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# ON SOME OSCILLATION CRITERIA FOR A CLASS OF NEUTRAL TYPE FUNCTIONAL DIFFERENTIAL EQUATIONS

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**Abstract**

The present paper proves criteria for oscillation of the solutions of functional differential equations of the type

$$x^{(n)}(t) + \lambda x^{(n)}(t-\tau) + p(t)\mathrm{f}(x(t-\tau)) = 0,$$

where $\lambda$, $\tau > 0$.

## 1. Introduction

The theory of oscillations has a wide range of applications to various areas of chemistry, biochemistry, biology, etc. An extensive reference on these subjects is given in [5], [7], [6]. Together with the classical models, there is an increasing implementation of models with aftereffect governed by functional-differential equations. This approach made possible the theoretical explanation of the 10-year cycle of oscillation of the mammalian populations in Canada and Jakutija, as well as of some other experimental phenomena [10], [3], [4].

The present paper sets down some oscillation criteria for the solutions of functional differential equations of the type

$$x^{(n)}(t) + \lambda x^{(n)}(t-\tau) + p(t)\mathrm{f}(x(t-\tau)) = 0,\ n \geqslant 1 \tag{1}$$

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where $\tau > 0$ is a constant delay and $\lambda > 0$ is an arbitrary constant. An analogous result for ordinary differential equations without delay is obtained in [1], and for equations with a retarded argument in [8].

## 2. Definitions

Suppose the following conditions (D) are fulfilled:

D1. The function $f(u)$: $\mathbf{R}^1 \to \mathbf{R}^1$ is continuous, $uf(u) > 0$ for $u \neq 0$ and $\liminf_{|u| \to +\infty} |f(u)| > 0$.

D2. The function $p(t)$: $\mathcal{T} \to [0, +\infty)$ is continuous, where $\mathcal{T} = [t_0 - \tau, +\infty)$, $t_0 \in \mathbf{R}^1$.

The alternative $\mathcal{L}$ will be said to hold for equation (1) if for $n$ even all of its solutions oscillate, while for $n$ odd, they either oscillate or tend to zero for $t \to +\infty$.

Let the operator $L$ be defined by the equality

$$(L\psi)(t) = \psi(t) + \lambda\psi(t - \tau) \tag{2}$$

and let us denote by $\tilde{C}^k$ the space of functions $\psi(t)$: $\mathcal{T} \to \mathbf{R}^1$ locally having absolutely continuous derivatives of order up to $k$.

A function $x(t) \in \tilde{C}^{n-1}$ is said to be a regular solution of equation (1) if it satisfies (1) almost everywhere for $t \geqslant t_0$, and for each $t \geqslant t_0$,

$$\sup_{s \in [t, +\infty)} |x(s)| > 0.$$

A solution $x(t)$ of equation (1) is said to be oscillatory if it has a sequence of zeros which tends to $+\infty$.

## 3. The main theorem

Lemma 1 ([2], p. 243). *Let the following conditions be fulfilled*:

1. *The function $\psi(t) \in \tilde{C}^{n-1}$ has a constant sign together with its derivatives of order up to n in the interval $[t_0, +\infty)$.*

2. *For each $t \geqslant t_0$, the following inequality is valid*

$$\psi(t)\psi^{(n)}(t) \leqslant 0 \qquad \left(\psi(t)\psi^{(n)}(t) \geqslant 0\right).$$

*Then there exists an integer $l$, $0 \leqslant l \leqslant n$, such that $l + n$ is odd (even) and for $t \geqslant t_0$, the inequalities*

$$\psi(t)\psi^{(i)}(t) \geqslant 0, \; i = 0, \dots, l;$$

$$(-1)^{l+i}\psi(t)\psi^{(i)}(t) \geqslant 0, \; i = l + 1, \dots, n;$$

$$|\psi^{(l-i)}(t)| \leqslant \frac{i!}{j!}(t - t_0)^{i-j}|\psi^{(l-j)}(t)|, \; j = 0, \dots, l; \; i = 0, \dots, j$$

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*take place. Moreover, if $l \neq 0$, then*

$$|\psi(t)| \geqslant \sum_{i=l+1}^{n} \frac{1}{l!(i-l)!}(t-t_0)^{i-1}|\psi^{(i-1)}(t)|.$$

