THE UNIQUENESS OF BOUNDED OR MEASURABLE SOLUTIONS OF SOME FUNCTIONAL EQUATIONS

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In this paper we are concerned with the uniqueness of solutions of functional equations of the form

$$(1) \qquad g(F(x, y)) = H(f(x), f(y); x, y),$$

$$(2) \qquad f(F(x, y)) = H(f(x), f(y); x, y).$$

Some conditions for (1) or (2) to have at most one real continuous solution f which satisfies two given initial conditions are contained in [2], [3], [4] and [7]. Conditions sufficient for the equation

$$f(x+y) = G(f(x), f(y))$$

to determine at most one continuous solution f with values in a Banach algebra are contained in [5]. It is well known (see [1] ch. 2) that one initial condition suffices for Cauchy's equation

$$f(x+y) = f(x)+f(y)$$

or two for Jensen's equation

$$f(\tfrac{1}{2}x+\tfrac{1}{2}y) = \tfrac{1}{2}f(x)+\tfrac{1}{2}f(y)$$

to uniquely determine a real solution f which is bounded on an interval or majorized on a set of positive measure by a measurable function. We place conditions on F and H so that similar statements can be made about solutions of (1) or (2). The corresponding results for solutions which are functions of many real variables follow as for Cauchy's and Jensen's equations (see [1] ch. 5).

In the following (Δ, d) is a metric space; $D \subset \Delta$; C is the complex plane; R is the real line; R^n is n-dimensional Euclidean space; I is a real interval or a convex set in R^n. If $f : I \to \Delta$ is bounded in a neighbourhood of each point of I we say f is locally bounded in I. If $D = R$ or C we take $d(u, v) = |u-v|$, $u, v \in D$. We use a square (round) bracket to denote a closed (open) end of an interval.

THEOREM 1. *Let I be an interval. Let $F : I \times I \to R$ be continuous and* *strictly increasing (or decreasing). Let* $H : D \times D \times I \times I \to \Delta$ *be such that*

(H$_1$) $$H(u, v; x, y) = H(U, v; x, y) \text{ iff } u = U,$$

or

$$H(u, v; x, y) = H(u, V; x, y) \text{ iff } v = V;$$

(H$_2$) *corresponding to every bounded set* $J \subset I$ *there exists* $r \in [\frac{1}{2}, 1)$ *and* $s > 0$ *such that in* $D \times D \times J \times J$

$$d\{H(u, v; x, y), H(U, V; x, y)\} \leqq rs\{d(u, U) + d(v, V)\}, \tag{3}$$

$$d\{H(u, u; x, x), H(v, v; x, x)\} \geqq sd(u, v). \tag{4}$$

Let $a, b \in I$ *and* $a < b$. *Then corresponding to any given* $f(a)$ *and* $f(b)$ *there is at most one pair of functions* $f : I \to D$ *and* $g : \{F(x, y) : x, y \in I\} \to \Delta$ *such that* f *is locally bounded in* I *and* (1) *holds.*

Proof. Let $z = G(x, y)$ be the unique solution of $F(z, z) = F(x, y)$. If $F(x, x) \equiv x$ (F is reflexive), then $G = F$. Let $\alpha_1(x) = G(a, x)$ and $\alpha_2(x) = G(x, b)$, $x \in I$; then α_1 and α_2 are continuous, strictly increasing, $\alpha_1(a) = a$, $\alpha_1(b) = \alpha_2(a)$, $\alpha_2(b) = b$. Hence α_1 and α_2 together map $[a, b]$ onto $[a, b]$.

Let $H(u, u; x, x) = h(u; x)$. If (1) holds in I the substitution of $G(x, y)$ for both x and y shows that

$$h\{f(G(x, y)); G(x, y)\} = H(f(x), f(y); x, y), \; x, y \in I.$$

If $f(a) = A$ and $f(b) = B$ then

$$h\{f(\alpha_1(x)); \alpha_1(x)\} = H(A, f(x); a, x), \; x \in I, \tag{5}$$

$$h\{f(\alpha_2(x)); \alpha_2(x)\} = H(f(x), B; x, b), \; x \in I. \tag{6}$$

Let f_1 and f_2 satisfy (5) and (6) and be locally bounded in I. If $x_0 \in [a, b]$ and n is a positive integer there exist $p_1, \cdots, p_n \in \{1, 2\}$ and $x_1, \cdots, x_n \in [a, b]$ such that $x_0 = \alpha_{p_1}(x_1)$, $x_k = \alpha_{p_{k+1}}(x_{k+1})$, $k = 1, \cdots, n-1$, and hence (5), (6) and (H$_2$) imply that there exist $r \in [\frac{1}{2}, 1)$ and $s > 0$ such that

