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SMOOTHNESS IN SPACES OF COMPACT OPERATORS

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We prove that if X and Y are two (real) Banach spaces such that $\dim X \geqslant 2$ and $\dim Y \geqslant 2$, then the space $K(X,Y)$ contains a convex compact subset C with $\dim C \leqslant 2$ (in the affine sense) which fails to be an intersection of balls. This improves two results of Ruess and Stegall.

INTRODUCTION

In this paper we consider only real Banach spaces. This is not a restriction since all the properties considered in this paper depend only on the real structure of the space.

It was proved by Ruess and Stegall [2] that if H is a subspace of $K_{w^*}(X^*, Y)$ (that is, the space of compact $w^* - w$ continuous operators) which contains $X \otimes Y$, then whenever $\dim X \geqslant 2$ and $\dim Y \geqslant 2$, the space H never has the Mazur intersection property (that is, every bounded closed convex set is the intersection of the (closed) balls containing it), and is never Gateaux-smooth (see also [1] Theorem 2.1).

In this paper we improve the results of Ruess and Stegall by proving that under the same assumptions as above, the space H never has the property (CI): every convex compact set is the intersection of the balls containing it. (This property in clearly weaker than the Mazur intersection property and is also weaker than the Gateaux-smoothness of the norm [3] and [5].) Moreover we give a more precise result: we prove that the convex compact subset of H which fails to be an intersection of balls can be choosen to be of affine dimension at most equal to 2.

This result cannot - in general - be improved, since every line segment of the space $\ell_2^2 \otimes_\varepsilon \ell_2^2$ is an intersection of balls.

NOTATION

For a finite dimensional convex set C we denote by $\dim C$ the affine dimension of C.

A point x of a Banach space X is said to be an extreme point if $x = 0$ or $x/\|x\|$ is an extreme point of the unit ball $B(X)$ of X. The set of extreme points of X will be denoted by $\mathrm{Ext}(X)$,

By a ball we always mean a closed ball.

RESULTS

The proof of our main result starts with a numerical lemma:

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LEMMA 1. *Let*

$$U_\varepsilon = \{(\alpha_1, \alpha_2, \lambda_1, \lambda_2) \in \mathbb{R}^4 : \sup_{1 \leqslant i,j \leqslant 2} |\alpha_i \lambda_j + \alpha_j \lambda_i - 2\delta_{ij}| < \varepsilon\}$$

(δ_{ij} are the Kronecker symbols). Then U_ε is non-empty if and only if $\varepsilon > 1$.

PROOF: Suppose $\varepsilon \leqslant 1$, and let $\varepsilon_0, \varepsilon_1, \varepsilon_2$ be such that $\alpha_1\lambda_1 = 1 + \varepsilon_1$, $\alpha_2\lambda_2 = 1 + \varepsilon_2$, $\alpha_1\lambda_2 + \alpha_2\lambda_1 = \varepsilon_0$, and $\sup(|\varepsilon_0|, 2|\varepsilon_1|, 2|\varepsilon_2|) < \varepsilon$.

Observe first that the first two equations imply that all the scalars $\alpha_1, \alpha_2, \lambda_1, \lambda_2$ are non zero.

So we deduce that $\frac{\alpha_1}{\alpha_2}(1 + \varepsilon_2) + \frac{\alpha_2}{\alpha_1}(1 + \varepsilon_1) = \varepsilon_0$. But this equation cannot be satisfied since:

$$\begin{aligned}
\left|\frac{\alpha_1}{\alpha_2}(1+\varepsilon_2) + \frac{\alpha_2}{\alpha_2}(1+\varepsilon_1)\right| &= \left|\frac{\alpha_1}{\alpha_2}\right|(1+\varepsilon_2) + \left|\frac{\alpha_2}{\alpha_1}\right|(1+\varepsilon_1) \\
&> (1 - \varepsilon/2)\left(\left|\frac{\alpha_1}{\alpha_2}\right| + \left|\frac{\alpha_2}{\alpha_1}\right|\right) \\
&\geqslant 2 - \varepsilon \\
&\geqslant 1 > |\varepsilon_0|.
\end{aligned}$$

This proves that $U_\varepsilon = \emptyset$ if $\varepsilon \leqslant 1$.