THEOREM 1. *Let the following conditions be fulfilled:*

1. *Conditions* $(D)$ *hold.*

2. *For each function* $\psi(t) \in \tilde{C}^{n-1}$ *such that* $|\psi(t)| > 0$ *for sufficiently large values of* $t$ *and* $\liminf_{t \to +\infty} |(L\psi)(t)| > 0$, *the inequality*

$$\liminf_{t \to +\infty} |\mathrm{f}(\psi(t-\tau))/(L\psi)(t)| > 0$$

*is valid.*

3. *There exists an absolutely continuous and non-decreasing function* $\varphi(t)$: $\mathscr{T} \to (0, +\infty)$ *such that for each measurable and closed set* $E$ *having the property* $\operatorname{meas}(E \cap [t, t+2\tau]) \geqslant \tau$, $t \in \mathscr{T}$, *the following relations hold:*

$$\int_E \left[(t-\tau)^{n-1} p(t)/\varphi(t-\tau)\right] dt = +\infty, \tag{3}$$

$$\int_{t_0}^{+\infty} \frac{dt}{t\varphi(t)} < +\infty. \tag{4}$$

*Then the alternative* $\mathscr{L}$ *holds for equation* (1).

PROOF. Let $x(t)$ be a non-oscillatory solution of equation (1), the operator $L$ be defined by equation (2), and for the sake of definiteness suppose that $x(t) > 0$ for $t \geqslant \bar{t}$, $\bar{t} \in \mathscr{T}$.

Let $n$ be an even number. Then equation (1) implies that for $t \geqslant \bar{t}$, $[(Lx)(t)]^{(n)} \leqslant 0$ and by virtue of Lemma 1 there exists a point $t_1 \geqslant \bar{t}$ and an integer $l$, $1 \leqslant l \leqslant n-1$, such that for $t \geqslant t_1$ the following inequalities hold:

$$(Lx)(t)[(Lx)(t)]^{(i)} \geqslant 0, \; i = 0, \dots, l; \tag{5}$$

$$(-1)^{l+i}(Lx)(t)[(Lx)(t)]^{(i)} \geqslant 0, \; i = l+1, \dots, n; \tag{6}$$

$$[(Lx)(t)]^{(l-i)} \leqslant \frac{i!}{j!}(t-t_1)^{i-j}[(Lx)(t)]^{(l-j)}. \tag{7}$$

Multiplying both sides of equation (1) by the function

$$\frac{(t-\tau)^{n-l}}{\varphi(t)[(Lx)(t)]^{(l-1)}}$$

and taking into account that $t - \tau > 0$, one obtains

$$\frac{t^{n-l}[(Lx)(t)]^{(n)}}{\varphi(t)[(Lx)(t)]^{(l-1)}} + \frac{p(t)\mathrm{f}(x(t-\tau))(t-\tau)^{n-l}}{\varphi(t)[(Lx)(t)]^{(l-1)}} \leqslant 0. \tag{8}$$

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Inequality (7) for $i = 1$, $j = l$ yields the following inequality for $t \geqslant t_1$:

$$(t - t_1)^{l-1}[(Lx)(t)]^{(l-1)} \leqslant l!(Lx)(t). \tag{9}$$

On the other hand, there exists a point $t_2 \geqslant t_1$, such that for $t \geqslant t_2$, $t - \tau \geqslant 2t_1$ holds. Hence, if $t \geqslant t_2$, inequalities (8) and (9) imply the inequality

$$\frac{t^{n-l}[(Lx)(t)]^{(n)}}{\varphi(t)[(Lx)(t)]^{(l-1)}} + \frac{cp(t)\mathrm{f}(x(t-\tau))(t-\tau)^{n-1}}{\varphi(t)(Lx)(t)} \leqslant 0, \tag{10}$$

where $c > 0$ is a constant.

