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ESTIMATES OF THE SECOND DERIVATIVE OF BOUNDED ANALYTIC FUNCTIONS

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Abstract

Assume a point z lies in the open unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$ and f is an analytic self-map of $\mathbb{D}$ fixing 0. Then Schwarz's lemma gives $|f(z)| \le |z|$, and Dieudonné's lemma asserts that $|f'(z)| \le \min\{1, (1+|z|^2)/(4|z|(1-|z|^2))\}$. We prove a sharp upper bound for $|f''(z)|$ depending only on $|z|$.

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1. Introduction

Let $\mathbb{D}$ be the open unit disk $\{z : |z| < 1\}$ in the complex plane $\mathbb{C}$. The set of all analytic self-maps of $\mathbb{D}$ is denoted by $\mathcal{H}$, and its subspace $\mathcal{H}_0$ consists of those $f \in \mathcal{H}$ such that $f(0) = 0$. We recall some classical growth estimates for the functions in these spaces.

Schwarz's lemma asserts that $|f(z)| \le |z|$ for all $f \in \mathcal{H}_0$ and $z \in \mathbb{D}$. Rogosinski [8] gave the generalisation: if $f \in \mathcal{H}_0$ and $f'(0)$ is fixed, then for $z \in \mathbb{D} \setminus \{0\}$, the region of values of $f(z)$ is the closed disk $\{\zeta \in \mathbb{C} : |\zeta - c| \le r\}$, where

$$c = \frac{zf'(0)(1-z^2)}{1-|z|^2|f'(0)|^2}, \quad r = |z|^2 \frac{1-|f'(0)|^2}{1-|z|^2|f'(0)|^2}.$$

Rivard [7] gave another version of Rogosinski's lemma for $f \in \mathcal{H}$, called the Rogosinski–Pick lemma. The Schwarz–Pick lemma states that

$$|f'(z)| \le \frac{1-|f(z)|^2}{1-|z|^2}, \quad f \in \mathcal{H},\ z \in \mathbb{D},$$

and equality holds for some $z \in \mathbb{D}$ if and only if f is an automorphism of $\mathbb{D}$. The Schwarz–Pick lemma has a higher-order version. Ruscheweyh [9] proved that, for $f \in \mathcal{H}$ and $n \in \mathbb{N}$,

$$|f^{(n)}(z)| \le \frac{n!(1-|f(z)|^2)}{(1-|z|)^n(1+|z|)}, \quad z \in \mathbb{D}, \tag{1.1}$$

and the inequality is sharp (see also [1] and [5]).

The Schwarz–Pick lemma implies the sharp inequality $|f'(z_0)| \leq 1/(1-|z_0|^2)$ for $f \in \mathcal{H}$ and $z_0 \in \mathbb{D}$. This inequality has upper bound depending only on $|z_0|$, and equality occurs only for $f(z) = e^{i\theta}(z-z_0)/(1-\bar{z}_0 z)$, $\theta \in \mathbb{R}$. Szász [10] extended this inequality to odd-order derivatives of $f \in \mathcal{H}$ and also obtained the following sharp upper bound for $|f''|$ (see also [1]):

$$|f''(z_0)| \leq \frac{(8+|z_0|^2)^2}{32(1-|z_0|^2)^2}, \quad f \in \mathcal{H},\ z_0 \in \mathbb{D}.$$

Equality occurs only for

$$f(z) = e^{i\theta}\frac{u^2 + \frac{1}{2}z_0 u - \frac{1}{8}z_0^2}{1 + \frac{1}{2}\bar{z}_0 u - \frac{1}{8}\bar{z}_0^2 u^2}, \quad u = \frac{z-z_0}{1-\bar{z}_0 z},\ z \in \mathbb{D},\ \theta \in \mathbb{R}.$$

Dieudonné [3] proved the following estimate for the derivative of $f \in \mathcal{H}_0$ depending only on $|z|$:

$$|f'(z)| \leq \begin{cases} 1 & \text{if } |z| \leq \sqrt{2}-1, \quad (1.2) \\ \dfrac{(1+|z|^2)^2}{4|z|(1-|z|^2)} & \text{if } |z| > \sqrt{2}-1. \quad (1.3) \end{cases}$$

Equality holds in (1.2) for some z_0 with $r = |z_0|$ if and only if $f(z) = e^{i\theta}z$ for some real constant θ. Equality holds in (1.3) for some z_0 with $r = |z_0|$ if and only if

$$f(z) = e^{i\theta} z \frac{z-a}{1-\bar{a}z},$$

where $a = (3r^2-1)z_0/(r^2(1+r^2))$ and $\theta \in \mathbb{R}$ is arbitrary. This result is known as Dieudonné's lemma and can be seen as Schwarz's lemma for f'. Another version of Dieudonné's lemma for $f \in \mathcal{H}$ called the Dieudonné–Pick lemma was proved by Kaptanoğlu [4] (see also [2, 7]).

