Can. J. Math., Vol XXXVI, No. 6, 1984, pp. 961-972

OUTER DERIVATIONS AND CLASSICAL-ALBERT-ZASSENHAUS LIE ALGEBRAS

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1. Introduction. This paper is concerned with the structure of the derivation algebra Der L of the Lie algebra L with split Cartan subalgebra H. The Fitting decomposition

$$\text{Der } L = D_0(H) + D_*(H)$$

of Der $L$with respect to ad ad H leads to a decomposition

$$\text{Der } L = D_0(H) + \text{ad } L^\infty$$

where

$$L^\infty = \bigcap_{i=1}^{\infty} L^i.$$

This decomposition is studied in detail in Section 2, where the centralizer of ad L^∞ in $D_0(H)$ is shown to be

$$\text{Der}(L, H) = \text{Der}(L/L^\infty, C_H(L^\infty)),$$

which is Hom(L/L^2, Center L) when H is Abelian. When the root-spaces L_a (a nonzero) are one-dimensional, this leads to the decomposition of Der L as

$$\text{Der } L = T + \text{Der}(L/L^\infty, C_H(L^\infty)) + \text{ad } L$$

where T is any maximal torus of $D_0(H)$.

In Section 3, we determine Der L explicitly for extended classical-Albert-Zassenhaus Lie algebras (defined below) in terms of the dual

$$R^* = \text{Hom}(R, k)$$
$$= \{f:R \to k | f(a + b) = f(a) + f(b) \text{ for all } a, b, a + b \in R\}$$

of the rootsystem R of L with respect to H. For classical Lie algebras, this is a consequence of the Block [**4**] theory of trace forms. It is shown there that all derivations of a classical Lie algebra L are inner if and only if L has no component of type $A_r(p/r + 1)$; and for characteristic $p > 5$, if

Received May 7, 1980 and in revised form November 20, 1983 and May 19, 1984.

https://doi.org/10.4153/CJM-1984-054-7 Published online by Cambridge University Press

and only if L has a faithful representation with nondegenerate trace form.

Throughout the paper, L is a finite dimensional Lie algebra over a field k of characteristic $p > 3$ with split Cartan sub-algebra H and Cartan decomposition

$$L = \sum_{a \in R} L_a.$$

Following the conventions of Winter [**11**], we let $R = R_1 \cup \ldots \cup R_n$ be the *irreducible component decomposition* of R, that is:

1. $R = R_1 \cup \ldots \cup R_n$ with $R_i \neq \{0\}$ for $1 \leqq i \leqq n$;
2. $R_i \cap R_j = \{0\}$ for all $i \neq j$;
3. if $a_i \in R(1 \leqq i \leqq n)$ and $a_1 + \ldots + a_n \in R$,

then at most one of the $a_i(1 \leqq i \leqq n)$ is nonzero.

If $R = R_1$, R is *irreducible*. We say that R is *classical* if there exists an isomorphism $f:R \rightarrow \hat{R}$ from R to the rootsystem $\hat{R}$ of some complex semisimple Lie algebra. Here, $f:R \rightarrow \hat{R}$ is an *isomorphism* if:

1. f is a bijection from R to $\hat{R}$;
2. $a + b \in R$ if and only if $f(a) + f(b) \in \hat{R}$ for all $a, b, \in R$;
3. $f(a + b) = f(a) + f(b)$ for all $a, b \in R$.

If R is an additive subgroup of $ka(a \in R - \{0\})$, we say that R is *Albert-Zassenhaus*. If each irreducible component R_i of R is either classical or Albert-Zassenhaus, we say that R is *classical-Albert-Zassenhaus*.