Since $[(Lx)(t)] \geqslant 0$ and $x(t)$ is a regular solution, there exists a point $t_3 \geqslant t_2$ such that $(Lx)(t) \geqslant c_1 > 0$ for $t \geqslant t_3$ and by virtue of Lemma 1 from [9], there exists a closed and measurable set $E$ with the property meas$(E \cap [t, t + 2\tau]) \geqslant \tau$ for $t \geqslant t_3$, such that $x(t - \tau) \geqslant c_2 > 0$ for each $t \in E$. On integrating inequality (10) on the set $E \cap [t_3, t]$, $t > t_3$ and taking into account that for $t \geqslant t_3$, $[(Lx)(t)]^{(n)} \leqslant 0$ and $[(Lx)(t)]^{(l-1)} \geqslant 0$, we obtain the inequality

$$\int_{t_3}^{t} \frac{s^{n-l}[(Lx)(s)]^{(n)}\,ds}{\varphi(s)[(Lx)(s)]^{(l-1)}} + \int_{E\cap[t_3,t]} \frac{cp(s)\mathrm{f}(x(s-\tau))(s-\tau)^{n-1}\,ds}{\varphi(s)(Lx)(s)} \leqslant 0. \tag{11}$$

Integrating the first integral in (11) by parts, we obtain

$$\begin{aligned}
&\left.\frac{\sum_{i=0}^{n-l+1}(-1)^i\dfrac{(n-l)!}{(i+1)!}s^{i+1}[(Lx)(s)]^{(l+i)}}{\varphi(s)[(Lx)(s)]^{(l-1)}}\right|_{t_3}^{t} - \int_{t_3}^{t}\sum_{i=0}^{n-l+1}(-1)^i s^{i+1}\frac{(n-l)!}{(i+1)!}\\
&\quad\times[(Lx)(s)]^{(l+i)}d\left[\left(\varphi(s)[(Lx)(s)]^{(l-1)}\right)^{-1}\right]\\
&\quad+\int_{E\cap[t_3,t]}\frac{cp(s)\mathrm{f}(x(s-\tau))(s-\tau)^{n-1}\,ds}{\varphi(s)[(Lx)(s)]}\\
&\quad-(n-l)!\int_{t_3}^{t}\frac{[(Lx)(s)]^{(l)}\,ds}{\varphi(s)[(Lx)(s)]^{(l-1)}} \leqslant 0.
\end{aligned} \tag{12}$$

From inequalities (5) and (6) and condition 2 of Theorem 1, we can draw the conclusion that for $t \geqslant t_3$ all derivatives $[(Lx)(t)]^{(l)}$ of an even order will be non-negative and monotonically decreasing, while all derivatives of an odd order will be non-positive and monotonically increasing, and hence the sum participating in inequality (12) is non-negative.

For $t \geqslant t_3$ inequality (5) and condition 3 of Theorem 1 yield

$$d\left[\left(\varphi(t)[(Lx)(t)]^{(l-1)}\right)^{-1}\right] \leqslant 0.$$

Hence we can conclude that for $t \geqslant t_3$ the first two summands in the right-hand side of inequality (12) are non-negative and therefore the following inequality holds:

$$\int_{E\cap[t_3,t]} \frac{cp(s)f(x(s-\tau))(s-\tau)^{n-1}\,ds}{\varphi(s)(Lx)(s)} \leqslant (n-l)!\int_{t_3}^{t} \frac{[(Lx)(s)]^{(l)}\,ds}{\varphi(s)[(Lx)(s)]^{(l-1)}}. \tag{13}$$

On the other hand (7) implies that for $t \geqslant t_3$, $j = 1$, $i = 0$, the inequality

$$[(Lx)(t)]^{(l)}(t-t_3) \leqslant [(Lx)(t)]^{(l-1)}$$

holds. Furthermore, there exists a point $t_4 \geqslant t_3$, such that for $t \geqslant t_4$ the inequality $t - t_3 \geqslant \frac{1}{2}(t-\tau)$ holds and hence

$$\tfrac{1}{2}[(Lx)(t)]^{(l)}(t-\tau) \leqslant [(Lx)(t)]^{(l-1)}. \tag{14}$$

Taking into account that the condition 2 of Theorem 1 implies that there exists a point $t_5 \geqslant t_4$ and a constant $c_3 > 0$, such that for $t \geqslant t_5$ we have

$$f(x(t-\tau))/(Lx)(t) \geqslant c_3/2.$$

Inequalities (13) and (14) then yield the inequality

$$\frac{cc_3}{2}\int_{E\cap[t_5,t]} \frac{p(s)(s-\tau)^{n-1}\,ds}{\varphi(s)} \leqslant 2(n-l)!\int_{t_5}^{t}\frac{ds}{(s-\tau)\varphi(s)}.$$

Accomplishing a boundary transition in the above inequality, for $t \to +\infty$, and taking into account inequality (4), we get

$$\frac{cc_3}{2}\int_{E\cap[t_5,+\infty)} \frac{(t-\tau)^{n-1}p(t)\,dt}{\varphi(t)} \leqslant 2(n-l)!\int_{t_5}^{+\infty} \frac{dt}{(t-\tau)\varphi(t)} < +\infty,$$

which contradicts equality (3).

