu(y)} \\
&= \sum_{n\geq 1}\Big(\int f_n(x)\,d\nu(x)\Big)\Big(\int \overline{f_n(y)}\,\overline{d\nu(y)}\Big)/\lambda_n \\
&= \sum_{n\geq 1} |c_n|^2/\lambda_n \geq 0,
\end{aligned}$$

with equality if and only if $c_n = 0$ for all $n \geq 1$, as desired. ■

Application: Lower Bounds for $g_\mu(x, y)$-Discriminant Sums

From the uniform convergence of the expansion (13.1), we obtain Elkies-type lower bounds for discriminant sums formed using $g_\mu(x, y)$. Such sums occur naturally in arithmetic geometry (see for example [HS, Lemma 2.1], in which Γ is a circle).

Proposition 13.7 *There exists a constant $C = C(\Gamma,\mu) > 0$ such that for all $N \geq 2$ and all $x_1,\ldots,x_N \in \Gamma$. Then*

$$\frac{1}{N(N-1)}\sum_{i\neq j} g_\mu(x_i,x_j) \geq -\frac{C}{N}.$$

Proof Put $M = \sup_{x\in\Gamma} g_\mu(x,x)$. Then

$$\begin{aligned}
\sum_{i\neq j} g_\mu(x_i,x_j) &= \sum_{i\neq j}\sum_n f_n(x_i)\overline{f_n(x_j)}/\lambda_n \\
&= \sum_n \Big|\sum_{i=1}^N f_n(x_i)/\sqrt{\lambda_n}\Big|^2 - \sum_{i=1}^N g_\mu(x_i,x_i) \\
&\geq -M\cdot N,
\end{aligned}$$

which yields the desired conclusion. ■

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14 The Canonical Measure and the tau Constant

In this section we discuss some properties of a "canonical measure" discovered by Chinburg and Rumely. This will not only provide us with a nice application of the energy pairing, it will also illustrate why it is useful to allow general measures μ (rather than just multiples of dx, for example) when discussing eigenvalues of the Laplacian.

For $x, y \in \Gamma$, we denote by $r(x, y)$ the "effective resistance" between x and y, where Γ is considered as a resistive electric circuit as in §1.5. Equivalently, we have $r(x, y) = j_x(y, y)(= j_y(x, x))$.

For example, if $\Gamma = [0, 1]$, then $r(x, y)$ is simply $|x - y|$. Referring back to §1.8, note that $-\frac{1}{2}|x - y|$ differs by a constant factor of $\frac{1}{4}$ from the Arakelov-Green's function $g_\mu(x, y)$ for the measure $\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$.

This observation generalizes to arbitrary metrized graphs as follows (see [CR, Theorem 2.11], [BF]):

Theorem 14.1

1. *The probability measure* $\mu_{\mathrm{can}} = \Delta_x\big(\frac{1}{2}r(x, y)\big) + \delta_y(x)$ *is independent of* $y \in \Gamma$.
2. μ_{can} *is the unique measure* μ *of total mass* 1 *on* Γ *for which* $g_\mu(x, x)$ *is a constant independent of* x.
3. *There is a constant* $\tau(\Gamma) \in \mathbb{R}$ *such that* $g_{\mu_{\mathrm{can}}}(x, y) = -\frac{1}{2}r(x, y) + \tau(\Gamma)$.

We call μ_{can} the *canonical measure* on Γ, and we call $\tau(\Gamma)$ the *tau constant* of Γ. In [CR, Theorem 2.11], the following explicit formula is given:

$$\mu_{\mathrm{can}} = \sum_{\text{vertices } p} (1 - \frac{1}{2}\mathrm{valence}(p))\delta_p(x) + \sum_{\text{edges } e} \frac{dx}{L(e) + R(e)}, \tag{14.1}$$

where $L(e)$ is the length of edge e and $R(e)$ is the effective resistance between the endpoints of e in the graph $\Gamma\backslash e$, when the graph is regarded as an electric circuit with resistances equal to the edge lengths. In particular, μ_{can} has a point mass of negative weight at each branch point p, so it is a positive measure only if Γ is a segment or a loop.

The following result is an immediate consequence of Theorem 10.4.

Corollary 14.2 *The canonical measure is the unique measure* ν *of total mass* 1 *on* Γ *maximizing the integral*

$$\iint_{\Gamma\times\Gamma} r(x, y)\, d\nu(x)\overline{d\nu(y)}. \tag{14.2}$$

One can therefore think of the canonical measure as being like an "equilibrium measure" on Γ (in the sense of capacity theory), and of $\tau(\Gamma)$ as the corresponding "capacity" of Γ (with respect to the potential kernel $\frac{1}{2}r(x, y)$). Note, however, that μ_{can}, unlike equilibrium measures in capacity theory, is not necessarily a positive measure.