Our main result gives a sharp upper bound for the modulus of the second derivative of $f \in \mathcal{H}_0$ and improves the upper bound given by Szasz [10].

Theorem 1.1. *If $f \in \mathcal{H}_0$, then*

$$|f''(z)| \leq \begin{cases} \dfrac{4}{1-9|z|^2+(1+3|z|^2)^{3/2}} & |z| \leq \dfrac{1+\sqrt{3}}{4}, \quad (1.4) \\ \dfrac{(1+8|z|^2)^2}{32|z|^3(1-|z|^2)^2} & |z| > \dfrac{1+\sqrt{3}}{4}. \quad (1.5) \end{cases}$$

Equality holds in (1.4) for some z_0 with $r = |z_0| \leq (1+\sqrt{3})/4$ if and only if

$$f(z) = e^{i\theta} z \frac{z-a}{1-\bar{a}z},$$

where

$$a = \frac{3}{1+\sqrt{1+3r^2}} z_0, \quad \theta \in \mathbb{R}.$$

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Equality holds in (1.5) for some z_0 *with* $r = |z_0| > (1+\sqrt{3})/4$ *if and only if*

$$f(z) = e^{i\theta} z \frac{z-a_1}{1-\overline{a}_1 z} \cdot \frac{z-a_2}{1-\overline{a}_2 z},$$

where

$$a_1 = \Big(\frac{2r^2-1}{r^2} + \frac{2(1-r^2)}{\sqrt{3}r^2}\Big) z_0, \quad a_2 = \Big(\frac{2r^2-1}{r^2} - \frac{2(1-r^2)}{\sqrt{3}r^2}\Big) z_0, \quad \theta \in \mathbb{R}.$$

Remark 1.2. As $r \downarrow (1+\sqrt{3})/4$,

$$a_1 \to 6(3\sqrt{3}-5)z_0 = a_0 \quad \text{and} \quad |a_2| \to 1.$$

When $r = (1+\sqrt{3})/4$, we obtain $a = 6(3\sqrt{3}-5)z_0 = a_0$. Thus

$$f(z) = e^{i\theta} z \frac{z-a_1}{1-\overline{a}_1 z} \frac{z-a_2}{1-\overline{a}_2 z} \to e^{i\gamma} z \frac{z-a_0}{1-\overline{a}_0 z}$$

for some $\gamma \in \mathbb{R}$ as $r \downarrow (1+\sqrt{3})/4$.

Remark 1.3. The upper bound of $|f''(z_0)|$ is continuous but not real analytic. Note that $4/(1-9r^2+(1+3r^2)^{3/2})$ is increasing with respect to r on $[0,(1+\sqrt{3})/4]$ and $(1+8r^2)^2/(32r^3(1-r^2)^2)$ is increasing with respect to r on $((1+\sqrt{3})/4, 1)$.

The remainder of this paper is organised as follows. In Section 2, we present some auxiliary results on the space $\mathcal{H}_0$. Section 3 consists of the proof of Theorem 1.

2. Auxiliary results on the space $\mathcal{H}_0$

In this section, we state and prove some auxiliary results related to the space $\mathcal{H}_0$. These results are needed for the proof of Theorem 1.1. Before that we fix some notation. For $c \in \mathbb{C}$ and $\rho > 0$, the discs $\mathbb{D}(c,\rho)$ and $\overline{\mathbb{D}}(c,\rho)$ are defined by

$$\mathbb{D}(c,\rho) := \{\zeta \in \mathbb{C} : |\zeta - c| < \rho\}$$

and

$$\overline{\mathbb{D}}(c,\rho) := \{\zeta \in \mathbb{C} : |\zeta - c| \le \rho\}.$$

For $n \in \mathbb{N}$, $\{z_j\}_{j=1}^n \subset \mathbb{D}$ and a point $\theta \in \mathbb{R}$, a Blaschke product of degree n with zeros $\{z_j\}$ takes the form

$$B(z) = e^{i\theta} \prod_{j=1}^{n} \frac{z-z_j}{1-\overline{z_j} z}, \quad z \in \mathbb{D}.$$