Let $n$ be odd and assume that the equation has a non-oscillatory solution $x(t)$. Without any loss of generality, we can suppose that $x(t) > 0$ for $t \geqslant \bar{t}$, $\bar{t} \in \mathscr{T}$. Then, since for $t \geqslant \bar{t}$

$$(Lx)(t) \leqslant 0,\ [(Lx)(t)]^{(n)} \leqslant 0,$$

Lemma 1 implies that there exists a point $t_1 \geqslant \bar{t}$ and an integer $l$, $0 \leqslant l < n$, $l + n$ being odd, such that for $t \geqslant t_1$ inequalities (5) and (6) hold, while if $l \neq 0$, the inequality (7) holds.

If $l \geqslant 2$, then our reasoning goes on as in the case when $n$ is even. In the case $l = 0$ and $\lim_{t\to+\infty}(Lx)(t) = 0$ we have $\lim_{t\to+\infty} x(t) = 0$.

Let $l = 0$ and $\lim_{t\to+\infty}(Lx)(t) = c_4 > 0$.

Since $\lim_{t\to+\infty}(Lx)(t) = c_4$, then there exists a point $t_2 \geqslant t_1$, such that $(Lx)(t) \geqslant c_4/2$ for $t \geqslant t_2$. By virtue of Lemma 1 from [9] there exists a measurable and closed set $E$, $E \subseteq [t_2, +\infty)$ with the property

$$\operatorname{meas}(E\cap[t,t+2\tau]) \geqslant \tau,\ t \geqslant t_2$$

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such that $x(t-\tau) \geqslant c_5 > 0$ *for* $t \in E$. Hence there exists a constant $c_6 > 0$, such that

$$\inf_{t \in E} f(x(t-\tau)) \geqslant c_6. \tag{15}$$

Multiplying equation (1) by $t^{n-1}$ and integrating within the bounds from $t_2$ to $t \geqslant t_2$, we obtain

$$\int_{t_2}^{t} s^{n-1}[(Lx)(s)]^{(n)}\,ds + \int_{t_2}^{t} s^{n-1}p(s)f(x(s-\tau))\,ds = 0. \tag{16}$$

Integrating the first integral in (16) $n-1$ times by parts, we get

$$\begin{aligned} &s^{n-1}[(Lx)(s)]^{(n-1)}\Big|_{t_2}^{t} - (n-1)s^{n-2}[(Lx)(s)]^{(n-2)}\Big|_{t_2}^{t} + \cdots \\ &\qquad \cdots + (n-1)!(Lx)(s)\Big|_{t_2}^{t} + \int_{t_2}^{t} s^{n-1}p(s)f(x(s-\tau))\,ds = 0. \end{aligned} \tag{17}$$

Since $[(Lx)(t)]^1 \leqslant 0$, inequality (6) implies that all derivatives of $(Lx)(t)$ of even order are non-negative and hence inequalities (15) and (17) yield the inequality

$$\begin{aligned} c_6 \int_{E \cap [t_2, t]} s^{n-1}p(s)\,ds \leqslant\ & t_2^{n-1}[(Lx)(t_2)]^{(n-1)} - (n-1)t_2^{n-2}[(Lx)(t_2)]^{(n-2)} \\ & + \cdots + (n-1)!(Lx)(t_2). \end{aligned}$$

The last inequality, after passing to the bound for $t \to +\infty$, gives the inequality

$$\int_{E \cap [t_2, +\infty)} t^{n-1}p(t)\,dt < +\infty. \tag{18}$$

Taking into account that $t - \tau < t$ and $\varphi(t)$ is a non-decreasing function, (18) yields

$$\int_{E \cap [t_2, +\infty)} [(t-\tau)p(t)/\varphi(t-\tau)]\,dt < +\infty,$$

which contradicts equality (3).

Thus, Theorem 1 is proved.

## 4. An alternative theorem

Since condition 2 of Theorem 1 is difficult to verify, then we proceed to prove, by means of an indirect criterion, the validity of the alternative $\mathscr{L}$ for equation (1) in the case when $f(u)$ is differentiable.