It can also be shown using Theorem 10.4(i) that there is a unique *probability* measure maximizing (14.2) over all positive measures ν of total mass 1. However, we do not know an explicit formula for this measure analogous to (14.1).

The next result follows immediately from Corollary 13.2.

Corollary 14.3 *If $\ell(\Gamma)$ denotes the total length of Γ, then*

$$\ell(\Gamma) \cdot \tau(\Gamma) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n},$$

where the λ_n's are the eigenvalues of the Laplacian with respect to the canonical measure, i.e., *the eigenvalues of Δ on $\mathrm{Dir}_{\mu_{\mathrm{can}}}(\Gamma)$. In particular, if $\ell(\Gamma) = 1$, then $\tau(\Gamma)$ is the trace of the operator $\varphi_{\mu_{\mathrm{can}}}$.*

Especially, $\tau(\Gamma) > 0$. Another description of $\tau(\Gamma)$, which shows this directly, is as follows.

Lemma 14.4 *For any fixed $y \in \Gamma$, we have*

$$\tau(\Gamma) = \frac{1}{4} \int_{\Gamma} \Big(\frac{\partial}{\partial x} r(x, y) \Big)^2 dx.$$

Proof Fix $y \in \Gamma$, and set $f(x) = \frac{1}{2} r(x, y)$. Since $\int g_{\mu_{\mathrm{can}}}(x, y)\, d\mu_{\mathrm{can}}(x) = 0$, we have $\tau(\Gamma) = \int f(x)\, d\mu_{\mathrm{can}}(x)$. Substituting $\mu_{\mathrm{can}} = \delta_y(x) + \Delta_x f(x)$ and noting that $f(y) = 0$, we obtain

$$\tau(\Gamma) = \int f(x) \Delta f(x) = \langle f, f \rangle_{\mathrm{Dir}} = \int (f'(x))^2\, dx. \qquad \blacksquare$$

Using Lemma 14.4, it is not hard to see that if we multiply all lengths on Γ by a positive constant β (obtaining a graph $\Gamma(\beta)$ of total length $\beta \cdot \ell(\Gamma)$), then $\tau(\Gamma(\beta)) = \beta \cdot \tau(\Gamma)$. Thus it is natural to consider the ratio $\tau(\Gamma)/\ell(\Gamma)$, which is scale-independent.

One can show using Lemma 14.4 that for a metrized graph Γ with n edges,

$$\frac{1}{16n} \ell(\Gamma) \le \tau(\Gamma) \le \frac{1}{4} \ell(\Gamma), \tag{14.3}$$

with equality in the upper bound if and only if Γ is a tree. The lower bound is not sharp.

Experience shows that it is hard to construct graphs of total length 1 and small tau constant. The smallest known example was found by Phil Zeyliger, with $\tau(\Gamma) \cong .021532$. We therefore pose the following conjecture.

Conjecture 14.5 *There is a universal constant $C > 0$ such that for all metrized graphs Γ, $\tau(\Gamma) \ge C \cdot \ell(\Gamma)$.*

We remark that by Corollary 14.3, a universal positive lower bound for $\tau(\Gamma)$ over all metrized graphs of length 1 would be implied by a universal upper bound for λ_1, the smallest nonzero eigenvalue of the Laplacian with respect to the canonical measure, on such graphs.

Application: The Operator φ_{dx} Has Minimal Trace

Given a measure μ on Γ of total mass 1, the trace of the operator φ_μ is

$$\mathrm{Tr}(\varphi_\mu) = \int_\Gamma g_\mu(x,x)\,dx = \sum_{n=1}^{\infty} \frac{1}{\lambda_n},$$

where the λ_n are the eigenvalues of Δ on $\mathrm{Dir}_\mu(\Gamma)$. We can compare the traces for different measures using the energy pairing (see Theorem 10.4):

Proposition 14.6 *Let $\mu_1, \mu_2 \in \mathrm{Meas}(\Gamma)$ be measures of total mass 1. Then*

$$\mathrm{Tr}(\varphi_{\mu_1}) = \mathrm{Tr}(\varphi_{\mu_2}) - \langle dx, dx\rangle_{\mu_2} + \langle dx - \mu_1, dx - \mu_1\rangle. \tag{14.4}$$

Proof We first claim that

$$g_{\mu_1}(x,y) = g_{\mu_2}(x,y) - \phi_{\mu_2}(\mu_1)(x) - \phi_{\mu_2}(\mu_1)(y) + \langle \mu_1, \mu_1\rangle_{\mu_2}. \tag{14.5}$$

Indeed, it follows from Proposition 7.1 that the Laplacian with respect to x of both sides of (14.5) is $\delta_y - \mu_1$, so the two sides differ by a constant depending only on x. By symmetry, the left- and right-hand sides of (14.5) also differ by a constant depending only on y. Thus (14.5) is true up to an additive constant. Integrating both sides with respect to $d\mu_1(x)d\mu_1(y)$ shows that the constant is zero, proving the claim.

