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A CLASS OF HARMONICALLY CONVERGENT SETS

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1.

Following Craven (1965) we say that a set M of natural numbers is harmonically convergent if

$$\mu(M) = \sum_{n \in M} \frac{1}{n} \tag{1}$$

converges, and we call $\mu(M)$ the harmonic sum of M. (Craven defined these concepts for sequences rather than sets, but we found it convenient to work with sets.) Throughout this paper, lower case italics denote non-negative integers.

Let $r > 1$, $1 \leqq m \leqq r$, and $0 \leqq d_1 < d_2 < \cdots < d_m < r$. We define

$$M(t) = M(r; d_1, d_2, \cdots, d_m; t_1, t_2, \cdots, t_m) \tag{2}$$

to be the set of natural numbers which contains the digit d_i exactly t_i times $(i = 1, 2, \cdots, m)$ when expressed in the scale of r. Further, let

$$M_\lambda(t) = \{[n^\lambda] \mid n \in M(t)\},$$

$$M_\lambda^0(t) = \{[n^\lambda] \mid n \in M(t),\ n < r\},$$

$$T = \sum_{i=1}^{m} t_i,$$

$$T^* = \prod_{i=1}^{m} (t_i!),$$

where $[x]$ denote the least integer $\geqq x$ and $0! = 1$. Note that $M_\lambda^0(t)$ is empty if $T > 1$. We prove the following theorems.

THEOREM 1. *If* $d_1 > 0$ *and* $\lambda > \log(r-m)/\log r$, *then*

$$\mu(M_\lambda(t)) \leqq \mu(M_\lambda^0(t)) + \frac{T!r^\lambda}{T^*(r^\lambda - r + m)^{T+1}}\left(\sum_{n=1}^{r-1} \frac{1}{n^\lambda} - \sum_{i=1}^{m} \frac{1}{d_i^\lambda}\right) + \frac{(T-1)!r^\lambda}{T^*(r^\lambda - r + m)^T} \sum_{i=1}^{m} \frac{t_i}{d_i^\lambda}.$$

THEOREM 2. *If* $d_1 > 0, m < r$, *and* $0 \leqq \lambda \leqq \log(r-m)/\log r$, *then* $M_\lambda(t)$ *is not harmonically convergent, i.e.* $\mu(M_\lambda(t)) = \infty$.

2.

To prove the theorems, we first prove some lemmata. We assume throughout that $d_1 > 0$.

LEMMA 1. *For* $l \geqq 1$ *we have*

$$\sum_{\substack{0 \leqq b < r^l \\ b \in M(t)}} 1 = \frac{l!(r-m)^{l-T}}{T^*(l-T)!} \quad \text{if } l \geqq T,$$
$$= 0 \quad \text{if } l < T.$$

PROOF. The case $l < T$ is obvious. Let $l \geqq T$. If $b < r^l$ then

$$b = \sum_{j=0}^{l-1} b_j r^j \text{ where } 0 \leqq b_j < r.$$

The sum in the lemma equals the number of ways we may choose $(b_0, b_1, \cdots, b_{l-1})$ such that $b_j = d_i$ for exactly t_i values of $j (i = 1, 2, \cdots, m)$. The T element of which t_i have value d_i may be chosen in

$$\frac{l!}{t_1!t_2!\cdots t_m!(l-T)!}$$

ways, and for the remaining $(l-T)$ elements there are $(r-m)$ possible choices. This proves the lemma.

LEMMA 2. *For* $\lambda > \log(r-m)/\log r$ *we have*

$$\sum_{l=1}^{\infty} \frac{1}{r^{\lambda l}} \sum_{\substack{0 \leqq b < r \\ b \in M(t)}} 1 = \frac{T!r^\lambda}{T^*(r^\lambda - r + m)^{T+1}}.$$

PROOF. By Lemma 1 we get

$$\sum_{l=1}^{\infty} \frac{1}{r^{\lambda l}} \sum_{\substack{0 \leqq b < r^l \\ b \in M(t)}} 1 = \sum_{l=1}^{\infty} \frac{1}{r^{\lambda l}} \cdot \frac{l!(r-m)^{l-T}}{T^*(l-T)!}$$

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$$= \frac{T!}{T^*(r-m)^T} \sum_{l=T}^{\infty} \binom{l}{T} \left(\frac{r-m}{r^\lambda}\right)^l$$

$$= \frac{T!}{T^*(r-m)^T} \cdot \frac{\left(\frac{r-m}{r^\lambda}\right)^T}{\left(1-\frac{r-m}{r^\lambda}\right)^{T+1}}$$

$$= \frac{T!r^\lambda}{T^*(r^\lambda - r + m)^{T+1}}.$$

We now prove Theorem 1. For $k \geqq 1$ and $\lambda > \log(r-m)/\log r$ we have, with $\mathbf{t}_i = (t_1, \cdots, t_{i-1}, \ t_i - 1, \ t_{i+1}, \cdots, t_m)$, and $\Delta = \{a \mid 1 \leqq a < r$ and $a \neq d_i$ for $i = 1, 2, \cdots, m\}$,