For $f \in \mathcal{H}$, Peschl [6] defined the so-called Peschl's invariant derivatives $D_n f(z)$ by the Taylor series expansion

$$\frac{f((z+z_0)/(1+\overline{z}_0 z)) - f(z_0)}{1-\overline{f(z_0)} f((z+z_0)/(1+\overline{z}_0 z))} = \sum_{n=1}^{\infty} \frac{D_n f(z_0)}{n!} z^n, \quad z, z_0 \in \mathbb{D}.$$

For example, precise forms of $D_n f(z)$, $n = 1, 2$, are given by

$$D_1 f(z) = \frac{(1-|z|^2)f'(z)}{1-|f(z)|^2},$$
$$D_2 f(z) = \frac{(1-|z|^2)^2}{1-|f(z)|^2}\Big[f''(z) - \frac{2\bar{z}f'(z)}{1-|z|^2} + \frac{2\overline{f(z)}f'(z)^2}{1-|f(z)|^2}\Big].$$

In addition, we write

$$T_a(z) = \frac{z+a}{1+\bar{a}z}, \quad z, a \in \mathbb{D},$$

and define

$$\Delta(z_0, w_0) = \overline{\mathbb{D}}\Big(\frac{w_0}{z_0}, \frac{|z_0|^2 - |w_0|^2}{|z_0|(1-|z_0|^2)}\Big).$$

With these preparations we are ready to state a classical theorem of Dieudonné [3] which gives a description of the region of values of $f'(z_0)$.

Lemma 2.1 [3]. *Suppose that z_0 and w_0 are points in $\mathbb{D}$ with $|w_0| < |z_0|$. If $f \in \mathcal{H}_0$ satisfies $f(z_0) = w_0$, then the region of values of $f'(z_0)$ is the closed disc $\Delta(z_0, w_0)$.*

Further, $f'(z_0) \in \partial\Delta(z_0, w_0)$ if and only if $f(z) = z\, T_{u_0}(e^{i\theta}T_{-z_0}(z))$, where $u_0 = w_0/z_0$ and $\theta \in \mathbb{R}$.

Cho *et al.* [2] gave a similar result to Lemma 2.1 for the second derivative (see also [7]). We refine their original version in an appropriate way. We also characterise f when $|f''(z_0) - c| = \rho$, where z_0, c, and ρ are as in Lemma 2.2. This result may look technical but it is needed for the argument of Theorem 1.1. Before the statement of Lemma 2.2, we define c and ρ by

$$\begin{cases} c = c(z_0, w_0, w_1) = \dfrac{2(r^2-s^2)\beta(1-\overline{w_0}\beta)}{z_0^2(1-r^2)^2}, \\ \rho = \rho(z_0, w_0, w_1) = \dfrac{2(r^2-s^2)(1-|\beta|^2)}{r(1-r^2)^2}. \end{cases}$$

Lemma 2.2 [2]. *Suppose that z_0 and w_0 are points in $\mathbb{D}$ with $|w_0| = s < r = |z_0|$, $w_1 \in \Delta(z_0, w_0)$, and that $f \in \mathcal{H}_0$ satisfies $f(z_0) = w_0$ and $f'(z_0) = w_1$. Let β be given by*

$$w_1 = \frac{w_0}{z_0} + \frac{r^2-s^2}{z_0(1-r^2)}\beta, \quad \text{with } |\beta| \le 1.$$

Set $u_0 = w_0/z_0$ and $v_0 = \bar{z}_0^2\beta/|z_0|^2$.

(1) *If $|\beta| = 1$, then $f''(z_0) = c$ and $f(z) = zT_{u_0}(e^{i\theta}T_{-z_0}(z))$, where $\theta = \arg(\bar{z}_0^2\beta)$.*

(2) *If $|\beta| < 1$, then the region of values of $f''(z_0)$ is the closed disc $\overline{\mathbb{D}}(c, \rho)$. Further, $f''(z_0) \in \partial\mathbb{D}(c, \rho)$ if and only if $f(z) = zT_{u_0}(T_{-z_0}(z)T_{v_0}(e^{i\theta}T_{-z_0}(z)))$, for some $\theta \in \mathbb{R}$. When $\beta \neq 0$, $f''(z_0) \in \partial\mathbb{D}(c, \rho)$ and $\arg f''(z_0) = \arg c$ if and only if $f(z) = zT_{u_0}(T_{-z_0}(z)T_{v_0}(e^{i\theta}T_{-z_0}(z)))$, where $\theta = \arg(\bar{z}_0^3\beta(1-\overline{w_0}\beta))$.*