THEOREM 2. *Let the following conditions be fulfilled*:

1. *Conditions* $(D)$ *hold.*
2. *The function* $f \in C^1(\mathbf{R}^1, \mathbf{R}^1)$ *and* $f'(u) \geqslant 0$, $u \in \mathbf{R}^1$.
3. *There exists a function* $\varphi \in C^1(\mathscr{T}, \mathbf{R}^1)$, $\varphi(t) > 0$, $\varphi'(t) \geqslant 0$ *for* $t \in \mathscr{T}$, *such that for each closed measurable set* $E \subseteq \mathscr{T}$ *with the property* $\operatorname{meas}(E \cap [t, t + 2\tau]) \geqslant \tau$, $t \in \mathscr{T}$, *the following relations hold*

$$\int_E \frac{(t-\tau)^{n-1} p(t)}{\varphi(t)}\, dt = +\infty \tag{19}$$

$$\int_\varepsilon^{+\infty} \frac{du}{f(u)\varphi(u^{1/(n-1)})} < +\infty, \quad \int_{-\varepsilon}^{-\infty} \frac{du}{f(u)\varphi\big((-u)^{1/(n-1)}\big)} < +\infty, \ \varepsilon > 0. \tag{20}$$

*Then, the alternative* $\mathscr{L}$ *holds for equation* (1).

PROOF. Let $x(t)$ be a non-oscillatory solution of equation (1), supposing for the sake of definiteness that $x(t) > 0$ for $t \geqslant \bar{t}$, $\bar{t} \in \mathscr{T}$, and that the operator $L$ is defined by equality (2). Then Lemma 1 implies that there exists a point $t_1 \geqslant \bar{t}$ and a number $l$, $0 \leqslant l \leqslant n$, $l + n$ odd, such that for $t \geqslant \bar{t}$ inequalities (5) and (6) hold, while if $l \neq 0$, then inequality (7) also holds.

Let $n$ be an even number. Then, since $l \geqslant 1$ by virtue of Lemma 1 from [9], there exists a set $E \subseteq \mathscr{T}$, such that $\operatorname{meas}(E \cap [t, t + 2\tau]) \geqslant \tau$, $t \in \mathscr{T}$, and $x(t - \tau) \geqslant c_7 > 0$ for $t \in E$. Besides, $l \geqslant 1$ and (7) implies that for $t \geqslant t_1$ the following inequality holds:

$$[(Lx)(t)]^{(l)} \leqslant j!(t - t_1)^{-j}[(Lx)(t)]^{(l-j)}, \ j = 0, \ldots, l.$$

If we choose a point $t_2 \geqslant t_1$, such that for $t \geqslant t_2$ we have $t - \tau \geqslant 2t_1$, then the last inequality implies the inequality

$$[(Lx)(t)]^{(l)} \leqslant 2^{l-1}(l-1)!(t-\tau)^{1-l}[(Lx)(t)], \ t \geqslant t_2. \tag{21}$$

There exists also a point $t_3 \geqslant t_2$ and a constant $C_8$, such that

$$\inf_{t \in E \cap [t_3, +\infty)} f(x(t-\tau))/f((Lx)(t)) \geqslant c_8. \tag{22}$$

Let us multiply equation (1) by the function $(t-\tau)^{n-1}/\varphi(t)f((Lx)(t))$ and integrate from $t_3$ to $t > t_3$. We get

$$\int_{t_3}^{t} \frac{p(s)(s-\tau)^{n-1} f(x(s-\tau))\, ds}{\varphi(s) f((Lx)(s))} = \int_{t_3}^{t} \frac{(s-\tau)^{n-1}[(Lx)(t)]^{(n)}\, ds}{\varphi(s) f((Lx)(s))},$$

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whence, integrating by parts the integral in the rigth-hand side and taking into account equality (22), we have

$$c_8\int_{E\cap[t_3,t]}\frac{(s-\tau)^{n-1}p(s)\,ds}{\varphi(s)}=-\frac{(s-\tau)^{n-1}[(Lx)(t)]^{(n-1)}}{\varphi(s)f((Lx)(s))}\Bigg|_{t_3}^{t}$$
$$+(n-1)\int_{t_3}^{t}\frac{(s-\tau)^{n-2}[(Lx)(s)]^{(n-1)}\,ds}{\varphi(s)f((Lx)(s))}$$
$$+\int_{t_3}^{t}(s-\tau)^{n-1}[(Lx)(s)]^{(n-1)}\,d\left[(\varphi(s)f((Lx)(s)))^{-1}\right]. \tag{23}$$