$$\sum_{\substack{1 \leqq n < r^{k+1} \\ n \in M(\mathbf{t})}} \frac{1}{[n^\lambda]} \leqq \sum_{\substack{1 \leqq n < r \\ n \in M(\mathbf{t})}} \frac{1}{[n^\lambda]} + \sum_{l=1}^{k} \sum_{r^l \leqq n < r^{l+1}} \frac{1}{n^\lambda}$$

$$= \mu(M_\lambda^0(t)) + \sum_{l=1}^{k} \left\{ \sum_{a \in \Delta} \sum_{\substack{0 \leqq b < r^l \\ b \in M(\mathbf{t})}} \frac{1}{(ar^l + b)^\lambda} + \sum_{i=1}^{m} \sum_{\substack{0 \leqq b < r^l \\ b \in M(\mathbf{t}_i)}} \frac{1}{(d_i r^l + b)^\lambda} \right\}$$

$$\leqq \mu(M_\lambda^0(t)) + \sum_{a \in \Delta} \frac{1}{a^\lambda} \sum_{l=1}^{k} \frac{1}{r^{l\lambda}} \sum_{\substack{0 \leqq b < r^l \\ b \in M(\mathbf{t})}} 1 + \sum_{i=1}^{m} \frac{1}{d_i^\lambda} \sum_{l=1}^{k} \frac{1}{r^{l\lambda}} \sum_{\substack{0 \leqq b < r^l \\ b \in M(\mathbf{t}_i)}} 1.$$

Letting $k \to \infty$, Theorem 1 follows from this by Lemma 2.

To prove Theorem 2, we first note that $[x] < 2x$ for $x > 1$. Hence, by Lemma 1,

$$\sum_{\substack{1 \leqq n < r^{k+1} \\ n \in M(\mathbf{t})}} \frac{1}{[n^\lambda]} > \frac{1}{2} \sum_{l=1}^{k} \sum_{\substack{r^l \leqq n < r^{l+1} \\ n \in M(\mathbf{t})}} \frac{1}{n^\lambda} > \frac{1}{2} \sum_{l=1}^{k} \frac{1}{r^{\lambda(l+1)}} \sum_{\substack{r^l \leqq n < r^{l+1} \\ n \in M(\mathbf{t})}} 1$$

$$= \frac{1}{2} \sum_{l=T}^{k} \frac{1}{r^{\lambda(l+1)}} \left\{ \frac{(l+1)!(r-m)^{l+1-T}}{T^*(l+1-T)!} - \frac{l!(r-m)^{l-T}}{T^*(l-T)!} \right\}$$

$$\geqq \frac{1}{2} \sum_{l=T}^{k} \frac{T!}{T^*(r-m)^T r^\lambda} \left\{ \binom{l+1}{T} \left(\frac{r-m}{r^\lambda}\right)^{l+1} - \binom{l}{T} \left(\frac{r-m}{r^\lambda}\right)^{l} \right\}$$

$$= \frac{T!}{2T^*(r-m)^T r^\lambda} \left\{ \binom{k+1}{T} \left(\frac{r-m}{r^\lambda}\right)^{k+1} - \left(\frac{r-m}{r^\lambda}\right)^{T} \right\} \to \infty$$

when $k \to \infty$.

3.

In Kløve (1971) we treated the case: λ integer, $d_1 > 0$, and $t_i = 0$, $i = 1, 2, \cdots, m$. The estimates given there are better then those given by Theorem 1. Better estimates may be given in the general case, but the expressions seem to be very complicated.

Craven (1965) and Alexander (1971) gave estimates for $\mu(\bigcup_{t=0}^{T} M(t))$ in the special case $m = 1$. Improved estimates for this sum may be obtained from Theorem 1. In general, if M_j, $j = 1, 2, \cdots, s$ are harmonically convergent sets, then so is $\bigcup_{j=1}^{s} M_j$ and

$$\mu\left(\bigcup_{j=1}^{s} M_j\right) \leqq \sum_{j=1}^{s} \mu(M_j).$$

In fact, μ is a measure on the σ-algebra of all subsets of the set of natural numbers. If we consider the special case $\lambda = m = 1$ in Theorem 1, we get, for $t > 0$,

$$\mu(M(t)) \leqq \mu(M_1^0(t)) + r\left(\sum_{n=1}^{r-1} \frac{1}{n} - \frac{1}{d}\right) + \frac{r}{d}$$

$$\leqq \mu(M_1^0(t)) + r(\log r + 1).$$

Further

$$\mu(M(0)) \leqq r(\log r + 1),$$

$$\mu(M_1^0(t)) = \frac{1}{d} \text{ if } t = 1.$$

$$= 0 \text{ if } t > 1.$$

Hence, for $T \geqq 1$,

$$\mu\left(\bigcup_{t=0}^{T} M(t)\right) \leqq (T+1) r \log r + (T+1) r + \frac{1}{d}.$$

References

R. Alexander (1971), 'Remarks about the digits of integers', *J. Austral. Math. Soc.* **12**, 239–241.

B. D. Craven (1965), 'On digital distribution in some integer sequences', *J. Austral. Math. Soc.* **5** 325–330.

T. Kløve (1971), 'Power sums of integers with missing digits', *Math Scand.* **28**, 247–251.

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