Note on an Abstract Inversion Principle

By L. C. Hsu

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In a recent paper [1] of Bell, an abstract inversion principle has been formulated for inverting a type of finite series by employing operators. Bell's result involves Baker's general principle of cross-classification [2], Dedekind-Möbius inversion, L.C.M. inversion and some known generalisations. The purpose of this note is to introduce operators of negative degree and to formulate an inversion principle which covers more cases than Bell's.

1. With Bell's notation, X_t denotes a subset of precisely t distinct elements taken from an abstract set $\{x_1, x_2, \ldots, x_n\}$, and X'_{n-t} is the complementary subset; $f_t(X_t, X'_{n-t})$, $F_t(X_t, X'_{n-t})$ denote arbitrary single-valued functions of (X_t, X'_{n-t}) whose values belong to a module M. $P(T)$ denotes an arbitrary polynomial in operators T_s which are defined by

$$T_s f_t(X_t, X'_{n-t}) = \begin{cases} f_{t-1}(X_{t-1}, X'_{n-t+1}) & \text{if } x_s \varepsilon X_t, \\ 0 & \text{otherwise,} \end{cases}$$

where X_{t-1} is the result of deleting x_s from X_t, and T_0 is the identity operation. The product $T_t T_s$ means operating first with T_s and then with T_t. Obviously, both the commutative law and the associative law hold for T operators, and for $s > 0$

$$T_s^2 f_t(X_t, X'_{n-t}) = 0 \;\; (0 \text{ in } M), \quad T_s^2 = O \;\; (\text{zero operator}).$$

2. Whenever $x_s \varepsilon X_t$, Bell's operator T_s is just an operation of transferring an element from X_t into X'_{n-t}. Thus it is also possible and quite natural to introduce operators of negative degree by the following reverse relation:

$$T_s^{-1} f_t(X_t, X'_{n-t}) = \begin{cases} f_{t+1}(X_{t+1}, X'_{n-t-1}) & \text{if } x_s \varepsilon X'_{n-t}, \\ 0 & \text{otherwise.} \end{cases}$$

Here T_s^{-1} means transferring the element x_s from X'_{n-t} back to X_t whenever $x_s \varepsilon X'_{n-t}$. In particular, $T_0^{-1} = T_0$, $T_s^{-1} T_s^{-1} = T_s^{-2} = O$ $(s > 0)$.

Now evidently the commutative law does not generally hold, *e.g.* for $s>0$, $T_s^{-1}T_s \neq T_s T_s^{-1}$. But except in the case just mentioned, the commutative law always holds. The fact that the associative law generally holds is easily seen.

Let $P(T)$ and $Q(T)$ be any two polynomials in operators with coefficients in a commutative ring C. If T_s and T_s^{-1} (with coefficients in C) appear in $P(T)$ and $Q(T)$ respectively, then we say that there is a pair in P, Q, where s is called the index of T_s or T_s^{-1}. The following commutative law may sometimes be useful:

If there is no pair occurring in P, Q, then we always have

$$P(T)\,Q(T)=Q(T)\,P(T).$$

3. Let $P(T)$ be an arbitrary polynomial in operators T_s, T_s^{-1}, etc., with coefficients in C. We shall say that $P^{-1}(T)$ *is the inverse* (*polynomial*) *of* $P(T)$ if and only if

$$P(T)\,P^{-1}(T)=T_0.$$

Simple examples are[1]:

(i) $P(T)=T_s+T_s^{-1}$, $\quad P^{-1}(T)=T_s+T_s^{-1} \quad (s>0)$.

(ii) $P_1(T)=T_0+T_1+T_2^{-1}+T_3^{-1}$,

$$P_1^{-1}(T)=T_0-T_1-T_2^{-1}-T_3^{-1}+2(T_1T_2^{-1}+T_1T_3^{-1}+T_2^{-1}T_3^{-1}) \\ -6T_1T_2^{-1}T_3^{-1}.$$

$P(T)$ may have no inverse, *e.g.*

$$(T_0+T_1+T_1^{-1})(T_0-T_1-T_1^{-1})=T_0^2-(T_1+T_1^{-1})^2=T_0-T_0=O$$

so that both factors on the left-hand side are zero divisors having no inverses.

In particular, if $P(T)=T_0+Q(T)$, where $Q(T)$ does not contain T_0 and the indexes of all T's in $Q(T)$ are distinct, then the inverse polynomial $P^{-1}(T)$ always exists and may easily be determined by expanding $\big(T_0+Q(T)\big)^{-1}$ formally by the binomial theorem. In fact, in the present case both the associative law and the commutative law do hold.

4. Now Bell's abstract inversion principle can easily be extended to the following form:

Let $P(T)=P(T_1, T_1^{-1}, \ldots, T_n, T_n^{-1})$ be an arbitrary polynomial operator such that its inverse $P^{-1}(T)=\big(P(T_1, T_1^{-1}, \ldots, T_n, T_n^{-1})\big)^{-1}$ exists. Then

[1] Example (i) follows immediately from the fact that $T_s^{-1}T_s+T_sT_s^{-1}=T_0$ $(s>0)$.

the following two sets of equations are equivalent:

$$F_t(X_t, X'_{n-t}) = P(T) f_t(X_t, X'_{n-t}) \quad (t = 0, 1, \ldots, n), \tag{4.1}$$

$$f_t(X_t, X'_{n-t}) = P^{-1}(T) F_t(X_t, X'_{n-t}) \quad (t = 0, 1, \ldots, n). \tag{4.2}$$

To see that the extension is genuine, it suffices to observe the simple inversion

$$F_t(X_t, X'_{n-t}) = \left\{ \prod_{s=1}^{n} (T_s + T_s^{-1}) \right\} f_t(X_t, X'_{n-t});$$

$$f_t(X_t, X'_{n-t}) = \left\{ \prod_{s=1}^{n} (T_s + T_s^{-1}) \right\} F_t(X_t, X'_{n-t}),$$

which is obviously not obtainable from the original inversion principle of Baker and Bell.

Note that in our case the commutative law has been sacrificed, so that the relation $P^{-1}(T) P(T) = P(T) P^{-1}(T) = T_0$ requires justification. Evidently (4.1) is obtainable by operating with $P(T)$ on both sides of (4.2) and using the definition of $P^{-1}(T)$. Since (4.2) is a system of 2^n simultaneous equations having the solution (inverse) (4.1) it is clear that the corresponding matrices of (4.2) and of (4.1) are non-singular. Thus it follows that (4.2) must be also the unique solution of (4.1). This shows that (4.1) and (4.2) are equivalent, and consequently

$$P^{-1} P = P P^{-1} = T_0.$$

5. The inversion principle may easily be further extended by considering k subsets of X_n $(k \geqslant 2)$ and defining operators of transference between each pair of subsets. Finally, it may be worthy of mention that P. Hall's enumeration principle [3] or its special case Weisner's inversion formulae [4] cannot be deduced from the abstract inversion principle here discussed. On the other hand, Hall's inversion formula also does not include the type of inversion considered in this note. Detailed explanations will be omitted here.

REFERENCES.

[1] Bell, E. T., *Duke Math. Journal*, 15 (1948), 79–85.

[2] Baker, H. F., *Proc. London Math. Soc.*, 21 (1891), 30–32.

[3] Hall, P., *Quarterly Journal of Math.* (Oxford), 7 (1936), 134–151.

[4] Weisner, L., *Trans. American Math. Soc.*, 38 (1935), 485–492, 474–484.

MAGDALENE COLLEGE,
CAMBRIDGE,