Setting $x = y$ in (14.5) and integrating both sides with respect to dx now gives $\mathrm{Tr}(\varphi_{\mu_1}) = \mathrm{Tr}(\varphi_{\mu_2}) - 2\langle dx, \mu_1\rangle_{\mu_2} + \langle \mu_1, \mu_1\rangle_{\mu_2}$, which is equivalent to (14.4). ■

Suppose Γ has total length 1. Setting $\mu_1 = \mu$ and $\mu_2 = dx$ in Proposition 14.6, we find that of all measures μ on Γ, the dx measure is the one for which φ_μ has minimal trace:

Corollary 14.7 *If $\ell(\Gamma) = 1$ and $\mu \in \mathrm{Meas}(\Gamma)$ has total mass 1, then*

$$\mathrm{Tr}(\varphi_\mu) = \mathrm{Tr}(\varphi_{dx}) + \langle dx - \mu, dx - \mu\rangle \geq \mathrm{Tr}(\varphi_{dx}),$$

with equality if and only if $\mu = dx$.

Proof By Theorem 10.4 $\langle dx - \mu, dx - \mu\rangle \geq 0$, with equality if and only if $\mu = dx$. ■

Since $\mathrm{Tr}(\varphi_{dx})$ is minimal, it is useful to have a formula for it.

Corollary 14.8 *If $\ell(\Gamma) = 1$, then* $\mathrm{Tr}(\varphi_{dx}) = \frac{1}{2}\iint_{\Gamma\times\Gamma} r(x,y)\,dxdy$.

Proof Taking $\mu_1 = dx$ and $\mu_2 = \mu_{\mathrm{can}}$ in Proposition 14.6 gives

$$\mathrm{Tr}(\varphi_{dx}) = \mathrm{Tr}(\varphi_{\mu_{\mathrm{can}}}) - \langle dx, dx\rangle_{\mu_{\mathrm{can}}}. \tag{14.6}$$

Here $\mathrm{Tr}(\varphi_{\mu_{\mathrm{can}}}) = \tau(\Gamma)$ by Corollary 14.3, while by the definition of $\langle dx, dx\rangle_{\mu_{\mathrm{can}}}$ and Theorem 14.1(iii),

$$\langle dx, dx\rangle_{\mu_{\mathrm{can}}} = \iint_{\Gamma\times\Gamma} g_{\mu_{\mathrm{can}}}(x,y)\,dxdy = \iint_{\Gamma\times\Gamma} \tau(\Gamma) - \frac{1}{2}r(x,y)\,dxdy. \tag{14.7}$$

Combining (14.6) and (14.7) gives the result. ■

In contrast with Conjecture 14.5, which asserts that $\mathrm{Tr}(\varphi_{\mu_{\mathrm{can}}}) \geq C > 0$ for all graphs of total length 1, $\mathrm{Tr}(\varphi_{dx})$ can be arbitrarily small. For example, if $\Gamma = B_n$ is the "banana graph" with two vertices connected by n edges of length $1/n$, then using Corollary 14.8 one computes that

$$\mathrm{Tr}(\varphi_{dx}) = \frac{n+2}{12n^2}.$$

15 Regularity and Boundedness

In this section, we will study the smoothness and boundedness of eigenfunctions.

Proposition 15.1 *Suppose μ has the form $\mu = g(x)dx + \sum_{i=1}^{N} c_i\delta_{p_i}(x)$ where $g(x)$ is piecewise continuous and belongs to $L^1(\Gamma)$. Assume that $X = \{p_1,\dots,p_N\}$ contains a vertex set for Γ, as well as all points where $g(x)$ is discontinuous. Then each eigenfunction of Δ in $\mathrm{BDV}_\mu(\Gamma)$ belongs to $\mathrm{Zh}(\Gamma)$. If $g(x)$ is $\mathcal{C}^m$ on $\Gamma\backslash X$, then each eigenfunction is $\mathcal{C}^{m+2}$ on $\Gamma\backslash X$, and satisfies the differential equation*

$$\frac{d^2 f}{dx^2} + \lambda f(x) = \lambda C g(x),$$

Proof If $0 \neq f \in \mathrm{BDV}_\mu(\Gamma)$ is an eigenfunction of Δ with eigenvalue λ, then by Proposition 9.1,

$$\Delta(f) = \lambda\cdot(f(x)dx - C\mu) = \lambda(f(x) - Cg(x))dx - C\lambda\sum_{i=1}^{N} c_i\delta_{p_i}(x) \tag{15.1}$$

with $C = \int_\Gamma f(x)\,dx$.