Conditions 2 and 3 of Theorem 2 and (6) yield the result that

$$[(Lx)(t)]^{(n-1)}\geqslant 0,\ d\left[(\varphi(t)f((Lx)(t)))^{-1}\right]\leqslant 0,$$

and hence from (23) we obtain the inequality

$$c_8\int_{E\cap[t_3,t]}\frac{(s-\tau)^{n-1}p(s)\,ds}{\varphi(s)}\leqslant c_9+(n-1)\int_{t_3}^{t}\frac{(s-\tau)^{n-1}[(Lx)(x)]^{(n-1)}\,ds}{\varphi(s)f((Lx)(s))} \tag{24}$$

where $c_9>0$ is a constant. By integrating the right-hand side of (24) $n-l$ times by parts, we obtain

$$c_8\int_{E\cap[t_3,t]}\frac{(s-\tau)^{n-1}p(s)\,ds}{\varphi(s)}$$
$$\leqslant c_9+(n-1)\frac{\sum_{i=l}^{n-2}(-1)^i\frac{(n-2)!}{i!}(s-\tau)^i[(Lx)(s)]^{(i)}}{\varphi(s)f((Lx)(s))}\Bigg|_{t_3}^{t}$$
$$-(n-1)\int_{t_3}^{t}\sum_{i=l}^{n-2}(-1)^i\frac{(n-2)!}{i!}(s-\tau)^i[(Lx)(s)]^{(i)}d\left[(\varphi(s)f((Lx)(s)))^{-1}\right]$$
$$+(-1)^{n-l-1}\frac{(n-1)!}{(l-1)!}\int_{t_3}^{t}\frac{(s-\tau)^{l-1}[(Lx)(s)]^{(l)}\,ds}{\varphi(s)f((Lx)(s))}. \tag{25}$$

Since (5) and (6) imply for $s\geqslant t_3$ the inequality

$$\sum_{i=l}^{n-2}(-1)^i\frac{(n-2)!}{i!}(s-\tau)^i[(Lx)(s)]^{(i)}\leqslant 0$$

holds, then (21) and (25) yield

$$c_8\int_{E\cap[t_3,t]}\frac{(s-\tau)^{n-1}p(s)\,ds}{\varphi(s)}\leqslant c_{10}+2^{l-1}(n-1)!\int_{t_3}^{t}\frac{[(Lx)(s)]'\,ds}{\varphi(s)f((Lx)(s))}. \tag{26}$$

Taylor's theorem and the fact that $[(Lx)(t)]^{(n)} \leqslant 0$ for $t \geqslant t_3$ imply that there exists a constant $a \geqslant 1$, such that $(Lx)(t) \leqslant at^{n-1}$ for $t \geqslant t_3$. Then conditions 2 and 3 of Theorem 2 and (5) imply the inequality

$$\int_{t_3}^{t} \frac{[(Lx)(s)]'\,ds}{\varphi(s)f((Lx)(s))} \leqslant \int_{t_3}^{t} \frac{d[(Lx)(s)]}{f((Lx)(s))\varphi\left(\left[\frac{(Lx)(s)}{a}\right]^{1/n-1}\right)}$$

$$= \int_{a^{-1}(Lx)(t_3)}^{a^{-1}(Lx)(t)} \frac{du}{f(au)\varphi(u^{1/(n-1)})}.$$

The last inequality and inequalities (20) and (26) yield the inequality

$$\int_{E\cap[t_3,+\infty)} \frac{(t-\tau)^{n-1}p(t)\,dt}{\varphi(t)} \leqslant a\int_{a^{-1}(Lx)(t_3)}^{\infty} \frac{du}{[f(u)\varphi(u^{1/(n-1)})]} < +\infty,$$

which contradicts condition (19).