On the other hand it is easy to see that there exist scalars $\alpha_1, \alpha_2, \lambda_1, \lambda_2$ such that $\alpha_i\lambda_j = 1/2$, so $U_\varepsilon \neq \emptyset$ if $\varepsilon > 1$. ∎

THEOREM 2. *Let X and Y be two Banach spaces such that $\dim X \geqslant 2$, and $\dim Y \geqslant 2$. And let H be a subspace of $K_{w^*}(X^*, Y)$ which contains $X \otimes Y$. Then there exists a convex compact subset C of H with $\dim C \leqslant 2$, and such that C is not an intersection of balls.*

PROOF: Let $(x_i, x_i^*)_{i=1,2}$, and $(y_i, y_i^*)_{i=1,2}$ be two biorthogonal systems in X and Y respectively.

Consider the w^*-open set:

$$W_\varepsilon = \{h^* \in H^* : \sup_{1 \leqslant i,j \leqslant 2} |\langle x_i \otimes y_j + x_j \otimes y_i ; x_1^* \otimes y_1^* + x_2^* \otimes y_2^* - h^* \rangle| < \varepsilon\}$$

and let us prove that $W_\varepsilon \cap \{x^* \otimes y^* : x^* \in X^*, y^* \in Y^*\} = \emptyset$ for every $\varepsilon \leqslant 1$. This is an easy consequence of Lemma 1 since:

$$\begin{aligned}
&\langle x_i \otimes y_j + x_j \otimes y_i ; x_1^* \otimes y_1^* + x_2^* \otimes y_2^* - x^* \otimes y^* \rangle \\
&= 2(\delta_{i1}\delta_{j1} + \delta_{i2}\delta_{j2}) - x^*(x_i)y^*(y_j) - x^*(x_j)y^*(y_i) \\
&= 2\delta_{ij} - x^*(x_i)y^*(y_j) - x^*(x_j)y^*(y_i).
\end{aligned}$$

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This completes the proof of Theorem 2 in view of these two known results:

1) Under our assumptions on H we have [2]

$$\mathrm{Ext}(H^*) = \{x^* \otimes y^* : x^* \in \mathrm{Ext}(X^*), y^* \in \mathrm{Ext}(Y^*)\}.$$

2) For every Banach space E and every natural number n, the following properties are equivalent [4], Theorem 1:

(i) every convex compace subset C of E with $\dim C \leqslant n$ is an intersection of balls.

(ii) for every $\varepsilon > 0$, every $f \in E^*$, and every $(n+1)$ points $(x_i)_{0 \leqslant i \leqslant n}$ of E there exists a $g \in \mathrm{Ext}(E^*)$ such that $\sup_{0 \leqslant i \leqslant n} |x_i(f-g)| < \varepsilon$.

∎

REMARK: Theorem 2 can be applied in particular for the spaces $H = X \hat{\otimes}_\varepsilon Y$, and $H = K(X,Y) = K_{w^*}(X^{**}, Y)$.

The result of Theorem 2 can not be improved in view of the following example:

PROPOSITION 3. *Every line segment of $\ell_2^2 \otimes_\varepsilon \ell_2^2$ is an intersection of balls.*

PROOF: By the above mentioned result of Ruess and Stegall [2], for $H = \ell_2^2 \otimes_\varepsilon \ell_2^2$, we have:

$$\mathrm{Ext}(H^*) = \{x \otimes y : x \in \ell_2^2,\, y \in \ell_2^2\}.$$

The main part in the proof of Proposition 3 is the following result:

CLAIM: The cone $\mathrm{Ext}(H^*)$ intersects all the two dimensional affine subspaces of H^*.

Before proving this claim, let us deduce from it the conclusion of Proposition 3.

Let $f \in H^*$, $u_1, u_2 \in H$, and consider the set $E = \{g \in H^* : u_i(f-g) = 0$ for $i = 1, 2\}$.

Then E is an affine subspace of H^*, whose dimension is at least equal to 2. By the claim, E contains an extreme point of H^*.