Here $f(x)$ is continuous, so $\lambda(f(x) - Cg(x))$ is continuous on $\Gamma\backslash X_g$ and belongs to $L^1(\Gamma)$. By Proposition 5.2, $f \in \mathrm{Zh}(\Gamma)$ and

$$\Delta(f) = \Delta_{\mathrm{Zh}}(f) = -f''(x)dx - \sum_p\Big(\sum_{\vec{v}\in\mathrm{Vec}(p)} d_{\vec{v}}f(p)\Big)\,\delta_p(x).$$

Comparing this with (15.1), we see that $\Delta(f)$ has no point masses except at points of X, and that on each subinterval of $\Gamma\backslash X$, $f''(x) = \lambda(Cg(x) - f(x))$. Differentiating this inductively shows that on each such subinterval, $f(x)$ has continuous derivatives up to order $m+2$. Furthermore, $f(x)$ satisfies the differential equation $f''(x) + \lambda f(x) = \lambda C g(x)$. ■

Under somewhat stronger hypotheses, the eigenfunctions are uniformly bounded.

Proposition 15.2 *Suppose $\mu = g(x)dx + \sum_{i=1}^{N} c_i\delta_{p_i}(x)$, where $g(x)$ is piecewise $\mathcal{C}^1$ on Γ and $g'(x) \in L^1(\Gamma)$. Let $\{f_n\}_{1\le n<\infty}$ be the eigenfunctions of Δ in $\mathrm{BDV}_\mu(\Gamma)$, normalized so that $\|f_n\|_2 = 1$ for each n. Then there is a constant B such that $\|f_n\|_\infty \le B$ for all n.*

Proof Write $X = \{p_1, \dots, p_N\}$. After enlarging X, we can assume that X contains a vertex set for Γ, as well as all points where $g'(x)$ fails to exist. Then $\Gamma\backslash X$ is a finite union of open segments; let their closures be denoted $e_1, \dots, e_m$. Without loss of generality, assume e_ℓ is isometrically parametrized by $[0, L_\ell]$, and identify it with that interval.

Let $f = f_n$ be an eigenfunction of Δ in $\mathrm{BDV}_\mu(\Gamma)$, normalized so that $\|f\|_2 = 1$. Let $\lambda = \lambda_n$ be the corresponding eigenvalue. By Proposition 9.1, $f \in \mathrm{Zh}(\Gamma)$ and on each segment e_ℓ it satisfies the differential equation

$$\frac{d^2 f}{dx^2} + \lambda f = \lambda C g, \tag{15.2}$$

where $C = C_f = \int_\Gamma f(x)\,dx$. Let $\gamma > 0$ be such that $\gamma^2 = \lambda$, and write f_ℓ for the restriction of f to e_ℓ.

By the method of variation of parameters (see [Si, p. 92]),

$$f_\ell(x) = A_\ell e^{i\gamma x} + B_\ell e^{-i\gamma x} + C \cdot h_{\gamma,\ell}(x)$$

for certain constants $A_\ell, B_\ell \in \mathbb{C}$, where

$$h_{\gamma,\ell}(x) = \lambda e^{i\gamma x}\int_{x_\ell}^{x} \frac{-e^{-i\gamma t}g(t)}{-2i\gamma}\,dt + \lambda e^{-i\gamma x}\int_{x_\ell}^{x} \frac{e^{i\gamma t}g(t)}{-2i\gamma}\,dt$$

for some fixed $x_\ell \in e_\ell$. Integrating by parts gives

$$h_{\gamma,\ell}(x) = g(x) - g(x_\ell) + \frac{1}{2}e^{i\gamma x}\int_{x_\ell}^{x} e^{-i\gamma t}g'(t)\,dt - \frac{1}{2}e^{-i\gamma x}\int_{x_\ell}^{x} e^{i\gamma t}g'(t)\,dt.$$

Here $g(x)$ is bounded since $g'(x) \in L^1(\Gamma)$, and for the same reason the last two terms are also bounded, independent of γ and ℓ. Hence the $h_{\gamma,\ell}(x)$ are uniformly bounded. Let M be such that $|h_{\gamma,\ell}(x)| \le M$, for all ℓ, γ, and x.

