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VARIETIES WITH MODULAR SUBALGEBRA LATTICES

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A classification is begun of varieties of algebras with the property that each algebra in the variety has a modular lattice of subalgebras. This turns out to be a very restrictive condition. Such a variety is hamiltonian. If the algebras in it are idempotent, then it is a variety of sets. A variety is subalgebra-modular if and only if it is hamiltonian and satisfies certain conditions on the terms in its three generator free algebras.

The lattice of subalgebras of an abelian group is modular, the lattice of subalgebras of any unary algebra is distributive, and there are varieties $\underline{\underline{V}}$ of algebras, for example, semigroups satisfying $xy = x$ and natural central groupoids [1], [5], where the lattice of subalgebras of any $\underline{\underline{V}}$-algebra is a boolean algebra. On the other hand, for the variety of groups, no non-trivial identity is common to all lattices of subalgebras. Is it possible to classify those (finitary) varieties in which every algebra has a modular lattice of subalgebras? In this paper, we take a first step towards such a classification by showing that subalgebra-modularity is indeed a very restrictive condition on a variety. Our main results are

(i) a subalgebra-modular variety is hamiltonian (hence, a

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subalgebra-modular variety of loops is a variety of abelian groups),

(ii) a subalgebra-modular variety of idempotent algebras is a variety of sets,

(iii) a variety is subalgebra-modular if and only if it is hamiltonian and satisfies certain "Malcev-type" conditions.

We remark that, where necessary, we allow the empty subalgebra.

LEMMA. *Let* $(A;\Omega)$ *be an algebra with a modular lattice of subalgebras, let* B *be a subalgebra of* A *and* y *an element such that* $A = \langle B \cup \{y\}\rangle$.

Then, for every element $p \in A$ *, we have*

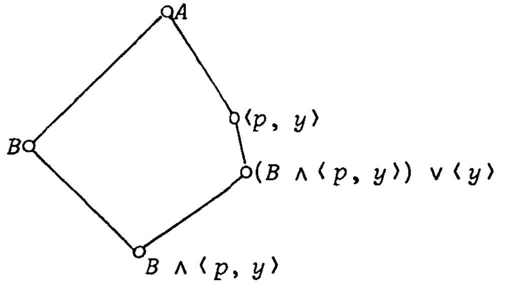
$$p \in (B \wedge \langle p, y\rangle) \vee \langle y\rangle .$$

(Here $\langle X\rangle$ denotes the subalgebra generated by the subset X .)

Proof. Consider the subalgebras

$$A,\ B,\ \langle p, y\rangle,\ B \wedge \langle p, y\rangle,\ (B \wedge \langle p, y\rangle) \vee \langle y\rangle .$$

They form a sublattice of the lattice of subalgebras, which cannot be the nonmodular lattice N_5 . Thus $\langle p, y\rangle = (B \wedge \langle p, y\rangle) \vee \langle y\rangle$, which proves the lemma.



A variety is *hamiltonian* if every nonempty subalgebra is a class of some congruence.

THEOREM 1. *If the lattice of subalgebras of every free algebra in a variety* $\underline{\underline{V}}$ *is modular, then* $\underline{\underline{V}}$ *is hamiltonian.*

Proof. Klukovits [3] has characterized the hamiltonian varieties as the varieties satisfying the following conditions: for every $(n+1)$-ary term $p(x_1, x_2, \ldots, x_n, y)$ there exists a ternary term $c_p(x, y, z)$ such that

$$c_p(p(x_1, x_2, \ldots, x_n, z), z, y) = p(x_1, \ldots, x_n, y) .$$

We construct such a $c_p(x, y, z)$, given $p(x_1, \ldots, x_n, y)$, by applying the lemma to $F_{n+1}(\underline{\underline{V}}) = \langle x_1, \ldots, x_n, y \rangle$, where $B = \langle x_1, \ldots, x_n \rangle$. Since $p \in (B \wedge \langle p, y \rangle) \vee \{y\}$, there are finitely many terms $a_1, \ldots, a_m$ in $B \wedge \langle p, y \rangle$ and an $(m+1)$-ary term q such that

$$p = q(a_1, \ldots, a_m, y) .$$

Each a_i is in $\langle p, y \rangle$, thus we find binary terms $q_1, \ldots, q_m$ such that $a_i = q_i(p, y)$. More precisely, we have

$$a_i(x_1, \ldots, x_n) = q_i(p(x_1, \ldots, x_n, y), y) ,$$

and since the a_i do not depend on y, we get

$$a_i(x_1, x_2, \ldots, x_n) = q_i(p(x_1, x_2, \ldots, x_n, z), z) .$$

Now let

$$c_p(x, z, y) = q(q_1(x, z), \ldots, q_m(x, z), y) ,$$

then

$$c_p(p(x_1, \ldots, x_n, z), z, y) = p(x_1, \ldots, x_n, y) .$$

COROLLARY. *A subalgebra-modular variety of loops is a variety of abelian groups.*

This follows from the fact that every hamiltonian variety of loops is a variety of abelian groups; Evans [2].

