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Rational Homogeneous Algebras

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Abstract. An algebra A is *homogeneous* if the automorphism group of A acts transitively on the one-dimensional subspaces of A. The existence of homogeneous algebras depends critically on the choice of the scalar field. We examine the case where the scalar field is the rationals. We prove that if A is a rational homogeneous algebra with $\dim A > 1$, then $A^2 = 0$.

1 Introduction

The algebras to be discussed are assumed to be finite dimensional over a field $\mathbf{k}$ and are not necessarily associative. We let $\operatorname{Aut}(A)$ denote the group of algebra automorphisms of A. Then A is *homogeneous* if $\operatorname{Aut}(A)$ acts transitively on the one-dimensional subspaces of A. This is a very strong condition indeed and the only known examples fall into two easily described classes.

The existence of homogeneous algebras depends critically on the choice of $\mathbf{k}$, the field of scalars, and a number of results are known classifying these algebras according to the scalar field. Kostrikin [6] showed how to construct homogeneous algebras of any dimension over the finite field $GF(2)$. Work by Shult [9], Gross [4], and Ivanov [5] showed that if $\mathbf{k}$ is finite, then there are no algebras other than those constructed by the method of Kostrikin. Djoković [1] completely classified homogeneous algebras over the reals and found only three examples, one each in dimensions 3, 6, and 7. In a recent paper [3], the automorphism groups of these algebras are determined.

The first general study of homogeneous algebras was carried out by Sweet [11], and subsequently the authors [8, 12] have completely classified the non-trivial algebras of dimensions 2, 3, and 4 over any field. There it has been shown that no examples exist other than those found by Kostrikin and by Djoković. Subsequently, motivated by the examples over the reals, Djoković and Sweet [2] have shown that when the field is infinite, all non-trivial homogeneous algebras satisfy $x^2 = 0$ for all $x \in A$, and hence are anti–commutative. It was shown by Sweet [10] that there are no non-trivial examples whatsoever when the scalar field is algebraically closed. In this paper we investigate the opposite extreme, namely, the case where the scalar field is $\mathbb{Q}$. We show that for every dimension greater than 1, a rational homogeneous algebra A is trivial in the sense that $A^2 = 0$.

2 Results

The proof of our main result depends on a rather technical property of rational polynomials. This result follows from a more general theorem due to [7], but we give an

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independent proof that is completely elementary.

Theorem 1 *Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and $g(x) = b_n x^n$ be rational polynomials such that $f(\mathbb{Q}) \subseteq g(\mathbb{Q})$. If a_n/b_n is an n-th power in $\mathbb{Q}$, then $f(x) = c(ax+b)^n$ for suitable a, b, and c in $\mathbb{Q}$.*

Proof The hypothesis is equivalent to the statement that

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = b_n y^n$$

is solvable for all $x \in \mathbb{Q}$, where y depends on x. Divide both sides by b_n and replace $\sqrt[n]{a_n/b_n}x$ by w to get

$$w^n + a'_{n-1}w^{n-1} + \cdots + a'_1 w + a'_0 = y^n.$$

Let d be the product of the denominators of $a'_{n-1}, \ldots, a'_0$ and multiply both sides by d^n. Finally replace dw by X and dy by Y to get

$$F(X) = X^n + A_{n-1}X^{n-1} + \cdots + A_0 = Y^n,$$

where the coefficients are now integers. This equation must be solvable for all integers X. Clearly Y must also be an integer. We write A_{n-1} as $A_{n-1} = nk + r$ for $0 \le r < n$. Also let

$$G(X) = (X+k)^n = X^n + B_{n-1}X^{n-1} + \cdots + B_0.$$

We wish to show that $F(X) = G(X)$. If $F(X) \neq G(X)$, then consider the largest value of i for which $A_i \neq B_i$. Either $A_i > B_i$ or $A_i < B_i$ and we consider the two possibilities separately.

Case 1: Assume $A_i > B_i$. Then there exists $X_1 \in \mathbb{Z}$ such that $F(X) > G(X)$ for all $X > X_1$. Let

$$L(X) = (X + (k+1))^n = X^n + C_{n-1}X^{n-1} + \cdots + C_0.$$

Then $C_{n-1} = n(k+1) > A_{n-1}$, and so there exists $X_2 \in \mathbb{Z}$ so $L(X) > F(X)$ for $X > X_2$. Now for X_0 larger than both X_1 and X_2, we must have

$$(X_0 + k)^n = G(X_0) < F(X_0) < L(X_0) = (X_0 + (k+1))^n,$$

which is impossible.