We say that a Lie algebra L with Cartan subalgebra H is *classical Albert-Zassenhaus* with *classical-Albert-Zassenhaus Cartan subalgebra H* if:

1. $L = L^2$ and Center $L = \{0\}$;
2. dim $L_a = 1$ and $a([L_a, L_{-a}]) \neq 0$ for all $a \in R - \{0\}$;
3. R is a classical-Albert-Zassenhaus rootsystem and

$$[L_a, L_b] = L_{a+b} \text{ for all } a, b, a + b \in R - \{0\}.$$

If L satisfies only conditions (2) and (3) above, L is called an *extended classical-Albert-Zassenhaus Lie algebra* with *extended classical-Albert-Zassenhaus Cartan subalgebra H*. Henceforth, we abbreviate "classical-Albert-Zassenhaus" by "CAZ" and "Albert-Zassenhaus" by "AZ".

Those CAZ Lie algebras for which R is classical are the classical Lie algebras of Seligman [**8**], by Mills-Seligman [**6**] and Mills [**5**].

Let us next consider the CAZ Lie algebras (L, H) for which R is AZ. Letting R be an additive subgroup of ka for some $a \in R - \{0\}$, observe that a is linear on H, since the root-spaces are one dimensional; and that

$$H_a = \{h \in H | a(h) = 0\}$$

is $\{0\}$, since it centralizes all root-spaces and since the latter generate

$L^\infty = L$. It follows that dim $H = 1$ and (L, H) is of rank 1. We may conclude that those CAZ Lie algebras for which R is AZ are the AZ Lie algebras of Albert-Frank [**1**] relative to one dimensional Cartan subalgebras, by Block [**2**]. Although Albert Zassenhaus Lie algebras have Cartan subalgebras of dimension greater than 1, the above considerations show that such other Cartan subalgebras do not arise as CAZ Cartan subalgebras when R is AZ.

For $p > 5$, the class of CAZ Lie algebras is the class of Lie algebras satisfying conditions (1) and (2), by Block [**3**].

The class of extended CAZ Lie algebras coincides with the class of those symmetric Lie algebras of Winter [**11**] whose root-spaces L_a (a nonzero) are one-dimensional.

On concluding the introduction, we note that although the main results of this paper are proved only for characteristics $p > 3$, all results up to Corollary 2.8 are valid over fields of any characteristic.

2. The derivation algebra of a Lie algebra with given Cartan subalgebra. To determine the derivation algebra $D = \text{Der } L$ of a Lie algebra L, we consider its Fitting decomposition $D = D_0(H) + D_*(H)$ with respect to ad ad H, H being a given Cartan subalgebra of L. Since the ideal ad L of D contains $[D, \text{ad } H]$, ad L contains $D_*(H)$. Consequently,

$$D_*(H) = \text{ad } L_*(\text{ad } H) \subset \text{ad } L^\infty$$

where $L_*(\text{ad } H)$ is the Fitting one space of L with respect to ad H and

$$L^\infty = \bigcap_{i=1}^{\infty} L^i.$$

This proves the following proposition, which reduces determination of Der L to that of $D_0(H)$.

2.1 PROPOSITION.

$$\text{Der } L = D_0(H) + \text{ad } L_*(\text{ad } H) = D_0(H) + \text{ad } L^\infty.$$

The *outer derivation algebra* Der L/ad L can now be described as

$$(D_0(H) + \text{ad } L)/\text{ad } L = D_0(H)/(D_0(H) \cap \text{ad } L).$$

Since $D_0(H) \cap \text{ad } L = \text{ad } H$, which is an ideal of the algebra $D_0(H)$, this establishes the following corollary, which characterizes the outer derivation algebra up to isomorphism of algebras.

2.2. COROLLARY Der L /ad $L = D_0(H)/\text{ad } H$.

The ideal

$$C(\text{ad } L^\infty) = \{d \in D | [d, \text{ad } L^\infty] = \{0\}\}$$

of Der L plays an important role in what follows. Its Fitting decomposition relative to ad ad H, obtained by restriction to $C(\text{ad } L^\infty)$ of that of Der L, is

$$C(\text{ad } L^\infty) = C(\text{ad } L^\infty) \cap D_0(H) + C(\text{ad } L^\infty) \cap \text{ad } L_*(\text{ad } H).$$

This decomposition leads to the following description of $C(\text{ad } L^\infty)$ in terms of

$$D(L, H) = \{d \in D | d(L) \subset H\},$$

the center Center ad L^∞ of ad L^∞ and the centralizer

$$C_H(L^\infty) = \{h \in H | [h, L^\infty] = 0\}$$

of L^∞ in H.