This proves Proposition 3 in view of the above mentioned characterisations for the intersection properties. ∎

PROOF OF THE CLAIM:

Let $u, v, w \in H^*$ be such that u and v are linearly independent. We want to find two vectors x and $y \in \ell_2^2$, and two scalars α and β such that:

$$(*) \qquad x \otimes y = \alpha u + \beta v + w.$$

Let $x = a_1 e_1 + a_2 e_2$, $y = b_1 e_1 + b_2 e_2$, $u = \sum_{i,j=1}^2 u_{ij} e_i \otimes e_j$, $v = \sum_{i,j=1}^2 v_{ij} e_i \otimes e_j$ and $w = \sum_{i,j=1}^2 w_{ij} e_i \otimes e_j$.

We will distinguise two cases:

CASE 1: One of the matrices $\begin{pmatrix} u_{11} & v_{11} \\ u_{12} & v_{12} \end{pmatrix}$, $\begin{pmatrix} u_{11} & v_{11} \\ u_{21} & v_{21} \end{pmatrix}$, $\begin{pmatrix} u_{12} & v_{12} \\ u_{22} & v_{22} \end{pmatrix}$, or $\begin{pmatrix} u_{21} & v_{21} \\ u_{22} & v_{22} \end{pmatrix}$ is invertible.

Suppose that $M = \begin{pmatrix} u_{11} & v_{11} \\ u_{12} & v_{12} \end{pmatrix}$ is invertible and let $N = \begin{pmatrix} u_{21} & v_{21} \\ u_{22} & v_{22} \end{pmatrix}$.
The equation (*) implies:

$$(**) \qquad \left(a_2 Id - a_1 NM^{-1}\right)\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} w_{21} \\ w_{22} \end{pmatrix} - NM^{-1}\begin{pmatrix} w_{11} \\ w_{12} \end{pmatrix}.$$

Take now $a_2 = 1$, if $|a_1| < \|NM^{-1}\|^{-1}$, then the matrix $\left(Id - a_1 NM^{-1}\right)$, is invertible, and the equation (**) gives the values of b_1 and b_2.

The computations are similar in the other three subcases.

CASE 2: One of the matrices $\begin{pmatrix} u_{11} & v_{11} \\ u_{22} & v_{22} \end{pmatrix}$ or $\begin{pmatrix} u_{12} & v_{12} \\ u_{21} & v_{21} \end{pmatrix}$ is invertible.

Suppose that $M = \begin{pmatrix} u_{11} & v_{11} \\ u_{22} & v_{22} \end{pmatrix}$ is invertible and let $N = \begin{pmatrix} u_{12} & v_{12} \\ u_{21} & v_{21} \end{pmatrix}$. The equation (*), implies:

$$(***) \qquad \left[\begin{pmatrix} b_2 & 0 \\ 0 & b_1 \end{pmatrix} - NM^{-1}\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}\right]\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} w_{12} \\ w_{21} \end{pmatrix} - NM^{-1}\begin{pmatrix} w_{11} \\ w_{22} \end{pmatrix}.$$

At this stage we need the following subclaim:

SUBCLAIM: For every 2×2 -matrix T, there exists scalars p, q such that the matrix $\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} - T\begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix}$ is invertible, except for the case when $T_{11} = T_{22} = 0$ and $T_{12}T_{21} = 1$.

PROOF: Take $p = 1$ and $q = 0$ if $T_{22} \neq 0$, $p = 0$ and $q = 1$ if $T_{11} \neq 0$ and $p = q = 1$ if $T_{11} = T_{22} = 0$ and $T_{12}T_{21} \neq 1$. ∎

We also distinguish two cases:

(i) $NM^{-1} = \begin{pmatrix} 0 & \lambda \\ 1/\lambda & 0 \end{pmatrix}$ which implies that $u_{12} = \lambda u_{22}$ and $v_{12} = \lambda u_{22}$, and then the matrix $\begin{pmatrix} u_{11} & v_{11} \\ u_{12} & v_{12} \end{pmatrix}$ is invertible. So we can apply the technique of Case 1 to solve the equation (*).

(ii) If not, by the preceeding subclaim find b_1 and b_2 such that $\begin{pmatrix} b_2 & 0 \\ 0 & b_1 \end{pmatrix} - NM^{-1}\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}$ is invertible, and use the equation (***) to find the values of a_1 and a_2.

The computations are similar in the case when $\begin{pmatrix} v_{12} & v_{12} \\ u_{21} & v_{21} \end{pmatrix}$ is invertible.

Since the vectors u and v are linearly independent, we are necessarily in either the situation of Case 1 or of Case 2. The claim is then proved. ∎

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