Case 2: Assume $A_i < B_i$. Then there exists $X_1 \in \mathbb{Z}$ such that $F(X) < G(X)$ for all $X > X_1$. Let

$$L(X) = (X + (k-1))^n = X^n + C_{n-1}X^{n-1} + \cdots + C_0.$$

Then $C_{n-1} = n(k-1) < A_{n-1}$, and so there exists $X_2 \in \mathbb{Z}$ so that $L(X) < F(X)$ for $X > X_2$. Again if we choose X_0 larger than both X_1 and X_2, we have

$$(X_0 + (k-1))^n = L(X_0) < F(X_0) < G(X_0) = (X_0 + k)^n,$$

which is impossible.

Therefore $F(X) = G(X) = (X+k)^n$ and the result follows easily by undoing the substitutions. ■

Theorem 2 *Let V be a vector space of rational $n \times n$ matrices with $\dim V > 1$. If all the nonzero matrices in V are projectively similar, then they are similar and nilpotent.*

Proof Let A and B be any two independent matrices in V having characteristic polynomials $\sum_{i=0}^{n} a_{n-i}\lambda^i$ and $\sum_{i=0}^{n} b_{n-i}\lambda^i$, respectively. Note the ordering of the coefficients so that, for example, a_k is up to sign the sum of the principal $k \times k$ minors of A. Now for any $x \in \mathbb{Q}$ let $p(\lambda) = \sum_{i=0}^{n} c_{n-i}\lambda^i$ be the characteristic polynomial of $A + xB$ where now each c_k depends on x. In fact, c_k is a polynomial of degree $\leq k$ in x; we write $c_k = f(x) = \sum_{i=0}^{k} d_i x^i$.

For any $x \in \mathbb{Q}$, $A + xB$ is projectively similar to B. So there exists a $\mu_x \in \mathbb{Q}$ so that $A + xB$ is similar to $\mu_x B$. This implies that $c_k = (\mu_x)^k b_k$. Assume for contradiction that $b_k \neq 0$ and let $g(x) = b_k x^k$. Now note that $f(\mathbb{Q}) \subseteq g(\mathbb{Q})$ and $d_k = b_k$, and so $d_k/b_k = 1 \in \mathbb{Q}$. Now Theorem 1 implies that $f(x) = c(ax+b)^k$ for suitable choices of $a, b, c \in \mathbb{Q}$. Since $d_k = b_k \neq 0$, f is a polynomial of degree k and so $a \neq 0$. But then $f(-b/a) = 0$ and so for $x = -b/a$, $c_k = 0$ which implies that $b_k = 0$ which is a contradiction. Hence $b_k = 0$. But since k was an arbitrary integer, $0 < k \leq n$, this implies that B is nilpotent. But all the nonzero matrices in V are projectively similar to B and so all the matrices in V are nilpotent. Finally it is well known that if λ is a nonzero scalar and A is any nilpotent matrix, then λA is similar to A. So all the nonzero matrices in V are in fact similar. ■

Theorem 3 *Let A be a rational homogeneous algebra. If $\dim A > 1$, then $A^2 = 0$.*

Proof If $a \in A \backslash \{0\}$, then define $L_a \colon A \to A$ as $L_a(x) = ax$. Let $\mathcal{L} = \{L_a \mid a \in A\}$. For independent $a, b \in A \setminus \{0\}$ there exists an automorphism $\alpha \in \operatorname{Aut}(A)$ such that $\alpha(a) = \mu b$ for some $\mu \in \mathbb{Q}$. This implies that

$$\alpha L_a = \mu L_b \alpha$$

and so L_a and L_b are projectively similar. Assume that $L_x \neq 0$ for some $x \in A$. Then $\mathcal{L}$ is a space of rational projectively similar matrices, and so Theorem 2 implies that every L_a in $\mathcal{L}$ is nilpotent. Now [13, Corollary 2.3] implies that

$$A = \ker L_a \oplus \operatorname{Im} L_a.$$

But this is impossible since L_a is nilpotent. Hence $L_a = 0$ and so $A^2 = 0$. ■

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