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Proof. Although the proof of the assertion that $f''(z_0) \in \overline{\mathbb{D}}(c, \rho)$ can be found in [2, Theorem 3.7] and [7, Corollary 4.2], we re-prove it here to present a full discussion for the equality conditions and to show that $\overline{\mathbb{D}}(c, \rho)$ is covered, which is not explicitly given in [2, 7]. Let $g(z) = f(z)/z$, so that $g \in \mathcal{H}$. From [11, Theorem 2],

$$|D_2 g(z_0)| \le 2(1 - |D_1 g(z_0)|),$$

which is equivalent to

$$|f''(z_0) - c| \le \rho. \tag{2.1}$$

Here equality holds for some point z_0 if and only if $f(z) = zg(z)$, where g is a Blaschke product of degree 1 or 2 satisfying $g(z_0) = u_0$ and $g'(z_0) = (z_0 w_1 - w_0)/z_0^2$.

(1) If $|\beta| = 1$, then $f''(z_0) = c$ and $f(z) = zg(z)$, where g is an automorphism of $\mathbb{D}$ satisfying $g(z_0) = u_0$ and $g'(z_0) = (z_0 w_1 - w_0)/z_0^2$. Applying this fact, we determine the explicit form of g. Set

$$h(z) = T_{-u_0} \circ g \circ T_{z_0}(z), \quad z \in \mathbb{D}.$$

It is obvious that h is an automorphism of $\mathbb{D}$ depending on g and satisfying

$$h(0) = 0 \quad \text{and} \quad h'(0) = \frac{\overline{z}_0^2}{|z_0|^2}\beta,$$

which means that $h(z) = e^{i\theta} z$ for $z \in \mathbb{D}$ and $\theta = \arg(\overline{z}_0^2 \beta)$. Now it is easy to check that

$$g(z) = T_{u_0} \circ h \circ T_{-z_0}(z) = T_{u_0}(e^{i\theta} T_{-z_0}(z)) = e^{i\gamma}\,\frac{z - a}{1 - \overline{a}z},$$

where

$$\gamma = \arg(\overline{z}_0^2 \beta (1 - w_0\overline{\beta})^2) \quad \text{and} \quad a = \frac{|z_0|^2 - w_0\overline{\beta}}{\overline{z}_0(1 - w_0\overline{\beta})}.$$

This completes the proof of (1).

(2) Inequality (2.1) means that $f''(z_0)$ lies in $\overline{\mathbb{D}}(c, \rho)$. To show that $\overline{\mathbb{D}}(c, \rho)$ is covered, let $\alpha \in \overline{\mathbb{D}}$, $u_0 = w_0/z_0$ and $v_0 = \overline{z}_0^2 \beta/|z_0|^2$ and set $f(z) = zg(z)$, where

$$g(z) = T_{u_0}(T_{-z_0}(z) T_{v_0}(\alpha T_{-z_0}(z))).$$

Then $f(0) = 0$ and $f(z_0) = w_0$. Next we show that $f'(z_0) = w_1$. A calculation shows that $f'(z_0) = g(z_0) + z_0 g'(z_0)$. Note that

$$T_{-u_0} \circ g(z) = T_{-z_0}(z) T_{v_0}(\alpha T_{-z_0}(z)).$$

Differentiating both sides,

$$(T_{-u_0})'(g(z))g'(z) = T'_{-z_0}(z) T_{v_0}(\alpha T_{-z_0}(z)) + T_{-z_0}(z) T'_{v_0}(\alpha T_{-z_0}(z))\alpha T'_{-z_0}(z) \tag{2.2}$$

for all $z \in \mathbb{D}$. Substituting $z = z_0$ into this equation,

$$(T_{-u_0})'(g(z_0))g'(z_0) = T'_{-z_0}(z_0) T_{v_0}(0),$$

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which gives

$$g'(z_0) = \frac{(r^2 - s^2)\bar{z}_0^2\beta}{(1 - r^2)r^4}.$$

Consequently, f also satisfies

$$f'(z_0) = \frac{w_0}{z_0} + \frac{|z_0|^2 - |w_0|^2}{z_0(1 - |z_0|^2)}\beta = w_1.$$