2.3. PROPOSITION. *$C(\text{ad } L^\infty) \cap D_0(H) = D(L, H)$, $D(L, H)$, is an ideal of* Der L *and*

$$C(\text{ad } L^\infty) = D(L, H) + \textit{Center} \text{ ad } L^\infty$$

with

$$D(L, H) \cap \text{ad } L = \text{ad } C_H(L^\infty).$$

Proof. We first show that

$$C(\text{ad } L^\infty) \cap D_0(H) = D(L, H).$$

For this, suppose that $d \in C(\text{ad } L^\infty) \cap D_0(H)$. Then $d(H) \subset H$ and

$$0 = [d, \text{ad } L^\infty] = \text{ad } d(L^\infty).$$

It follows that

$$d(L^\infty) \subset \text{Center } L \subset H \quad \text{and}$$

$$d(L) = d(H + L^\infty) = d(H) + d(L^\infty) \subset H.$$

Next, suppose conversely that $d \in D(L, H)$. Observe for $h \in H$ that

$$[d, \text{ad } h] = \text{ad } d(h) \in \text{ad } H,$$

since $d(h) \in H$. Then

$$[d, \text{ad } H] = \text{ad } d(H) \subset \text{ad } H$$

and, therefore, $d \in D_0(H)$. It follows that d stabilizes the $L_a (a \in R)$, since $d \in D_0(H)$; and maps them into H, since $d \in D(L, H)$. Thus, $d(L_a) = 0$ for $a \in R - \{0\}$. Since L^∞ is generated by the $L_a (a \in R - \{0\})$, $d(L^\infty) = \{0\}$. Thus,

$$d \in C(\text{ad } L^\infty) \cap D_0(H).$$

To see that $D(L, H)$ is an ideal of Der L, observe that

$$[D(L, H), \text{Der } L] = [D(L, H), D_0(H)] + [D(L, H), \text{ad } L^\infty]$$
$$= [D(L, H), D_0(H)],$$

by Proposition 2.1 and the foregoing discussion. Thus, it suffices to show that $D(L, H)$ is an ideal of $D_0(H)$, which is easily verified.

We now have

$$C(\text{ad } L^\infty) = C(\text{ad } L^\infty) \cap D_0(H) + C(\text{ad } L^\infty) \cap \text{ad } L_*(\text{ad } H)$$
$$= D(L, H) + \text{Center ad } L^\infty.$$

It remains only to show that

$$D(L, H) \cap \text{ad } L = \text{ad } C_H(L^\infty).$$

Clearly,

$$D(L, H) \cap \text{ad } L \supset \text{ad } C_H(L^\infty).$$

Thus, it suffices to show that ad $C_H(L^\infty)$ contains every element ad x of $D(L, H) \cap$ ad L. Since such an ad x stabilizes H, x is in H. Since ad x is in $D(L, H)$,

$$\text{ad } x(L^\infty) = 0,$$

as observed above. Thus, x is in $C_H(\text{ad } L^\infty)$ as asserted.

Proposition 2.3 shows that the contribution of $C(\text{ad } L^\infty)$ to the outer derivation algebra Der $L/$ad L is

$$(D(L, H) + \text{ad } L)/\text{ad } L = D(L, H)/(D(L, H) \cap \text{ad } L)$$
$$= D(L, H)/\text{ad } C_H(L^\infty).$$

The following proposition describes $D(L, H)$ as

$$D(L/L^\infty, C_H(L^\infty)) = \{d \in D/d(L^\infty) = 0 \text{ and } d(L) \subset C_H(L^\infty)\}.$$

Since the latter is canonically isomorphic to

$$D(H/H_\infty, C_H(L^\infty)) = \{d \in \text{Der } H | d(H_\infty) = 0 \text{ and }$$
$$d(H) \subset C_H(L^\infty)\}$$

where $H_\infty = H \cap L^\infty$, the problem of determining $D(L, H)$ is a problem concerning derivations of a nilpotent Lie algebra H annihilating a specified ideal H_∞ of H and taking on values in a specified subalgebra $C_H(L^\infty)$ of H centralizing H_∞.