A direct proof of the corollary may be easily obtained. Let $\underline{\underline{V}}$ be a subalgebra-modular variety of loops and consider the following forbidden subalgebra lattice diagram.

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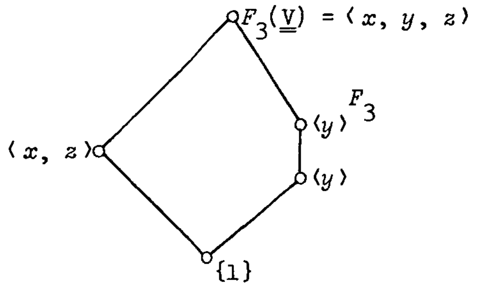


Here $\langle y \rangle^{F_3}$ denotes the normal closure of $\langle y \rangle$ in F_3. Since $\langle y \rangle^{F_3} = \langle y \rangle$, both xy/x and $xy \cdot z/x \cdot yz$ belong to $\langle y \rangle$. The commutative and associative laws follow immediately.

Not every hamiltonian variety is subalgebra-modular. Consider the variety of groupoids with operation $x \circ y$, defined by the identities

$$x \circ x = x ,$$

$$x \circ y = y \circ x ,$$

$$x \circ (x \circ y) = y ,$$

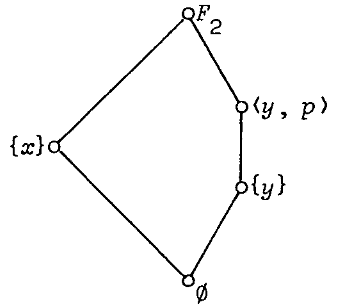
$$(x \circ y) \circ (z \circ w) = (x \circ z) \circ (y \circ w) .$$

The algebras in this variety are obtained from vector spaces over GF(3) where $x \circ y = 2x + 2y$ (see, for example, Ganter and Werner [*3*]). The variety is hamiltonian, but the free algebra on three generators is not subalgebra-modular.

Idempotency in an algebra is in fact an obstacle to subalgebra modularity and the example of semigroups satisfying $xy = x$ reflects the general situation.

THEOREM 2. *The only subalgebra-modular varieties of idempotent algebras are varieties of sets (that is, varieties where the only operations are projections).*

Proof. Let $\underline{\underline{V}}$ be a subalgebra-modular variety of idempotent algebras and let $F_2(\underline{\underline{V}}) = \langle x, y \rangle$ be the free $\underline{\underline{V}}$-algebra on two generators. If $F_2(\underline{\underline{V}})$ contains an element other than x and y, say $p(x, y)$, then the impossibility of the diagram below implies that F_2 is generated by y



and p. Hence, F_2 is generated by any two elements in it.

If $p \neq q$ are in F_2, then the endomorphism $x \to p$, $y \to q$ must have all of its congruence classes singletons (otherwise, one would be all of F_2) and so the endomorphism is actually an automorphism.

Continuing under the assumption that F_2 contains an element $p(x, y) \neq x, y$, the invertibility of the automorphism $x \to p(x, y)$, $y \to y$ implies that $p\big(r(x, y), y\big) = x$, $r\big(p(x, y), y\big) = x$ for some $r(x, y)$. Similarly, there is an element $s(x, y)$ such that $p\big(x, s(x, y)\big) = y$, $s\big(x, p(x, y)\big) = y$. Thus the derived operation $p(x, y)$ is a quasigroup operation on each algebra in $\underline{\underline{V}}$ and so congruences on $\underline{\underline{V}}$-algebras are regular; that is, congruence classes have the same cardinality.

Now consider the congruence on $F_3(\underline{\underline{V}}) = \langle x, y, z\rangle$ induced by the endomorphism $x \to x$, $y \to x$, $z \to z$. The congruence class containing x is the subalgebra generated by x and y and the congruence class containing z consists of one element only.

This is a contradiction and so we conclude that $F_2(\underline{\underline{V}})$ contains only x and y.

It is now easy to prove that $|F_n(\underline{\underline{V}})| = n$ by induction. If $F_{n+1}(\underline{\underline{V}}) = \langle x_1, x_2, \ldots, x_{n+1}\rangle$, map F_{n+1} onto $F_2 = \langle x, y\rangle$ by $x_1, x_2, \ldots, x_n \to x$, $x_{n+1} \to y$. The congruence classes are the subalgebras $\langle x_1, x_2, \ldots, x_n\rangle$ and $\{x_{n+1}\}$. Any element in F_{n+1} must lie in one of these two classes.