Next we find the form of $f''(z_0)$. By a straightforward computation,

$$f''(z_0) = 2g'(z_0) + z_0 g''(z_0). \tag{2.3}$$

Differentiating both sides of (2.2),

$$\begin{aligned}&(T_{-u_0})''(g(z))(g'(z))^2 + (T_{-u_0})'(g(z))g''(z)\\ &\quad = T''_{-z_0}(z)T_{v_0}(\alpha T_{-z_0}(z)) + 2T'_{-z_0}(z)T'_{v_0}(\alpha T_{-z_0}(z))\alpha T'_{-z_0}(z)\\ &\qquad + T_{-z_0}(z)\, T''_{v_0}(\alpha T_{-z_0}(z))(\alpha T'_{-z_0}(z))^2 + T_{-z_0}(z)T'_{v_0}(\alpha T_{-z_0}(z))\alpha T''_{-z_0}(z),\end{aligned}$$

for $z \in \mathbb{D}$. Substituting $z = z_0$ into this equation,

$$(T_{-u_0})''(g(z_0))(g'(z_0))^2 + (T_{-u_0})'(g(z_0))g''(z_0) = \frac{2\bar{z}_0^3}{(1 - r^2)^2 r^2}\beta + \frac{2(1 - |\beta|^2)\alpha}{(1 - r^2)^2}.$$

Consequently,

$$g''(z_0) = \frac{2(r^2 - s^2)}{r^2(1 - r^2)^2}\Big(\frac{\bar{z}_0^3\beta}{r^2} + \alpha(1 - |\beta|^2) - \frac{\overline{w}_0 r^2\beta^2}{z_0^3}\Big).$$

Together with (2.3), this gives

$$f''(z_0) = \frac{2(r^2 - s^2)\beta(1 - \overline{w}_0\beta)}{z_0^2(1 - r^2)^2} + \frac{2z_0(r^2 - s^2)(1 - |\beta|^2)}{r^2(1 - r^2)^2}\alpha = c + \rho\frac{z_0\alpha}{r}.$$

Since $\alpha \in \overline{\mathbb{D}}$ is arbitrary, it follows that the closed disc $\overline{\mathbb{D}}(c, \rho)$ is covered.

We know that $f''(z_0) \in \partial\mathbb{D}(c, \rho)$ if and only if $f(z) = zg(z)$, where g is a Blaschke product of degree 2 satisfying $g(z_0) = w_0/z_0$ and $g'(z_0) = (z_0 w_1 - w_0)/z_0^2$. Applying this fact, we determine the precise form of g. Set

$$h(z) = \frac{T_{-u_0} \circ g \circ T_{z_0}(z)}{z}, \quad z \in \mathbb{D}.$$

It is clear that h is an automorphism of $\mathbb{D}$ depending on g and satisfying

$$h(0) = (T_{-u_0} \circ g \circ T_{z_0})'(0) = \frac{(1 - |z_0|^2)g'(z_0)}{1 - |u_0|^2} = v_0.$$

Then $T_{-v_0} \circ h$ is an automorphism of $\mathbb{D}$ fixing 0, which means that $T_{-v_0} \circ h(z) = e^{i\theta}z$ for $z \in \mathbb{D}$ and $\theta \in \mathbb{R}$. Now it is easy to check that

$$g(z) = T_{u_0}(T_{-z_0}(z)T_{v_0}(e^{i\theta}T_{-z_0}(z))), \quad z \in \mathbb{D}.$$

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Conversely, if $f(z) = zT_{u_0}(T_{-z_0}(z)T_{v_0}(e^{i\theta}T_{-z_0}(z)))$, where $\theta \in \mathbb{R}$, then

$$f''(z_0) = c + \rho\frac{z_0}{r}e^{i\theta} \in \partial\mathbb{D}(c, \rho).$$

Next, we prove the last assertion in this case. By basic geometry, we note that $f''(z_0) \in \partial\mathbb{D}(c, \rho)$ and $\arg f''(z_0) = \arg c$ if and only if $f''(z_0) = tc$ for $t = 1 + \rho/|c|$. Hence it will be sufficient to show that $f''(z_0) = tc$ for $t = 1 + \rho/|c|$ if and only if $f(z) = zT_{u_0}(T_{-z_0}(z)T_{v_0}(e^{i\theta}T_{-z_0}(z)))$, where $\theta = \arg(\overline{z}_0^3\beta(1 - \overline{w}_0\beta))$.

