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Convergence criteria for Fourier series

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The following convergence criterion of Fourier series is due to M. Izumi, S. Izumi and the author:

THEOREM. *Let* $\Delta \geq 1$. *If*

(i) $\int_0^t \phi(u)du = o(t)$, *and*

(ii) $\int_{t^{1/\Delta}}^{\delta} \left|d\left(u^{-a}\phi(u)\right)\right| \leq At^{-a}$ *as* $t \to 0$

for an a , $0 < a < 1$ *and for a* δ , $0 < \delta < \pi$, *then the Fourier series of* $\phi(t)$ *is convergent at the origin.*

The object of this paper is to generalize the above theorem in the Hardy-Littlewood direction.

1.

Let $\phi(t)$ be an even periodic function which is integrable L and let

$$\phi(t) \sim \sum_{n=1}^{\infty} a_n \cos nt .$$

Sunouchi [4] generalized the Young-Pollard [3]·convergence criterion as follows:

THEOREM A. *The Fourier series of* $\phi(t)$ *converges at the point* $t = 0$ *to the value zero, provided that there is a* $\Delta \geq 1$ *such that*

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(1.1) $$\int_0^t \phi(u)du = o\left(t^\Delta\right)$$

and

(1.2) $$\int_0^t \left|d\left(u^\Delta\phi(u)\right)\right| = O(t) \, , 0 \le t \le \eta \, .$$

Recently we [2] proved the following theorem:

THEOREM B. *Let* $\Delta \ge 1$. *If the condition* (1.1) *holds and*

(1.3) $$\int_{t^{1/\Delta}}^{\delta} \left|d\left(u^{-a}\phi(u)\right)\right| = O\left(t^{-a}\right) \quad as \quad t \to 0$$

for an a *,* $0 < a < 1$ *and for a* δ *,* $0 < \delta < \pi$ *, then the Fourier series of* ϕ *is convergent at the origin.*

Hardy and Littlewood [1] generalized the condition (1.1) for the case $\Delta = 1$ in the form

$$\phi_\beta(t) = o\left(t^\gamma\right) \text{ , as } t \to 0$$

for any $\beta > 0$, where $\phi_\beta(t)$ is the β-th integral of $\phi(t)$. Corresponding to this result we prove the following theorem, which generalizes Theorem B in the Hardy-Littlewood direction.

THEOREM. *Let* $\Delta = \gamma/\beta \ge 1$ *and* $1 > \beta > 0$. *If*

(1.4) $$\phi_\beta(t) = o\left(t^\gamma\right)$$

where $\phi_\beta(t)$ *is the* β*-th integral of* $\phi(t)$ *, and further if*

(1.5) $$\int_{t^{1/\Delta}}^{\delta} \left|d\left(u^{-\eta}\phi(u)\right)\right| = O\left(t^{-\eta}\right) \, , \quad 1 > \eta > 0 \, ,$$

and $\Delta > 1$ *, then the Fourier series of* $\phi(t)$ *converges at* $t = 0$.

2.

Proof of the theorem. To prove our theorem we need to show that

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$$\int_0^\delta \phi(t)\,\frac{\sin nt}{t}\,dt = o(1) \quad \text{as} \quad n \to \infty .$$

Putting $n^{-1/\Delta} = \alpha$, we have

$$\int_0^\delta \phi(t)\,\frac{\sin nt}{t}\,dt = \int_0^\alpha \phi(t)\,\frac{\sin nt}{t}\,dt + \int_\alpha^\delta \phi(t)\,\frac{\sin nt}{t}\,dt$$
$$= I + J ,$$

say.

Putting $\theta(t) = t^{-\eta}\phi(t)$, then $\theta(t) = O\left(t^{-\eta\Delta}\right)$ by (1.5). Since

$$\Theta(t) = \int_t^\delta \frac{\sin nu}{u^{1-\eta}}\,du = O\left(\frac{1}{nt^{1-\eta}}\right) \quad \text{as} \quad t \to 0 ,$$

we get

$$J = \left[\theta(t)\Theta(t)\right]_\alpha^\delta - \int_\alpha^\delta \Theta(t)d\theta(t)$$
$$= O\left(\frac{1}{n^{(1-\eta)(1-1/\Delta)}}\right) + O\left(\frac{1}{n}\int_\alpha^\delta \frac{|d\theta(t)|}{t^{1-\eta}}\right)$$
$$= O\left(\frac{1}{n^{(1-\eta)(1-1/\Delta)}}\right) = o(1) \quad \text{as} \quad n \to \infty .$$

We shall now estimate I .

Putting $\Phi(t) = \int_0^t \phi(u)du$ and integrating by parts we have

$$I = \left[\Phi(t)\,\frac{\sin nt}{t}\right]_0^\alpha - \int_0^\alpha \Phi(t)\,\frac{nt\cos nt - \sin nt}{t^2}\,dt$$
$$= I_1 + I_2 ,$$

say. Since

$$\Phi(t) = o\left(t^{1+\gamma-\beta}\right) = o(t)$$

by (1.4), we get

$$I_1 = o(1) .$$

Finally

$$I_2 = \int_0^{\alpha} \frac{nt\cos nt - \sin nt}{t^2}\, dt \int_0^t \phi_\beta(t)(t-u)^{-\beta} du$$

$$= \int_0^{\alpha} \phi_\beta(u) du \int_u^{\alpha} \frac{nt\cos nt - \sin nt}{t^2} (t-u)^{-\beta} dt$$

where the inner integral becomes

$$n^{1+\beta} \int_{nu}^{n\alpha} \frac{\tau\cos\tau - \sin\tau}{\tau^2} (\tau - nu)^{-\beta} d\tau = O\left(\frac{n^\beta}{u}\right) .$$

Thus we get

$$I_2 = O\left(n^\beta \int_0^{\alpha} \frac{\phi_\beta(u)}{u} du\right)$$

$$= o\left(n^\beta \int_0^{\alpha} \frac{u^\gamma}{u} du\right)$$

$$= o(1) .$$

This completes the proof of the theorem.

References

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