It follows that the only operations in $\underline{\underline{V}}$ are projections and so $\underline{\underline{V}}$ is a variety of sets.

We conclude with a theorem giving, in terms of Malcev-type conditions, a sufficient condition for a hamiltonian variety to be subalgebra-modular. We need the following property of hamiltonian varieties (the *2-generation* property) implicit in Winkler [6, Theorems 1, 4].

PROPOSITION. *If* B, C *are non-empty subalgebras of an algebra in a hamiltonian variety and* $x \in B \vee C$, *then there are elements* $b_1, b_2 \in B$, $c_1, c_2 \in C$ *such that* $x \in \langle\{b_1, b_2, c_1, c_2\}\rangle$.

THEOREM 3. *For a variety* $\underline{\underline{V}}$, *the following conditions are equivalent:*

(i) $\underline{\underline{V}}$ *is subalgebra-modular;*

(ii) $\underline{\underline{V}}$ *is hamiltonian and satisfies the following condition: for every ternary term* $t(x, y, z) \notin \langle z\rangle$ *(that is, which involves essentially at least one of* x, y*) in* $\underline{\underline{V}}$, *there are binary terms* b_1, b_2, c_1, c_2 *and a ternary term* q *such that, for* $i = 1, 2$,

$$t(x, y, z) = q\big(b_1(x, y), b_2(x, y), z\big),$$

$$b_i(x, y) = c_i\big(t(x, y, z), z\big)$$

hold in $\underline{\underline{V}}$.

Proof. Let $\underline{\underline{V}}$ be a subalgebra-modular, thus hamiltonian, variety, and let $t(x, y, z) \in F_3(\underline{\underline{V}})$. By the lemma,

$$t \in (\langle x, y\rangle \wedge \langle t, z\rangle) \vee \langle z\rangle,$$

and by the 2-generation property there are elements

$$b_1, b_2 \in \langle x, y\rangle \wedge \langle t, z\rangle$$

such that $t \in \langle b_1, b_2, z\rangle$.

Now assume $\underline{\underline{V}}$ is hamiltonian and satisfies the property stated in the theorem. Let B, C, C' be subalgebras such that $C \subseteq C'$, $B \wedge C = B \wedge C'$ and $B \vee C = B \vee C'$. We have to prove that $C = C'$. Let $p \in C'$.

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Since $C' \subseteq B \vee C$, we can apply the 2-generation property to find elements $b_1, b_2 \in B$, $c_1, c_2 \in C$ such that

$$p \in \langle b_1, b_2, c_1, c_2 \rangle = \langle b_1, b_2, c_1 \rangle \vee \langle c_2 \rangle .$$

Using the property again, we obtain elements $d_1, d_2 \in \langle b_1, b_2, c_1 \rangle$ such that $p \in \langle d_1, d_2 \rangle \vee \langle c_2 \rangle$. Now we apply the condition in the theorem to obtain e_1, e_2 in $\langle d_1, d_2 \rangle$ such that $p \in \langle e_1, e_2 \rangle \vee \langle c_2 \rangle$ and $e_i \in \langle p, c_2 \rangle$ for $i = 1, 2$. Note that $e_i \in C'$.

Since $e_i \in \langle d_1, d_2 \rangle \subseteq \langle b_1, b_2 \rangle \vee \langle c_1 \rangle$, we can use the property in the theorem again to find elements $f_{i1}, f_{i2} \in \langle b_1, b_2 \rangle$ such that $e_i \in \langle f_{i1}, f_{i2} \rangle \vee \langle c_1 \rangle$ and $\langle f_{i1}, f_{i2} \rangle \subseteq \langle c_1, e_i \rangle \subseteq C'$, $i = 1, 2$. Thus $f_{ij} \in B \wedge C \subseteq C$, and since $c_1, c_2 \in C$ we have

$$e_i \in \langle f_{i1}, f_{i2} \rangle \vee \langle c_1 \rangle \subseteq C .$$

Hence,

$$p \in \langle e_1, e_2 \rangle \vee \langle c_2 \rangle \subseteq C ,$$

and so $p \in C$ and $C = C'$.

References

[1] Trevor Evans, "Products of points - some simple algebras and their identities", *Amer. Math. Monthly* 74 (1967), 362-372.

[2] T. Evans, "Properties of algebras almost equivalent to identities", *J. London Math. Soc.* 37 (1962), 53-59.

[3] Bernhard Ganter and Heinrich Werner, "Equational classes of Steiner systems", *Algebra Universalis* 5 (1975), 125-140.

[4] L. Klukovits, "Hamiltonian varieties of universal algebras", *Acta Sci. Math (Szeged)* 37 (1975), 11-15.

[5] Donald E. Knuth, "Notes on central groupoids", *J. Combin. Theory* 8 (1970), 376-390.

[6] P.M. Winkler, "Polynomial hyperforms", *Algebra Universalis* (to appear).

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