If $f''(z_0) = tc$ for $t = 1 + \rho/|c|$, then

$$f(z) = zg(z) = zT_{u_0}(T_{-z_0}(z)T_{v_0}(e^{i\theta}T_{-z_0}(z))), \quad z \in \mathbb{D}.$$

Next we determine the precise value of θ. A calculation shows that

$$f''(z_0) = c + \rho\frac{z_0}{r}e^{i\theta}.$$

Therefore, $f''(z_0) = tc$ implies that

$$e^{i\theta} = \frac{r^3\beta(1 - \overline{w}_0\beta)}{z_0^3|\beta|\,|1 - \overline{w}_0\beta|}.$$

Conversely, if

$$f(z) = zg(z) = zT_{u_0}(T_{-z_0}(z)T_{v_0}(e^{i\theta}T_{-z_0}(z))), \quad e^{i\theta} = \frac{r^3\beta(1 - \overline{w}_0\beta)}{z_0^3|\beta|\,|1 - \overline{w}_0\beta|},$$

then

$$f''(z_0) = c + \rho\frac{z_0}{r}e^{i\theta} = c + \rho\frac{r^2\beta(1 - \overline{w}_0\beta)}{z_0^2|\beta|\,|1 - \overline{w}_0\beta|} = c + \frac{c}{|c|}\rho = tc.$$

Hence (2) is proved. □

Based on Lemma 2.2, we give a sharp upper bound for $|f''(z)|$ depending only on $|z|$ and $|f(z)|$.

LEMMA 2.3. *Suppose that z_0 and w_0 are points in $\mathbb{D}$ with $|w_0| = s < r = |z_0|$. If $f \in \mathcal{H}_0$ satisfies $f(z_0) = w_0$, then*

$$|f''(z_0)| \le \begin{cases} \dfrac{2(1+s)(r^2 - s^2)}{r^2(1 - r^2)^2} & r - s \le \dfrac{1}{2}, \quad (2.4) \\[2ex] \dfrac{(r+s)(4r^2 - 4rs + 1)}{2r^2(1 - r^2)^2} & r - s > \dfrac{1}{2}. \quad (2.5) \end{cases}$$

Equality holds in (2.4) if and only if

$$f(z) = e^{i\theta}z\frac{z - a}{1 - \overline{a}z},$$

where

$$\theta = \arg(-\overline{z}_0^2 w_0), \quad a = \frac{r^2 + s}{r^2(1 + s)}z_0.$$

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If $w_0 = 0$, then $\theta \in \mathbb{R}$ is arbitrary. Equality holds in (2.5) if and only if

$$f(z) = e^{i\theta} z \frac{z - a_1}{1 - \overline{a}_1 z} \cdot \frac{z - a_2}{1 - \overline{a}_2 z},$$

where

$$\begin{aligned} \theta &= \arg(-\overline{z}_0^3 w_0) \quad (\textit{and } \theta \in \mathbb{R} \textit{ is arbitrary when } w_0 = 0), \\ a_1 &= \frac{-1 + 3r^2 - 4rs + (1 - r^2)\sqrt{1 + 16rs}}{2r^2(1 - 2rs)} z_0, \\ a_2 &= \frac{-1 + 3r^2 - 4rs - (1 - r^2)\sqrt{1 + 16rs}}{2r^2(1 - 2rs)} z_0. \end{aligned}$$

Proof. First we suppose that $w_0 \neq 0$. From Lemma 2.1,

$$f'(z_0) = \frac{w_0}{z_0} + \frac{|z_0|^2 - |w_0|^2}{z_0(1 - |z_0|^2)}\beta, \quad |\beta| \le 1.$$

Set $|\beta| = x$. From Lemma 2.2,

$$\begin{aligned} |f''(z_0)| \le |c| + \rho &= \frac{2(r^2 - s^2)}{r^2(1 - r^2)^2}(|\beta|\,|1 - \overline{w}_0\beta| + r(1 - |\beta|^2)) \\ &\le \frac{2(r^2 - s^2)}{r^2(1 - r^2)^2}(|\beta|(1 + s|\beta|) + r(1 - |\beta|^2)) \\ &= \frac{2(r^2 - s^2)\Psi(x)}{r^2(1 - r^2)^2}, \end{aligned}$$

where

$$\Psi(x) = (s - r)x^2 + x + r,$$

and equality holds in the second last inequality if and only if $-\overline{w}_0\beta = s|\beta|$.