2.4. PROPOSITION. $D(L, H) = D(L/L^\infty, C_H(L^\infty))$.

Proof. One inclusion is clear. For the other, let $d \in D(L, H)$. Then

$$H \supset d(L) = d(H + L^{\infty}) = d(H) + 0 = d(H).$$

Therefore,

$$[d(L), L_a] = [d(H), L_a] \subset d([H, L_a]) + [H, d(L_a)]$$
$$= 0 + 0 \ (a \in R - \{0\}).$$

It follows that $d(L) \subset C_H(L^{\infty})$, as asserted.

When H is Abelian, $C_H(L^{\infty})$ is the center of L, in which case

$$D(L, H) = D(L/L^{\infty}, C_H(L^{\infty}))$$

is just the space $\mathrm{Hom}(L/L^2, \text{Center } L)$ of homomorphisms of L to Center L vanishing on L^2.

2.5. COROLLARY. *If H is Abelian,*

$$D(L, H) = \mathrm{Hom}(L/L^2, \textit{Center } L).$$

When L has center 0, $D(L, H)$ is Abelian, as we show in Proposition 2.7 below, using the following result.

2.6. PROPOSITION. ([**7**]). *Let L have center* 0. *Then every element of L centralizing L^{∞} is contained in L^{∞}.*

2.7 PROPOSITION. *Let L have center* 0. *Then $D(L, H)$ is an Abelian ideal of* Der L *and $de = 0$ for all $d, e \in D(L, H)$.*

Proof. Since $C_H(L^{\infty}) \subset L^{\infty}$, by Proposition 2.6, and

$$D(L, H) = D(L/L^{\infty}, C_H(L^{\infty})),$$

by Proposition 2.4, we have

$$de(L) \subset d(C_H(L^{\infty})) \subset d(L^{\infty}) = 0$$

for all $d, e \in D(L, H)$.

2.8. COROLLARY. *Suppose that L is semisimple or $L^2 = L$. Then* $D(L, H) = 0$.

Proof. If $L^2 = L$, this follows from Proposition 2.4. If L is semisimple, then $D(L, H)$ is an Abelian ideal of Der L, by Proposition 2.7, so that

$$D(L, H) \cap \mathrm{ad}\, L = 0 \quad \text{and}$$

$$0 = [d, \mathrm{ad}\, L] = \mathrm{ad}\, d(L) \quad \text{for all } d \in D(L, H).$$

Since Center $L = 0$, it follows that $D(L, H) = 0$.

We now consider Lie algebras L whose root-spaces L_a ($a \in R - \{0\}$) are one-dimensional. We let T be a maximal torus of the Lie p-algebra $D_0(H)$ in the sense of Winter [**9**] and proceed to show that

$D_0(H) = T + D(L, H)$. By the methods of ascent and descent, we may assume with no loss of generality that T is split. Consider the eigenspace decompositions $D_0(H) = \sum D_0(H)_b$, $L = \sum L_c$ where