Note that $C = C_f$ is bounded, independent of γ: indeed

$$|C_f| = \Big|\int_\Gamma f(x)\,dx\Big| \le \|f\|_1 \le c(\Gamma)\|f\|_2 = c(\Gamma).$$

Now we will use the fact that $\|f\|_2 = 1$ to bound the A_ℓ and B_ℓ. Write $\langle f, f\rangle_\ell = \int_{e_\ell} f(x)\overline{f(x)}\,dx$. Then

$$\begin{aligned}(15.3)\qquad 1 = \langle f, f\rangle &= \sum_{\ell=1}^{m}\langle f, f\rangle_\ell \\ &= \sum_{\ell=1}^{m}\Big(|A_\ell|^2L_\ell + |B_\ell|^2L_\ell + |C|^2\langle h_{\gamma,\ell}, h_{\gamma,\ell}\rangle_\ell \\ &\quad + 2\,\mathrm{Re}\Big(A_\ell\overline{B_\ell}\frac{e^{2i\gamma x}-1}{2i\gamma}\Big) + 2\,\mathrm{Re}(A_\ell\overline{C}\langle e^{i\gamma x}, h_{\gamma,\ell}(x)\rangle_\ell) \\ &\quad + 2\,\mathrm{Re}(B_\ell\overline{C}\langle e^{-i\gamma x}, h_{\gamma,\ell}(x)\rangle_\ell)\Big)\ .\end{aligned}$$

The fourth term is $O(|A_\ell B_\ell|/\gamma)$, and by the Cauchy–Schwarz inequality, for sufficiently large γ it is bounded by $\frac{1}{3}(|A_\ell|^2L_\ell + |B_\ell|^2L_\ell)$. The fifth term is bounded by $2|A_\ell|\cdot c(\Gamma)ML_\ell$, and the sixth term is bounded by $2|B_\ell|\cdot c(\Gamma)ML_\ell$.

Suppose that for some ℓ we have $\max(|A_\ell|, |B_\ell|) \geq 12c(\Gamma)M$. Then

$$\begin{aligned}\frac{1}{3}(|A_\ell|^2L_\ell + |B_\ell|^2L_\ell) &\geq 4\max(|A_\ell|, |B_\ell|)\cdot c(\Gamma)ML_\ell \\ &\geq |2\,\mathrm{Re}(A_\ell\overline{C}\langle e^{i\gamma x}, h_{\gamma,\ell}(x)\rangle_\ell)| + |2\,\mathrm{Re}(B_\ell\overline{C}\langle e^{-i\gamma x}, h_{\gamma,\ell}(x)\rangle_\ell)|.\end{aligned}$$

If γ is large enough that

$$2\,\mathrm{Re}\Big(A_\ell\overline{B_\ell}\frac{e^{2i\gamma x}-1}{2i\gamma}\Big) \leq \frac{1}{3}(|A_\ell|^2L_\ell + |B_\ell|^2L_\ell),$$

then (15.3) shows that $\frac{1}{3}(|A_\ell|^2L_\ell + |B_\ell|^2L_\ell) \leq 1$, from which it follows that $\max(|A_\ell|, |B_\ell|) \leq \sqrt{3/L_\ell}$. Thus for all sufficiently large γ and all ℓ,

$$|A_\ell|, |B_\ell| \leq \max(12c(\Gamma)M, \sqrt{3/L_\ell}).$$

It follows that $\|f_n\|_\infty$ is uniformly bounded for all sufficiently large n, say $n > n_0$. Trivially $\|f_n\|_\infty < \infty$ for each $n \leq n_0$, and the proposition follows. ∎

16 Examples

In this section, we will explain how the theory developed above can be used to compute eigenvalues and eigenfunctions in certain cases, and we will illustrate this with several examples. When $\mu = dx$, the method we are about to describe is well known (see [Be1, N1]).

Throughout this section, we suppose that μ has the same form as in Proposition 15.1:

$$\mu = g(x)dx + \sum_{i=1}^{N} c_i\delta_{p_i}(x),$$

where $g(x)$ is piecewise continuous and belongs to $L^1(\Gamma)$. Let $X_g \subset \Gamma$ be a finite set containing a vertex set for Γ, all points where $g(x)$ is not continuous, and all points where μ has a nonzero point mass. Then $\Gamma \backslash X_g$ is a finite union of open segments. Let their closures in Γ be denoted e_ℓ, for $\ell = 1, \ldots, m$. Without loss of generality, assume e_ℓ is isometrically parametrized by $[a_\ell, b_\ell]$, and identify it with that interval. For each $p \in X_g$, let $v(p)$ be the valence of Γ at p, that is, the number of segments e_ℓ with p as an endpoint.

