Radical Axes in Spherical Geometry.

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It is evident that by an extension of the method of deriving from any triangle its polar triangle, it is possible to derive from any figure whatever another figure, the properties of which may be deduced at once from those of the first. This may be done either by imagining a point to move along the original figure and considering the envelope of the great circle of which the moving point is the pole; or by imagining a great circle to envelope the figure and considering the locus of its pole. In both cases two figures will be obtained; but these will be antipodal, and will therefore have like properties. Since the point of intersection of two great circular arcs is the pole of the great circular arc which joins the poles of these two arcs, the two methods of derivation mentioned above will lead to the same derived figure. These methods of transformation of figures are evidently closely analogous to that of reciprocal polars.

The ordinary polar triangle may be obtained as the locus of the pole of a great circular arc which is made to revolve round the original triangle; and the fact that this arc revolves through the exterior angle of the one triangle, while its pole moves along a side of the other, shows that the angles of the one triangle are the supplements of the sides of the other; and that a great circular arc, cutting an interior angle of the one, corresponds to an external point in a side of the other.

A point may, of course, be considered as enveloped by all the arcs that pass through it, just as an arc is traced out by all the points that lie in it.

It will be well to notice the following points connected with the transformation of circles and related points and lines, as bearing on the subject of this note:—

A circle transforms into a circle; its centre into the great circle parallel to (or concentric with) the derived circle.

A tangent transforms into a point on the circumference; the point of contact into the tangent at the point into which the tangent transforms. A secant transforms into a point on the sphere; the points of section into the tangents from that point.

If two circles be now considered, it will be seen that the central

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axis transforms into either of the (antipodal) points of intersection of the two great circles into which the two centres transform. Again, since a point of intersection is a point common to two circles, and to this corresponds a tangent common to two circles; it would seem that points of intersection and common tangents transform into one another. But two circles can only intersect in two points, while in a plane at least two circles may have four common tangents; and a little examination will show that this may also be the case on the surface of the sphere. Now, on examination of the figure it will be seen that if a great circular arc be made to revolve in the same direction round each of the circles, either of its poles will be in the same direction for the two circles when it is a direct common tangent; and in a different direction when it is an inverse common tangent. Hence to the two direct common tangents correspond the points of intersection; and to the inverse common tangents, antipodal points on the two circles. From this it follows that there can be only two pairs of such points, except when the circles are antipodal, in which case there are an infinite number, and likewise an infinite number of common tangents. (It is evident that a pair of antipodal circles transforms into a pair of antipodal circles).

The point of intersection of the direct common tangents (the external centre of similitude) transforms into the arc joining the points of intersection, that is, the radical axis; and since the great circular arc which passes through the two pairs of antipodal points corresponds to the internal centre of similitude, it may conveniently be called the second radical axis of the two circles.

If the potency of a point with respect to a circle be defined as $\tan\frac{1}{2}a\tan\frac{1}{2}b$, where a and b are the segments of a secant through the point, the potencies of any point on the first radical axis with respect to two circles will be the same, while those of any point on the second radical axis will be the reciprocals of one another. From this it follows at once that the six radical axis of three circles pass in sets of three through four points.

Corresponding to the potency of a point with regard to a circle, there is the following property of a line and a circle, suggesting a "potency of a line" with regard to a circle. If tangents be drawn from any point P in a fixed line PQ to a given circle, touching it at A and B, then $\tan\frac{1}{2}\text{APQ}\cot\frac{1}{2}\text{BPQ}$ will be constant. This property follows from the fact that it is the polar theorem to that which gives the corresponding property of a point and a circle; and the quantity $\tan\frac{1}{2}APQ\cot\frac{1}{2}BPQ$ may be called the potency of the line PQ with reference to the circle. The above theorem is also true *in plano*, and affords a simple proof that the six centres of similitude of three circles lie in sets of three on four straight lines. To the fact of the collinearity of the centres of two circles and their two centres of similitude corresponds the proposition that the two parallel great circles and the two radical axes are concurrent.

The following examples of properties of circles which correspond according to the above method of transformation may also be noted.

To the fact that the radius of a circle is of constant length, corresponds the fact that the angle which the tangent to a circle makes with the parallel great circle is constant; and the proposition that the tangents make equal angles with the chord of contact transforms into the proposition that the two tangents from the same point are equal.

Again, the centres of similitude of two circles divide the line joining the centres of the circles so that the ratio of the sines of the segments is equal to the ratio of the sines of the radii. This theorem gives on transformation the theorem that the radical axes divide the angle between the parallel circles so that the ratio of the sines of the parts is equal to the ratio of the sines of the angles which the tangents to the circles make with the parallel great circles. This theorem evidently cannot have any analogue in plane geometry, as must in general be the case with the polars of such theorems as refer to the centre of a circle, since the line in a plane corresponding to a parallel great circle is altogether at an infinite distance.

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Gilbert's Method of Treating Tangents to Confocal Conicoids.

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The method of treating tangents to confocal conicoids, of which I propose to give an account, is discussed in the number for December