$$D_0(H)_b = \{d \in D_0(H) \mid [t, d] = b(t)d \text{ for all } t \in T\},$$

$$L_c = \{x \in L \mid t(x) = c(t)x \text{ for all } t \in T\}$$

for k-valued functions b, c on T. Since d^{p^e} maps each L_c to $L_{c+p^e b} = L_c$ for $d \in D_0(H)_b$, $D_0(H)_0$ contains d^{p^e} *for* $e \geqq 1$. Taking $p^e \geqq \dim L$, d^{p^e} is semisimple and centralizes the maximal torus T of $D_0(H)$, so that $d^{p^e} \in T$. We consider the Jordan decomposition $d = d_s + d_n$ of d and properties of it discussed in [**9**]. If d is semisimple, that is, $d = d_s$, *then* d is contained in the span of $d^p, d^{p^2}, \ldots$; and, therefore, in the span of $d^{p^i}, d^{p^{i+1}}, \ldots$ for $i \geqq 1$. It follows that d_s is contained in the span of $d^{p^e}, d^{p^{e+1}}, \ldots$, since $d_s^{p^e} = d^{p^e}$. *Since* $d^{p^e} \in T$, it follows that $d_s \in T$. Since T is split, it follows that d is split and that T contains the semisimple part d_s of d. Letting $d_n = d - d_s$ be the nilpotent part of d, d_n stabilizes the one-dimensional spaces L_a ($a \in R - \{0\}$). By the nilpotency of d_n, we have $d_n(L_a) = 0$ ($a \in R - \{0\}$), so that $d_n(L^\infty) = 0$. But then

$$d_n(L) = d_n(H + L^\infty) = d_n(H) \subset H \text{ and } d_n \in D(L, H).$$

It follows that $d \in T + D(L, H)$. Thus, $D_0(H) = T + D(L, H)$, which establishes part of the following theorem.

2.9. THEOREM. *Suppose that the root-spaces L_a ($a \in R - \{0\}$) of L are all one-dimensional and let T be a maximal torus of $D_0(H)$. Then*

$$D_0(H) = T + D(L/L^\infty, C_H(L^\infty)) \quad \textit{and}$$

$$\operatorname{Der} L = T + D(L/L^\infty, C_H(L^\infty)) + \operatorname{ad} L.$$

Moreover:

1. *if H is Abelian, then*

$$D_0(H) = T + \operatorname{Hom}(L/L^2, \textit{Center } L),$$

2. *if L has center* 0 *or* $L^2 = L$, *then $D_0(H)$ is a torus T and*

$$\operatorname{Der} L = T + \operatorname{ad} L \quad \textit{with } T \cap \operatorname{ad} L = \operatorname{ad} H.$$

Proof. The foregoing discussion establishes that

$$D_0(H) = T + D(L, H),$$

so that

$$D_0(H) = T + D(L/L^\infty, C_H(L^\infty)),$$

by Proposition 2.4. From this, (1) follows immediately, as does (2) in the case $L^2 = L$. It remains to establish (2) when L has center 0. By (1), it suffices to show that H is abelian. Suppose to the contrary that it is not, and choose $x, y \in H$ with $[x, y] \neq 0$ and $[x, y]$ central in H. Since ad x, ad y stabilize the one-dimensional spaces L_a ($a \in R - \{0\}$), we have

$$0 = [\text{ad } x, \text{ad } y](L_a) = \text{ad } [x, y](L_a)(a \in R - \{0\})$$

and

$$[[x, y], L^\infty] = 0.$$

But then

$$[[x, y], L] = [[x, y], H] + [[x, y], L^\infty] = 0 + 0;$$

since $[x, y]$ centralizes H, so that $[x, y] \in \text{Center } L = \{0\}$, a contradiction. We conclude that H is Abelian, as asserted.

3. The derivation algebra of an extended classical-Albert Zassenhaus Lie algebra. Let

$$L = \sum_{a \in R} L_a$$

be an extended CAZ Lie algebra with extended CAZ Cartan subalgebra H. Let $d \in D_0(H)$, so that d is a derivation of L stabilizing the one dimensional spaces L_a ($a \in R - \{0\}$). Then d determines scalars $f(a) \in k$ ($a \in R - \{0\}$) such that

$$d(e_a) = f(a)e_a$$

where

$$L_a = ke_a \ (a \in R - \{0\}).$$

Let $f(0) = 0$. We then claim that

$$f \in R^* = \text{Hom}(R, k),$$

that is,

$$f(a + b) = f(a) + f(b) \quad \text{for all } a, b, a + b \in R.$$

If $a = 0$ or $b = 0$, this is trivial. Consider next the case $a, b, a + b \in R - \{0\}$. Then

$$L_{a+b} = k[e_a, e_b],$$

so that

$$\begin{aligned} f(a + b)[e_a, e_b] &= d([e_a, e_b]) \\ &= [d(e_a), e_b] + [e_a, d(e_b)] \\ &= (f(a) + f(b))[e_a, e_b]. \end{aligned}$$