Let $n$ be odd and let for the non-oscillatory solution $x(t)$ of (1) and the operator $L$ the same assumptions be made as in the case when the number $n$ is even. If for the numbers $l$, $0 \leqslant l \leqslant n$, existing by virtue of Lemma 1, we have the condition $l \geqslant 2$, then by the aid of reasoning analogous to that for the case when $n$ is even, we arrive at a contradiction. Therefore, $l = 0$ ($n$ odd) and since (6) implies that $[(Lx)(t)]' \leqslant 0$, then either $\lim_{t\to+\infty}(Lx)(t) = 0$, and hence $\lim_{t\to+\infty} x(t) = 0$, or $\lim_{t\to+\infty}(Lx)(t) = c_{11} > 0$.

Therefore, by virtue of Lemma 1 from [9], there exists a closed and measurable set $E \subseteq \mathscr{T}$, $\operatorname{meas}(E \cap [t, t+2\tau]) \geqslant \tau$, $t \geqslant \bar{t}$, such that $x(t-\tau) \geqslant c_{12} > 0$ for $t \in E$. We multiply equation (1) by $t^{n-1}$ and, integrate on the interval from $\bar{t}$ to $t > \bar{t}$ to obtain

$$\sum_{i=1}^{n} (-1)^{i+1}\frac{(n-1)!}{(n-i)!}s^{n-i}[(Lx)(s)]^{(n-i)}\Big|_{\bar{t}}^{t} + c_{13}\int_{E\cap[\bar{t},t]} s^{n-1}p(s)\,ds \leqslant 0,\ c_{13} > 0.$$

Since (6) implies that for $t \geqslant \bar{t}$ all derivatives of $(Lx)(t)$ of even order are non-positive and monotonically increasing, while those of odd order are non-negative and monotonically decreasing, then the last inequality, after passing to the limit $t \to +\infty$, yields the inequality

$$\int_{E\cap[\bar{t},+\infty)} t^{n-1}p(t)\,dt < +\infty$$

whence, since $\varphi'(t) \geqslant 0$, $\varphi(t) > 0$, we obtain the validity of the inequality

$$\int_{E\cap[\bar{t},+\infty)} [t^{n-1}p(t)/\varphi(t)]\,dt < +\infty$$

which contradicts (19).

Thus, Theorem 2 is proved.

REMARK. For $n = 1$ the proofs of Theorems 1 and 2 can be considerably simplified, since the integration by parts is omitted.

## 5. Necessity of equation (19)

We are going to show by a counterexample that equality (19) from condition 3 of Theorem 2 cannot be replaced by the weaker classical condition

$$\int_{t_0}^{+\infty} \left[(t-\tau)^{n-1} p(t)/\varphi(t)\right] dt = +\infty. \tag{27}$$

Consider the equation

$$x'(t+\pi/2) + x'(t) + p(t)x^3(t) = 0. \tag{28}$$

where $t \geqslant t_0 > 0$ and $p(t) = [t^2 + (t+\pi)^2]/[t^2(t+\pi)^2(t^{-1} + 1 - \cos t)^3]$. Here $f(u) = u^3$, and let $\varphi(t) \equiv 1$. After simple calculations one obtains

$$\begin{aligned} \int_{t_0}^{+\infty} p(t)\,dt &\geqslant \sum_{k=[t_0]+1}^{+\infty} \int_{2k\pi - k^{-1}}^{2k\pi + k^{-1}} \left(t^2 + (t+\pi)^2\right) t^{-2} \\ &\qquad \times (t+\pi)^{-2}(t^{-1} + 1 - \cos t)^{-3}\,dt \\ &\geqslant \sum_{k=[t_0]+1}^{+\infty} 4k^{-1}(2k\pi + k^{-1} + \pi)^{-2}\left([2k\pi + k^{-1}]^{-1} + 1 - \cos k^{-1}\right), \end{aligned}$$

which yields

$$\int_{t_0}^{+\infty} p(t)\,dt = +\infty.$$

On the other hand if

$$E = \bigcup_{k=[t_0]+1}^{+\infty} \{t \mid t \geqslant t_0,\ \pi/4 + 2k\pi \leqslant t \leqslant 3\pi/4 + 2k\pi\},$$

then

$$\int_E p(t)\,dt \leqslant \int_E \frac{\left[t^2 + (t+\pi)^2\right] dt}{t^2(t+\pi)^2\left(t^{-1} + 1 - 1/\sqrt{2}\right)} < +\infty$$

which shows that $p(t)$ satisfies the classical condiiton (27) but does not satisfy (19).

It can be easily verified that equation (28) has a solution $x(t) = t^{-1} + 1 - \cos t$. Thus condition (19) is substantial.

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