If f is an eigenfunction of Δ in $\mathrm{BDV}_\mu(\Gamma)$, then by Proposition 15.1, f belongs to $\mathrm{Zh}(\Gamma)$ and satisfies the differential equation

$$\frac{d^2 f}{dx^2} + \lambda f(x) = \lambda C g(x) \tag{16.1}$$

on each e_ℓ. Necessarily $\lambda > 0$; write $\lambda = \gamma^2$. Suppose that for each e_ℓ we are able to find a particular solution $h_{\gamma,\ell}(x)$ to $d^2 f/dx^2 + \lambda f(x) = \lambda g(x)$. Then the general solution to (16.1) has the form $f_\ell(x) = A_\ell e^{i\gamma x} + B_\ell e^{-i\gamma x} + C h_{\gamma,\ell}(x)$ for some constants A_ℓ, B_ℓ.

Proposition 9.1 shows that for a collection of solutions $\{f_\ell(x)\}_{1 \le \ell \le m}$ to be the restriction of an eigenfunction $f(x)$, it is necessary and sufficient that

$$\{A_1, \ldots, A_m, B_1, \ldots, B_m, C\}$$

satisfy the following system of linear equations:

(i) For each $p \in X_g$, there are $v(p) - 1$ "continuity" conditions, each of the form $f_i(q_i) = f_j(q_j)$, where e_i is a fixed edge containing p, e_j runs over the remaining edges containing p, and q_i (or similarly q_j) is the endpoint a_i or b_i of $e_i \cong [a_i, b_i]$ corresponding to p.

(ii) For each $p \in X_g$, there is one "derivative" condition, corresponding to the fact that $\Delta_{Zh}(f)$ and $-\lambda C \mu$ must have the same point mass at p. This condition reads $\sum_i \pm f_i'(q_i) = \lambda C \cdot c_p$, where the sum runs over the edges e_i containing p, q_i is the endpoint of e_i corresponding to p, the sign $\pm$ is + if $q_i = a_i$ and $-$ if $q_i = b_i$, and $c_p = \mu(\{p\})$.

(iii) There is a global "integral" condition coming from the fact that f belongs to $\mathrm{BDV}_\mu(\Gamma)$: $\int_\Gamma f(x)\, d\mu(x) = 0$.

Each condition gives a homogeneous linear equation in the A_ℓ, B_ℓ and C, whose coefficients depend on γ. There are $1 + \sum_{p \in X_g} v(p)$ such equations; since each edge has two endpoints, $\sum_{p \in X_g} v(p) = 2m$. Hence the number of equations is the same as the number of variables, $2m + 1$. For an eigenfunction to exist, it is necessary and sufficient that this system of equations should have a nontrivial solution.

Let $M(\gamma)$ be the $(2m+1) \times (2m+1)$ matrix of coefficients. It follows that $\lambda = \gamma^2$ is an eigenvalue of Δ in $\mathrm{BDV}_\mu(\Gamma)$ if and only if $\gamma > 0$ satisfies the characteristic equation

$$\det(M(\gamma)) = 0. \tag{16.2}$$

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In all examples known to the authors, the multiplicity of γ as a root coincides with the dimension of the nullspace of $M(\gamma)$, which is the multiplicity of the corresponding eigenvalue.[1]

Example 16.1 Let Γ be the segment $[0,1]$, and take $\mu = dx$.

In this case, the eigenvalues are $\lambda_n = n^2\pi^2$ for $n = 1, 2, 3, \dots$ and the eigenfunctions (normalized to have L^2-norm 1) are $f_n(x) = \sqrt{2}\cos(n\pi x)$. One can show that

$$g_\mu(x,y) = \begin{cases} \frac{1}{2}x^2 + \frac{1}{2}(1-y)^2 - \frac{1}{6} & \text{if } x < y \\ \frac{1}{2}(1-x)^2 + \frac{1}{2}y^2 - \frac{1}{6} & \text{if } x \geq y \end{cases}$$

and the eigenfunction expansion in Proposition 13.1 reads

$$g_\mu(x,y) = 2\sum_{n=1}^{\infty} \frac{\cos(n\pi x)\cos(n\pi y)}{n^2\pi^2}.$$

We have

$$\sum \frac{1}{\lambda_n} = \sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2} = \int_0^1 g_\mu(x,x)\,dx = \frac{1}{6}.$$

Example 16.2 Let $\Gamma = [0,1]$, and take $\mu = \delta_0(x)$.