Observe that $\Psi(x)$ takes its maximum at $x = 1/(2(r - s))$, which is less than 1 if and only if $r - s > 1/2$. In this case, the sharp upper bound for $|f''(z_0)|$ is

$$\frac{2(r^2 - s^2)\Psi(1/(2(r - s)))}{r^2(1 - r^2)^2} = \frac{(r + s)(4r^2 - 4rs + 1)}{2r^2(1 - r^2)^2}.$$

Moreover, from Lemma 2.2, the sharp upper bound for $|f''(z_0)|$ is obtained if and only if $f(z) = zT_{u_0}(T_{-z_0}(z)T_{v_0}(e^{i\theta}T_{-z_0}(z)))$, where $\theta = \arg(\overline{z}_0^3\beta)$, $u_0 = w_0/z_0$ and $\beta = -w_0/(2s(r - s))$. In other words, equality holds in (2.5) if and only if the form of f is

$$f(z) = e^{i\theta} z \frac{z - a_1}{1 - \overline{a}_1 z} \cdot \frac{z - a_2}{1 - \overline{a}_2 z},$$

where

$$\begin{aligned} \theta &= \arg(-\overline{z}_0^3 w_0), \\ a_1 &= \frac{-1 + 3r^2 - 4rs + (1 - r^2)\sqrt{1 + 16rs}}{2r^2(1 - 2rs)} z_0, \\ a_2 &= \frac{-1 + 3r^2 - 4rs - (1 - r^2)\sqrt{1 + 16rs}}{2r^2(1 - 2rs)} z_0. \end{aligned}$$

If $w_0 = 0$, then $\theta \in \mathbb{R}$ is arbitrary.

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For $r - s \le 1/2$, $\Psi(x) \le \Psi(1) = 1 + s$ in the interval $0 \le x \le 1$, so that

$$|f''(z_0)| \le \frac{2(r^2 - s^2)\Psi(1)}{r^2(1 - r^2)^2} = \frac{2(1 + s)(r^2 - s^2)}{r^2(1 - r^2)^2}.$$

Equality holds in the above inequality if and only if $f(z) = zT_{u_0}(e^{i\theta}T_{-z_0}(z))$, where $u_0 = w_0/z_0$, $\theta = \arg(-\bar{z}_0^2\beta)$ and $|\beta| = 1$. In other words, equality holds in (2.4) if and only if f is a Blaschke product of degree 2 of the form

$$f(z) = e^{i\theta}z\frac{z - a}{1 - \bar{a}z},$$

where

$$\theta = \arg(-\bar{z}_0^2 w_0), \quad a = \frac{r^2 + s}{r^2(1 + s)}z_0.$$

If $w_0 = 0$, then $\theta \in \mathbb{R}$ is arbitrary. □

We close this section by noting that from Ruscheweyh's inequality (1.1), for $f \in \mathcal{H}$,

$$|f''(z_0)| \le \frac{2(1 - |w_0|^2)}{(1 + |z_0|)^2(1 - |z_0|)},$$

where z_0 and w_0 are as in Lemma 2.3. Lemma 2.3 offers a smaller bound for $|f''(z_0)|$ when $f \in \mathcal{H}_0$.

3. Proof of Theorem 1.1

Fix $z_0 \in \mathbb{D}$ and take $f \in \mathcal{H}_0$, $w_0 = f(z_0)$, $s = |w_0|$, $r = |z_0|$. If $r = 0$, then equality in (1.4) holds if and only if

$$f(z) = e^{i\theta}z^2, \quad \theta \in \mathbb{R}.$$

Suppose that $r \ne 0$ and $s < r$. (If $s = r$, then $f(z) = e^{i\theta}z$ and $f''(z) = 0$.) From Lemma 2.3, we consider the two cases for $r - s \le 1/2$ and $r - s > 1/2$.

Case (i). For $r - s \le 1/2$,

$$|f''(z_0)| \le \frac{2(1 + s)(r^2 - s^2)}{r^2(1 - r^2)^2} = \frac{2\varphi(s)}{r^2(1 - r^2)^2},$$

where $\varphi(s) = -s^3 - s^2 + r^2 s + r^2$ and $s < r$. The values of s for which

$$\varphi'(s) = -3s^2 - 2s + r^2 = 0$$

are

$$s_1 = \frac{-1 - \sqrt{1 + 3r^2}}{3}, \quad s_2 = \frac{-1 + \sqrt{1 + 3r^2}}{3}.$$

Note that $s_1 < 0$, while $s_2 < r$ is equivalent to $6r^2 + r > 0$. Thus, $\varphi(s)$ is increasing with respect to s on $[0, s_2)$ and is decreasing on $(s_2, r]$. In this case, if $r - s_2 \le 1/2$, then $r \le (1 + \sqrt{3})/4$, so that

$$|f''(z_0)| \le \frac{2\varphi(s_2)}{r^2(1 - r^2)^2} = \frac{4}{1 - 9r^2 + (1 + 3r^2)^{3/2}}.$$