Since $[e_a, e_b] \neq 0$, it follows

$$f(a + b) = f(a) + f(b).$$

Finally, consider whether $f(a) + f(-a) = 0$ $(a \in R - \{0\})$. Let

$$h_a = [e_a, e_{-a}],$$

so that

$$[h_a, e_a] = a(h_a)e_a \quad \text{with } a(h_a) \neq 0.$$

Then

$$f(a)[h_a, e_a] = d([h_a, e_a]) = [d(h_a), e_a] + [h_a, d(e_a)].$$

Since

$$d(h_a) = d([e_a, e_{-a}]) = (f(a) + f(-a))[e_a, e_{-a}]$$

and

$$d(e_a) = f(a)e_a,$$

we conclude that

$$f(a)[h_a, e_a] = (f(a) + f(-a) + f(a))[h_a, e_a].$$

Since $[h_a, e_a] \neq 0$, it follows that $f(a) + f(-a) = 0$.

Observe next that each $f \in R^*$ determines a derivation d_f defined by

$$d_f(H) = 0 \quad \text{and} \quad d_f(e_a) = f(a)e_a.$$

We let $T = T(H) = \{d_f | f \in R^*\}$ and note that T is a torus in $D_0(H)$. We claim that

$$D_0(H) = T \oplus D(L, H),$$

where $D(L, H)$ is the ideal of Der L defined in Section 2. For this, let $d \in D_0(H)$ and take the corresponding $f \in R^*$ constructed above by the condition.

$$d(e_a) = f(a)e_a \ (a \in R - \{0\}).$$

Let $d_H = d - d_f$ and note that

$$d_H(e_a) = 0 \ (a \in R - \{0\}).$$

Then $d_H \in D(L, H)$, so that

$$d = d_f + d_H \in T + D(L, H).$$

We conclude that

$$D_0(H) = T + D(L, H).$$

Finally, suppose that $d_f \in T \cap D(L, H)$. Then

$$0 = d_f(e_a) = f(a)e_a \quad \text{and} \quad f(a) = 0 \ (a \in R - \{0\}),$$

so that $f = 0$ and $d_f = 0$. Thus,

$$D_0(H) = T \oplus D(L, H).$$

The foregoing discussion and Proposition 2.4, which characterizes $D(L, H)$, establish the following theorem, since

$$\text{Der } L = D_0(H) + \text{ad } L^\infty.$$

3.1. THEOREM. *Let* $L = \sum_{a \in R} L_a$ *be an extended* CAZ *with extended* CAZ H. *Then*

$$\text{Der } L = (T(H) \oplus D(L/L^\infty, C_H(L^\infty))) + \text{ad } L.$$

For H Abelian, $D(L/L^\infty, C_H(L^\infty))$ is given explicitly by

$$\text{Hom}(L/L^2, \text{Center } L).$$

3.2. COROLLARY. *Let* $L = \sum_{a \in R} L_a$ *be an extended* CAZ *with extended* CAZ H. *Suppose that* H *is Abelian and either* $L = L^2$ *or Center* $L = 0$. *Then*

$$\text{Der } L = T(H) + \text{ad } L \text{ and } \text{Der } L/\text{ad } L = T(H)/\text{ad } H.$$

3.3. COROLLARY. Der $L = T(H) + \text{ad } L$ *for any* CAZ *Lie algebra* L *with* CAZ *Cartan subalgebra* H.

Finally, we determine $T(H) = \{d_f | f \in R^*\}$. By construction, this reduces to determining $R^* = \text{Hom}(R, k)$. For this, we construct a base $a_1, \ldots, a_r$ for the CAZ rootsystem R by expressing R as union $R = R_1 \cup \ldots \cup R_n$ of its irreducible components and taking $a_1, \ldots, a_{r_1}$ to be base for R_1, $a_{r_1+1}, \ldots, a_{r_1+r_2}$ to be base for $R_2, \ldots,$ and $a_{r_1+\ldots+r_{n-1}+1}, \ldots, a_{r_1+\ldots+r_n}$ to be base for R_n. Here, a base for an irreducible CAZ rootsystem R is a subset $\pi = \{a_1, \ldots, a_r\}$ of R such that:

(1) if R is a classical and $R \to \hat{R}$ is an isomorphism from R to the rootsystem $\hat{R}$ of a complex semisimple Lie algebra, then $\hat{a}_1, \ldots, \hat{a}_r$ is a base for $\hat{R}$;

(2) if R is Albert-Zassenhaus, then R is the direct sum

$$R = \mathbf{Z}_p a_1 \oplus \ldots \oplus \mathbf{Z}_p a_r$$

where $\mathbf{Z}_p$ is the prime subfield of k. We let rank R denote the cardinality of a base π of a CAZ rootsystem R.

Each base $\pi = \{a_1, \ldots, a_r\}$ for a CAZ rootsystem R uniquely determines a dual base $\pi^* = \{f_1, \ldots, f_r\}$ satisfying the following conditions:

1. $\pi^* \subset R^* = \text{Hom}(R, k)$;

2. $f_i(a_j) = \delta_{ij}$.

Clearly, such a π^* is a basis for R^* over k. To see that π^* exists, note that it suffices to show that π^* exists for every irreducible component R_i of R, since any element $f \in \text{Hom}(R_i, k)$ can be regarded as an element of $\text{Hom}(R, k)$ by taking $f(R_j) = \{0\}$ for all $j \neq i$. Next, it is clear that π^* exists by decree for any Albert-Zassenhaus rootsystem. Finally, let R be a classical rootsystem and $\Lambda:R \rightarrow \hat{R}$ an isomorphism from R to the rootsystem of a complex semisimple Lie algebra. Then there exist homomorphisms

$$\hat{f}_i:\mathbf{Z}\hat{R} \rightarrow \mathbf{Z}$$

defined by the condition

$$\hat{f}_i(\hat{a}_j) = \delta_{ij}.$$

We may, therefore, define $f_i \in \text{Hom}(R, k)$ by letting $f_i(a)$ be $\hat{f}_i(\hat{a})$ reduced modulo p. The resulting $\pi^* = \{f_1, \dots, f_r\}$ is a dual base to $\pi = \{a_1, \dots, a_r\}$.

The foregoing discussion establishes the following theorem, since it shows that

$$\dim T(H) = \dim R^* = \text{cardinality of } \pi^* = \text{rank } R.$$

3.4. THEOREM. *Let L be an extended Lie algebra with extended* CAZ *Cartan subalgebra H. Then*

$$\text{Der } L = T(H) + D(L, H) + \text{ad } L$$

where

$$\dim T(H) = \text{rank } R \quad \textit{and} \quad D(L, H) = D(L/L^\infty, C_H(L^\infty)).$$

3.5. THEOREM. *Let L be a* CAZ *Lie algebra with* CAZ *Cartan subalgebra H. Then $D_0(H)$ is a torus and Cartan subalgebra of* Der L *of dimension equal to rank R.*

Let L be a CAZ Lie algebra with CAZ Cartan subalgebra H, as defined in Section 1. The dimension of H is one if L is Albert-Zassenhaus. The dimension of H is rank $R - 1$ if L is classical of type A_r where $p/r + 1$, and it is rank R otherwise, by [**8**]. Letting R_{AZ} be the union of those irreducible components of R which are Albert-Zassenhaus and R_{C} be the union of those irreducible components of R which are classical, these observations can be expressed as follows.

3.6. THEOREM. *Let L be a* CAZ *Lie algebra with* CAZ *Cartan subalgebra H. Then*

$$\dim \text{Der } L/\text{ad } L = a - b + \text{rank } R_{\text{AZ}}$$

where a is the number of irreducible components of R_C of type $A_r(p/r + 1)$ and b is the number of irreducible components of R_{AZ}.

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