In this case, the eigenvalues are $\lambda_n = \frac{n^2\pi^2}{4}$ for $n = 1, 3, 5, \dots$ and the corresponding normalized eigenfunctions are $f_n(x) = \sqrt{2}\sin\left(\frac{n\pi}{2}x\right)$.

One can show that $g_\mu(x,y) = \min(x,y)$, and its eigenfunction expansion is

$$g_\mu(x,y) = 8\sum_{\substack{n\geq 1 \\ n \text{ odd}}} \frac{\sin(n\pi x)\sin(n\pi y)}{n^2\pi^2}.$$

The trace of φ_μ is

$$\sum \frac{1}{\lambda_n} = \sum_{\substack{n\geq 1 \\ n \text{ odd}}} \frac{4}{\pi^2 n^2} = \int_0^1 g_\mu(x,x)\,dx = \frac{1}{2}.$$

Note that the upper bound (14.3) for the trace need not hold since $\mu \neq \mu_{\text{can}}$.

Example 16.3 Let $\Gamma = [0,1]$, and take $\mu = \frac{1}{2}\delta_0(x) + \frac{1}{2}\delta_1(x)$. This is the "canonical measure" for Γ, as defined in §14.

By a calculation similar to that in Examples 16.1 and 16.2, we see that the eigenvalues are $\lambda_n = n^2\pi^2$ for $n = 1, 3, 5, \dots$ and the eigenspace corresponding to each eigenvalue is two-dimensional; a normalized basis is given by

$$f_n^+(x) = e^{in\pi x}, \qquad f_n^-(x) = e^{-in\pi x}.$$

[1] It might be possible to explain this observation using the results of [Be1], which we learned about from the referee.

One can show that $g_\mu(x,y) = \frac{1}{4} - \frac{1}{2}|x-y|$ and its eigenfunction expansion is

$$g_\mu(x,y) = \sum_{n \text{ odd}} \frac{e^{in\pi x}\overline{e^{in\pi y}}}{n^2\pi^2}.$$

The tau constant $\tau(\Gamma)$ is

$$\sum \frac{1}{\lambda_n} = \sum_{\substack{n\geq 1 \\ n \text{ odd}}} \frac{2}{\pi^2 n^2} = \int_0^1 g_\mu(x,x)\,dx = \frac{1}{4}.$$

Here Γ is a tree, and the upper bound for $\tau(\Gamma)$ in (14.3) is achieved.

Example 16.4 Let $\Gamma = [0,1]$, and take $\mu = \delta_0(x) + \delta_1(x) - dx$.

This example illustrates phenomena which occur with more complicated measures. We have $g(x) = -1$. It will be convenient to write the general solution to (16.1) as $f(x) = A\cos(\gamma x) + B\sin(\gamma x) - C$. The derivative conditions say

$$f'(0) = B\gamma = \gamma^2 C,$$
$$f'(1) = -A\gamma\sin(\gamma) + B\gamma\cos(\gamma) = -\gamma^2 C,$$

and the integral condition says

$$A\Big(1+\cos(\gamma)+\frac{\sin(\gamma)}{\gamma}\Big) + B\Big(\sin(\gamma)+\frac{\cos(\gamma)-1}{\gamma}\Big) - C = 0.$$

One has $\det(M(\gamma)) = 2\gamma^3(1+\cos(\gamma)) - 3\gamma^2\sin(\gamma)$.

It is easy to show that for each odd integer $2k-1 > 0$, there are two solutions to $M(\gamma) = 0$ near $(2k-1)\pi$: one of these, which we will denote by γ_{2k-1}, is slightly less than $(2k-1)\pi$; the other is $\gamma_{2k} = (2k-1)\pi$. The first six eigenvalues $\lambda_n = \gamma_n^2$ are

$$\lambda_1 \cong 2.854280792, \quad \lambda_2 = \pi^2 \cong 9.869604401, \quad \lambda_3 \cong 82.77313456,$$
$$\lambda_4 = 9\pi^2 \cong 88.82643963, \quad \lambda_5 \cong 240.7215434, \quad \lambda_6 = 25\pi^2 \cong 246.7401101.$$

The normalized eigenfunctions corresponding to the eigenvalues $\lambda_{2k-1} = \gamma^2$ are

$$f_{2k-1}(x) = c_{2k-1}\Big(\cos\Big(\gamma\Big(x-\frac{1}{2}\Big)\Big) - \frac{1}{\gamma}\sin\Big(\frac{\gamma}{2}\Big)\Big)$$

for an appropriate constant c_{2k-1}, and the normalized eigenfunctions corresponding to the eigenvalues $\lambda_{2k} = (2k-1)^2\pi^2$ are $f_{2k}(x) = \sqrt{2}\cos((2k-1)\pi x)$.