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In addition, if $r - s_2 > 1/2$, then $r > (1 + \sqrt{3})/4$. Hence $\varphi(s) \leq \varphi(r - 1/2)$ and

$$|f''(z_0)| \leq \frac{2\varphi(r - \frac{1}{2})}{r^2(1 - r^2)^2} = \frac{(2r + 1)(4r - 1)}{4r^2(1 - r^2)}.$$

Case (ii). For $r - s > 1/2$,

$$|f''(z_0)| \leq \frac{(r + s)(4r^2 - 4rs + 1)}{2r^2(1 - r^2)^2} = \frac{\Phi(s)}{2r^2(1 - r^2)^2},$$

where

$$\Phi(s) = -4rs^2 + s + r + 4r^3.$$

But $\Phi(s)$ reaches its maximum at $s = 1/(8r)$, which is less than r if and only if $r > \sqrt{2}/4$. In this case, if $r - 1/(8r) > 1/2$, then $r > (1 + \sqrt{3})/4$, so that the sharp upper bound for $|f''(z_0)|$ is

$$\frac{\Phi(1/(8r))}{2r^2(1 - r^2)^2} = \frac{(8r^2 + 1)^2}{32r^3(1 - r^2)^2}.$$

Moreover, if $1/(8r) \leq r$ but $r - 1/(8r) \leq 1/2$, then $1/2 < r \leq (1 + \sqrt{3})/4$. Hence $\Phi(s) < \Phi(r - 1/2)$ and

$$|f''(z_0)| < \frac{(r + s)\Phi(r - \frac{1}{2})}{2r^2(1 - r^2)^2} = \frac{(2r + 1)(4r - 1)}{4r^2(1 - r^2)^2}.$$

From cases (i) and (ii), noting that

$$\frac{(2r + 1)(4r - 1)}{4r^2(1 - r^2)} < \frac{(8r^2 + 1)^2}{32r^3(1 - r^2)^2}, \quad \text{for } r > (1 + \sqrt{3})/4,$$

and

$$\frac{(2r + 1)(4r - 1)}{4r^2(1 - r^2)} < \frac{4}{1 - 9r^2 + (1 + 3r^2)^{3/2}}, \quad \text{for } 1/2 \leq r \leq (1 + \sqrt{3})/4,$$

we see that inequalities (1.4) and (1.5) hold.

From Lemma 2.3, equality holds in (1.4) at a point z_0 with $r = |z_0| \leq (1 + \sqrt{3})/4$ if and only if $f(z) = zT_{u_0}(e^{i\theta}T_{-z_0}(z))$, where $u_0 = w_0/z_0$, $\theta = \arg(-\bar{z}_0^2\beta)$, $\beta = -w_0/s$ and $s = (-1 + \sqrt{1 + 3r^2})/3$. In other words, equality holds in (1.4) at a point z_0 with $r = |z_0| \leq (1 + \sqrt{3})/4$ if and only if f is of the form

$$f(z) = e^{i\theta}z\frac{z - a}{1 - \bar{a}z},$$

where

$$a = \frac{3}{1 + \sqrt{1 + 3r^2}}z_0, \quad \theta \in \mathbb{R}.$$

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For such an f, we compute

$$|f''(z_0)| = \frac{2(1-|a^2|)}{(1-|a|r)^3} = \frac{4}{1-9r^2+(1+3r^2)^{3/2}}.$$

Further, equality holds in (1.5) at a point z_0 with $r = |z_0| > (1+\sqrt{3})/4$ if and only if $f(z) = zT_{u_0}(T_{-z_0}(z)T_{v_0}(e^{i\theta}T_{-z_0}(z)))$, where $u_0 = w_0/z_0$, $v_0 = \bar{z}_0^2\beta/|z_0|^2$, $\theta = \arg(\bar{z}_0^3\beta)$, $\beta = -w_0/(2s(r-s))$ and $s = 1/(8r)$. In other words, equality holds in (1.5) at a point z_0 with $r = |z_0| > (1+\sqrt{3})/4$ if and only if the form of f is

$$f(z) = e^{i\theta} z \frac{z-a_1}{1-\bar{a}_1 z} \cdot \frac{z-a_2}{1-\bar{a}_2 z},$$

where

$$a_1 = \frac{2-\sqrt{3}+2(\sqrt{3}-1)r^2}{\sqrt{3}r^2} z_0, \quad a_2 = \frac{-(2+\sqrt{3})+2(\sqrt{3}+1)r^2}{\sqrt{3}r^2} z_0, \quad \theta \in \mathbb{R}.$$

For such an f, we calculate

$$|f''(z_0)| = \frac{(1+8r^2)^2}{32r^3(1-r^2)^2}.$$

This completes the proof.

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