$$\begin{aligned} sd\{f_1(x_0), f_2(x_0)\} &\leqq d\{h(f_1(x_0); x_0), h(f_2(x_0); x_0)\} \\ &\leqq rs\, d\{f_1(x_1), f_2(x_1)\} \\ &\leqq r^n s\, d\{f_1(x_n), f_2(x_n)\} \end{aligned}$$

which decreases to 0 as $n \to \infty$ since f_1 and f_2 are bounded on $[a, b]$. Hence $f_1 = f_2$ in $[a, b]$. As in [7] the condition (H$_1$) ensures that $f_1 = f_2$ in I.

Remarks. If $I = [a, b]$ the condition (H$_1$) is unnecessary; and if also F is reflexive then F need only be non-decreasing (or non-increasing). On the other hand if (1) is replaced by (2) and $a \neq F(a, a) \in I$ then the condition $f(b) = B$ is redundant (cf. [4], [6]) — this remark also applies to the next theorem. If $H(u, u; x, x) \equiv u$ then (3) and (4) are equivalent to (3) with $s = 1$.

THEOREM 2. *Let I be an interval. Let F satisfy the hypotheses of Theorem 1 and corresponding to each bounded set $J \subset I$ let there exist positive constants k, K such that in $J \times J$*

$$k|x-y| \leqq |F(x, z)-F(y, z)| \leqq K|x-y|. \tag{7}$$

Let $D = R$ or C. Let H: $D \times D \times I \times I \to D$ be continuous, satisfy (H_2) *and*

(H_3) *there exists a continuous function G: $D \times D \times I \times I \to D$ such that*

$$G(u, H(u, v; x, y); x, y) \equiv v.$$

Let $a, b \in I$ and $a < b$. Then corresponding to any given $f(a)$ and $f(b)$ there is at most one pair of functions $f : I \to D$ and g: $\{F(x, y) : x, y \in I\} \to D$ such that $|f|$ has a measurable majorant on a subset of I of positive measure and (1) *holds.*

PROOF. Let f and g satisfy the given hypotheses. If $S \subset I$ and $t \in I$ let $F(S, t) = \{F(x, t) : x \in S\}$ and mS denote the Lebesgue measure of S (if measurable). There exists a closed and bounded set $E \subset I$ such that $mE > 0$ and f is bounded on E. Let I_1 be a closed interval such that $E \subset I_1 \subset I$. Then $F(E, a) \subset F(I_1, a)$ and $mF(E, a) > 0$. Hence there exists a closed interval $I_0 \subset I_1$ such that

$$m\{F(E, a) \cap F(I_0, a)\} \geqq \tfrac{2}{3} mF(I_0, a) > 0. \tag{8}$$

Let J be a closed interval, $I_0 \cup [a, b] \subset J \subset I$; then there exists $K > 0$ such that (7) holds. Also there exists $S \subset I$ such that $S \supset E$, S is a finite union of disjoint closed intervals, and

$$m(S-E) \leqq \tfrac{1}{6} K^{-1} mF(E \cap I_0, a). \tag{9}$$

If $x \in [a, b]$ then (7) and (9) imply

$$\begin{aligned} mF(S \cap I_0, x) - mF(E \cap I_0, x) &\leqq Km\{(S-E) \cap I_0\} \\ &\leqq \tfrac{1}{6} mF(E \cap I_0, a), \\ mF(E \cap I_0, x) &\geqq mF(S \cap I_0, x) - \tfrac{1}{6} mF(E \cap I_0, a). \end{aligned}$$

But since $F(S \cap I_0, x)$ is a finite sum of closed intervals, $mF(S \cap I_0, x)$ is continuous at $x = a$. Hence there exists $\delta_1 \in (0, b-a)$ such that if $x \in [a, a+\delta_1]$ then

$$mF(S \cap I_0, x) \geqq mF(S \cap I_0, a) - \tfrac{1}{6} mF(E \cap I_0, a)$$
$$\geqq \tfrac{5}{6} mF(E \cap I_0, a),$$
$$(10) \qquad mF(E \cap I_0, x) \geqq \tfrac{2}{3} mF(E \cap I_0, a).$$