One has

$$g_\mu(x,y) = \frac{7}{12} - \frac{1}{2}|x-y| - \frac{1}{2}(x-y)^2 - \Big(x-\frac{1}{2}\Big)\Big(y-\frac{1}{2}\Big),$$

and the trace of φ_μ is

$$\sum \frac{1}{\lambda_n} = \int_0^1 g_\mu(x,x)\,dx = \frac{1}{2}.$$

Example 16.5 Let Γ be a circle of length L, and take $\mu = \frac{1}{L}dx$ to be the canonical measure on Γ.

This is the setting of classical Fourier analysis, and our formalism leads to the familiar Fourier expansion of a periodic function.

In this case, the eigenvalues are $\lambda_n = 4n^2\pi^2/L^2$ for $n = 1, 2, 3, \dots$; each eigenspace is two-dimensional, with normalized basis

$$f_n^+(x) = \frac{1}{\sqrt{L}}e^{2\pi inx/L}, \qquad f_n^-(x) = \frac{1}{\sqrt{L}}e^{-2\pi inx/L}.$$

One has $g_\mu(x, y) = \frac{1}{2L}|x - y|^2 - \frac{1}{2}|x - y| + \frac{L}{12}$ and its eigenfunction expansion is

$$g_\mu(x, y) = \sum_{n\neq 0} \frac{e^{2\pi inx/L}\overline{e^{2\pi iny/L}}}{4n^2\pi^2/L}.$$

We have

$$\sum \frac{1}{\lambda_n} = \sum_{n\neq 0} \frac{L^2}{4\pi^2 n^2} = \int_0^L g_\mu(x, x)\, dx = \frac{L^2}{12},$$

and $\tau(\Gamma) = \frac{L}{12}$.

During summer 2003, the authors led an REU on metrized graphs at the University of Georgia. The student participants were Maxim Arap, Jake Boggan, Rommel Cortez, Crystal Gordan, Kevin Mills, Kinsey Rowe, and Phil Zeyliger. Among other things, the students wrote a MAPLE package implementing the above algorithm, and used it to investigate numerical examples, some of which are given in the following table. In this table, each graph has total length 1, with edges of equal length. The graph's name is followed by its tau constant, and then by the two smallest eigenvalues for the measures $\mu = dx$ and $\mu = \mu_{\mathrm{can}}$. The multiplicity of each eigenvalue is given in parentheses.

Name	$\tau(\Gamma)$	$\lambda_{1,dx}$	$\lambda_{2,dx}$	$\lambda_{1,\mathrm{can}}$	$\lambda_{2,\mathrm{can}}$
$K_{3,3}$	.0442	199.86(4)	799.44(5)	105.63(1)	199.86(4)
K_5	.0460	332.51(4)	986.96(5)	47.62(1)	332.51(4)
Petersen	.0353	340.93(5)	1190.79(4)	107.14(1)	340.93(5)
Tetrahedron	.0521	131.42(3)	355.31(2)	102.75(1)	131.42(3)
Cube	.0396	218.20(3)	525.67(3)	106.66(1)	218.20(3)
Octahedron	.0434	355.31(3)	631.65(2)	47.73(1)	355.31(3)
Dodecahedron	.0264	479.25(3)	1363.73(5)	107.78(1)	479.25(3)
Icosahedron	.0399	1103.20(3)	2826.48(5)	33.31(1)	1103.20(3)

Here $K_{3,3}$ denotes the complete bipartite graph on 6 vertices, K_5 denotes the complete graph on 5 vertices, and "Petersen" denotes the well-known Petersen graph, which is a 3-regular graph on 10 vertices. The other graph names are hopefully self-explanatory. For more data, together with some of the MAPLE routines used to calculate them, see www.math.uga.edu/~mbaker/REU/REU.html.

In all examples known to us, $\lambda_{1,\mathrm{can}}$ has multiplicity 1. For the graphs in the table, $\lambda_{1,dx}$ coincides with $\lambda_{2,\mathrm{can}}$. Factoring the characteristic equation for the corresponding eigenvalue problems shows that in fact for these graphs, all eigenvalues for dx are eigenvalues for μ_{can}, with multiplicities differing by at most 1 (μ_{can} has infinitely many other eigenvalues as well). The graphs in the table are highly symmetric: all their edges have equal weight under the canonical measure, and all their vertices have the same valence. For more general graphs, it seems that some but not all eigenvalues for dx are eigenvalues for μ_{can}. It would be interesting to understand the reasons for this. It would also be very interesting to understand the geometric/combinatorial significance of the the eigenvalues of the canonical measure.

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