Similarly there exists $\delta \in (0, \delta_1)$ such that if $x \in [a, a+\delta]$ then $F(I_0, x) \cap F(I_0, a)$ is not empty and

$$m\{F(I_0, x) \cup F(I_0, a)\} < \tfrac{10}{9} mF(I_0, a),$$
$$(11) \qquad m\{F(E \cap I_0, x) \cup F(E \cap I_0, a)\} < \tfrac{10}{9} mF(I_0, a).$$

Let $x \in [a, a+\delta]$. If $F(E \cap I_0, a) \cap F(E \cap I_0, x)$ is empty then from (10) and (8)

$$m\{F(E \cap I_0, a) \cup F(E \cap I_0, x)\} = mF(E \cap I_0, a) + mF(E \cap I_0, x)$$
$$\geqq \tfrac{5}{3} mF(E \cap I_0, a)$$
$$\geqq \tfrac{10}{9} mF(I_0, a)$$

which contradicts (11). Hence the intersection is not empty and there exist $s, t \in E \cap I_0$ such that $F(s, a) = F(t, x)$; (1) and $f(a) = A$ then imply

$$H(f(s), A; s, a) = H(f(t), f(x); t, x),$$
$$f(x) = G\{f(t), H(f(s), A; s, a); t, x\}.$$

The continuity of H and G imply that f is bounded on $[a, a+\delta]$. Also from (5)

$$f(\alpha_1^k(x)) = G(A, h\{f(\alpha_1^{k+1}(x)); \alpha_1^{k+1}(x)\}; a, \alpha_1^k(x)), x \in I, \qquad k = 0, 1, \cdots.$$

Hence $f(x)$ can be expressed as a continuous function of x and $f(\alpha_1^n(x))$, $n = 1, 2, \cdots$. Since $\alpha_1^n(b) \to a$ as $n \to \infty$, there exists an integer n such that $\alpha_1^n([a, b]) \subset [a, a+\delta]$. This implies that f is bounded on $[a, b]$ which is sufficient for the proof of the theorem.

THEOREM 3. *Let I be a convex subset of R^n. Let H: $D \times D \times I \times I \to \Delta$ be such that* (H_1) *and* (H_2) *hold.*

Let $\alpha, \beta \in R$, $\alpha\beta > 0$, and $\gamma \in R^n$. Let $a_0, \cdots, a_n$ be the vertices of a non-degenerate simplex in I. Then corresponding to any given $f(a_0), \cdots, f(a_n)$, there is at most one pair of functions $f : I \to D$ and

$$g : \{\alpha x + \beta y + \gamma : x, y \in I\} \to \Delta$$

such that f is locally bounded in I and

$$(12) \qquad g(\alpha x + \beta y + \gamma) = H(f(x), f(y); x, y).$$

PROOF. Let f and g satisfy the given hypotheses. Let S be the set of points in I where f is uniquely determined; then $a_0, \cdots, a_n \in S$. Let $x, y \in S$ and xy denote the straight line through x and y:

$$xy = \{x+s(y-x) : s \in R\}.$$

Then there exists a real interval $\mathscr{I}$ such that

$$xy \cap I = \{x+s(y-x) : s \in \mathscr{I}\}.$$

Let $f_1(s) = f(x+s(y-x)),\ s \in \mathscr{I}$,

$$g_1(\alpha s+\beta t) = g[(\alpha s+\beta t)(y-x)+\alpha x+\beta x+\gamma],\ s, t \in \mathscr{I}.$$

Then

$$g_1(\alpha s+\beta t) = H[f_1(s), f_1(t); x+s(y-x), x+t(y-x)].$$

But f_1 is locally bounded in $\mathscr{I}$, $f_1(0) = f(x)$ and $f_1(1) = f(y)$. By theorem 1 f_1 is uniquely determined in $\mathscr{I}$. Hence $xy \cap I \subset S$ and in particular S is convex. Let $z \in I$ and M be the linear manifold spanned by S. Then $M = R^n$ and $z \in M$. Hence (see [6] p. 16, Ex. 1.5.1) some straight line through z meets S in a non-degenerate segment, hence in at least two points. Hence $z \in S$ and $I = S$.

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